Solutions to Problem Set 14

ECON 772001 - Math for Economists Peter Ireland
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Practice for the Final – Not Collected or Graded

1. Stochastic Linear-Quadratic Dynamic Programming

The problem is to choose contingency plans for a flow variable z_t for all $t = 0, 1, 2, ...$ and a stock variable y_t for all $t = 1, 2, 3, ...$ to maximize the objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t (R y_t^2 + Q z_t^2),$$

subject to the constraints y_0 given and

$$A y_t + B z_t + \varepsilon_{t+1} \geq y_{t+1} \tag{1}$$

for all $t = 0, 1, 2, ...$ and all possible realizations of ε_{t+1}, where $0 < \beta < 1$, $R < 0$, $Q < 0$, A, and B are all constant, known parameters and ε_{t+1} is an independently and identically distributed random shock that satisfies $E_t(\varepsilon_{t+1}) = 0$ and $E_t(\varepsilon_{t+1}^2) = \sigma^2$, that is, which has zero mean and variance σ^2.

a. Using the guess that the value function for this problem depends only on y_t and not ε_t and takes the specific form

$$v(y_t, \varepsilon_t) = v(y_t) = P y_t^2 + d,$$

where P and d are unknown constants, the Bellman equation for this problem becomes

$$\begin{aligned} P y_t^2 + d &= \max_{z_t} R y_t^2 + Q z_t^2 + \beta P E_t[(A y_t + B z_t + \varepsilon_{t+1})^2] + \beta d \\ &= \max_{z_t} R y_t^2 + Q z_t^2 + \beta P (A y_t + B z_t)^2 + \beta P \sigma^2 + \beta d \end{aligned}$$

b. Using the expression on the last line from part (a), the first-order condition for z_t becomes

$$2 Q z_t + 2\beta B P (A y_t + B z_t) = 0$$

and the envelope condition for y_t becomes

$$2 P y_t = 2 R y_t + 2\beta A P (A y_t + B z_t).$$

Notably, the introduction of uncertainty into the linear-quadratic problem does not alter the form of the optimality conditions for z_t and y_t. This special feature of the stochastic LQ model is often referred to as the property of "certainty equivalence," and it does not apply more generally for problems that do not have the linear-quadratic structure.

c. Rewrite the first-order condition as

$$z_t = -\left(\frac{\beta ABP}{Q+\beta B^2 P}\right) y_t$$

and substitute this result into the envelope condition to obtain

$$Py_t = Ry_t + \beta A^2 P y_t - \left(\frac{\beta^2 A^2 B^2 P^2}{Q+\beta B^2 P}\right) y_t$$

or, after combining the last two terms and dividing through by y_t,

$$P = R + \frac{\beta A^2 QP}{Q+\beta B^2 P}.$$

which is the same Riccati equation that helps characterize the problem's solution in the nonstochastic case, and therefore another reflection of the certainly equivalent property.

d. Substituting the first-order condition for z_t back into the Bellman equation yields

$$Py_t^2 + d = Ry_t^2 + Q\left(\frac{\beta ABP}{Q+\beta B^2 P}\right)^2 y_t^2 + \beta P\left[A - B\left(\frac{\beta ABP}{Q+\beta B^2 P}\right)\right]^2 y_t^2 + \beta P\sigma^2 + \beta d.$$

However, the Riccati equation implies that all of the terms involving y_t^2 cancel out, leaving

$$d = \beta P\sigma^2 + \beta d,$$

which can be solved for

$$d = \left(\frac{\beta}{1-\beta}\right) P\sigma^2.$$

2. Stochastic Growth

The problem is to choose contingency plans for consumption c_t for all $t = 0, 1, 2, ...$ and physical capital k_t for all $t = 1, 2, 3, ...$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

with $0 < \beta < 1$, subject to the constraints k_0 given and

$$z_t k_t^{\alpha} \geq c_t + k_{t+1}, \tag{2}$$

for all $t = 0, 1, 2, ...$ and all possible realizations of z_t, where $E_t[\ln(z_{t+1})] = 0$ for all $t = 0, 1, 2, ...$

a. Using the guess that the value function for this problem takes the form

$$v(k_t, z_t) = E + F\ln(k_t) + G\ln(z_t),$$

where E, F, and G are unknown constants, the Bellman equation for this problem can be written as

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(z_t k_t^\alpha - c_t),$$

where the term involving $GE_t[\ln(z_{t+1})]$ drops out of the right-hand-side of the Bellman equation in light of the assumption that $E_t[\ln(z_{t+1})] = 0$.

b. Using the Bellman equation as it appears above, the first-order condition for c_t is

$$\frac{1}{c_t} - \frac{\beta F}{z_t k_t^\alpha - c_t} = 0$$

and the envelope condition for k_t is

$$\frac{F}{k_t} = \frac{\alpha\beta F z_t k_t^{\alpha-1}}{z_t k_t^\alpha - c_t}.$$

c. Rewrite the first-order condition as

$$c_t = \left(\frac{1}{1+\beta F}\right) z_t k_t^\alpha$$

and substitute this expression into the envelope condition to obtain

$$F z_t k_t^\alpha - F\left(\frac{1}{1+\beta F}\right) z_t k_t^\alpha = \alpha\beta F z_t k_t^\alpha,$$

which implies that

$$\frac{1}{1+\beta F} = 1 - \alpha\beta.$$

In light of this last expression, the first-order condition for c_t implies

$$c_t = (1-\alpha\beta) z_t k_t^\alpha$$

and the binding constraint for k_{t+1} implies

$$k_{t+1} = \alpha\beta z_t k_t^\alpha.$$

These last two expressions confirm that the key result from the perfect foresight case, that with complete depreciation it is optimal to consume the fixed fraction $1-\alpha\beta$ of output and to save the remaining fraction $\alpha\beta$, carries over to the stochastic case as well.

d. The expression
$$\frac{1}{1+\beta F} = 1 - \alpha\beta$$
from above implies that
$$F = \frac{\alpha}{1-\alpha\beta}.$$
Substituting this result, along with the solution for c_t, back into the Bellman equation yields
$$\begin{aligned} E + F\ln(k_t) + G\ln(z_t) &= \ln(1-\alpha\beta) + \ln(z_t) + \alpha\ln(k_t) \\ &\quad + \beta E + \beta F\ln(\alpha\beta) + \beta F\ln(z_t) + \alpha\beta F\ln(k_t). \end{aligned}$$
The solution for F implies that the terms involving k_t drop out of this last expression, leaving
$$E + G\ln(z_t) = \ln(1-\alpha\beta) + \ln(z_t) + \beta E + \beta F\ln(\alpha\beta) + \beta F\ln(z_t).$$
Since this last equation must hold for all possible realizations of z_t, it requires that
$$G\ln(z_t) = \ln(z_t) + \beta F\ln(z_t)$$
or, using the solution for F again,
$$G = \frac{1}{1-\alpha\beta}.$$
Hence, the Bellman equation also requires that
$$E = \ln(1-\alpha\beta) + \beta E + \beta F\ln(\alpha\beta)$$
or, using the solution for F one last time,
$$E = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta)\right].$$

3. Saving with a Random Return

The problem takes initial assets A_0 as given and chooses contingency plans for s_t for all $t = 0, 1, 2, ...$ and A_t for $t = 1, 2, 3, ...$ to maximize the expected utility function
$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t u(A_t - s_t),$$
with $0 < \beta < 1$, subject to the constraints
$$R_{t+1}s_t \geq A_{t+1},$$
which must hold for all $t = 0, 1, 2, ...$ and all possible realizations of R_{t+1}.

a. Allowing for the possibility of serial correlation in the random return on saving, the Bellman equation for this problem can be written as

$$v(A_t, R_t) = \max_{s_t} u(A_t - s_t) + \beta E_t[v(R_{t+1}s_t, R_{t+1})].$$

Write down the Bellman equation for this problem, using A_t as the state variable, s_t and the control variable, and allowing the value function for time t to depend on R_t as well as A_t.

b. Using the assumed form

$$u(A_t - s_t) = \frac{(A_t - s_t)^{1-\sigma}}{1-\sigma}$$

for the single-period utility function, the additional assumptions that the random interest rate R_{t+1} is independently and identically distributed with

$$E_t(R_{t+1}^{1-\sigma}) = 1$$

for all $t = 0, 1, 2, ...$, and the guess that under these conditions, the value function depends only on A_t and takes the specific form

$$v(A_t) = \frac{KA_t^{1-\sigma}}{1-\sigma},$$

where K is an unknown constant, the Bellman equation can be rewritten as

$$\frac{KA_t^{1-\sigma}}{1-\sigma} = \max_{s_t} \frac{(A_t - s_t)^{1-\sigma}}{1-\sigma} + \frac{\beta K s_t^{1-\sigma}}{1-\sigma}.$$

The first-order condition for s_t then becomes

$$-(A_t - s_t)^{-\sigma} + \beta K s_t^{-\sigma} = 0$$

and the envelope condition for A_t becomes

$$KA_t^{-\sigma} = (A_t - s_t)^{-\sigma}.$$

c. There are a number of ways to derive this result, but one is to start by rewriting the first-order condition for s_t as

$$s_t = \left[\frac{(\beta K)^{1/\sigma}}{1 + (\beta K)^{1/\sigma}}\right] A_t$$

and substitute this result into the envelope condition to obtain

$$KA_t^{-\sigma} = \left[\frac{1}{1 + (\beta K)^{1/\sigma}}\right]^{-\sigma} A_t^{-\sigma},$$

which can be solved for

$$K = \left(\frac{1}{1 - \beta^{1/\sigma}}\right)^{\sigma}.$$

d. Since the solution for K just derived implies that

$$(\beta K)^{1/\sigma} = \frac{\beta^{1/\sigma}}{1 - \beta^{1/\sigma}},$$

the first-order condition for s_t implies that the optimal choice for s_t is given by

$$s_t = \beta^{1/\sigma} A_t.$$

Hence, the optimal choice for c_t is

$$c_t = (1 - \beta^{1/\sigma}) A_t.$$