

Solutions to Problem Set 13

ECON 772001 - Math for Economists
Boston College, Department of Economics

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Human Capital Accumulation and Economic Growth

In this version of the Uzawa-Lucas model, the representative consumer or social planner chooses sequences $\{c_t\}_{t=0}^{\infty}$, $\{u_t\}_{t=0}^{\infty}$, $\{h_t\}_{t=1}^{\infty}$, and $\{k_t\}_{t=1}^{\infty}$, to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

subject to the constraints

$$\gamma u_t h_t \geq h_{t+1} \tag{1}$$

and

$$k_t^\alpha [(1 - u_t) h_t]^{1-\alpha} \geq c_t + k_{t+1} \tag{2}$$

for all $t = 0, 1, 2, \dots$, taking the initial stocks h_0 and k_0 as given.

1. The Bellman equation for the problem can be written as

$$v(k_t, h_t; t) = \max_{c_t, u_t} \ln(c_t) + \beta v(k_t^\alpha [(1 - u_t) h_t]^{1-\alpha} - c_t, \gamma u_t h_t; t + 1)$$

or, using the guess that the value function takes the time-invariant form

$$v(k_t, h_t; t) = v(k_t, h_t) = E + F \ln(k_t) + G \ln(h_t)$$

where E , F , and G are constants to be determined,

$$\begin{aligned} E + F \ln(k_t) + G \ln(h_t) &= \max_{c_t, u_t} \ln(c_t) + \beta E \\ &\quad + \beta F \ln(k_t^\alpha [(1 - u_t) h_t]^{1-\alpha} - c_t) + \beta G \ln(\gamma u_t h_t) \\ &= \max_{c_t, u_t} \ln(c_t) + \beta E + \beta F \ln(k_t^\alpha [(1 - u_t) h_t]^{1-\alpha} - c_t) \\ &\quad + \beta G \ln(\gamma) + \beta G \ln(u_t) + \beta G \ln(h_t). \end{aligned}$$

2. Continuing to use the form of the value function conjectured above, the first-order conditions for c_t and u_t can be written as

$$\frac{1}{c_t} - \frac{\beta F}{k_t^\alpha [(1 - u_t) h_t]^{1-\alpha} - c_t} = 0$$

and

$$-\frac{(1 - \alpha) \beta F k_t^\alpha (1 - u_t)^{-\alpha} h_t^{1-\alpha}}{k_t^\alpha [(1 - u_t) h_t]^{1-\alpha} - c_t} + \frac{\beta G}{u_t} = 0,$$

and the envelope conditions for k_t and h_t can be written as

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1} [(1-u_t)h_t]^{1-\alpha}}{k_t^\alpha [(1-u_t)h_t]^{1-\alpha} - c_t}$$

and

$$\frac{G}{h_t} = \frac{(1-\alpha)\beta F k_t^\alpha (1-u_t)^{1-\alpha} h_t^{-\alpha}}{k_t^\alpha [(1-u_t)h_t]^{1-\alpha} - c_t} + \frac{\beta G}{h_t}.$$

3. Together with the binding constraints (1) and (2), the four equations from above form a system of six equations in six unknowns: the unknown variables c_t , u_t , k_t , and h_t and the unknown constants F and G . Once all these unknowns have been solved for, they can be substituted back into the Bellman equation itself to solve for E . As the next step in characterizing the model's solution, rewrite the first-order condition for c_t as

$$c_t = \left(\frac{1}{1+\beta F} \right) k_t^\alpha [(1-u_t)h_t]^{1-\alpha}$$

and substitute this result into the envelope condition for k_t to solve for

$$\frac{1}{1+\beta F} = 1 - \alpha\beta$$

or

$$F = \frac{\alpha}{1-\alpha\beta}.$$

Next, substitute these expressions for c_t and F into the envelope condition for h_t and rearrange to obtain

$$\begin{aligned} & (1-\beta)Gk_t^\alpha [(1-u_t)h_t]^{1-\alpha} - (1-\beta)G(1-\alpha\beta)k_t^\alpha [(1-u_t)h_t]^{1-\alpha} \\ = & (1-\alpha)\beta \left(\frac{\alpha}{1-\alpha\beta} \right) k_t^\alpha [(1-u_t)h_t]^{1-\alpha} \end{aligned}$$

or, more simply,

$$G = \frac{1-\alpha}{(1-\beta)(1-\alpha\beta)}.$$

4. The solution for F , when substituted back into the first-order condition for c_t , implies that

$$c_t = (1-\alpha\beta)k_t^\alpha [(1-u_t)h_t]^{1-\alpha}.$$

Evidently, as in the more basic neoclassical growth model with complete depreciation of physical capital, it is optimal to set consumption as a fixed fraction $1-\alpha\beta$ of output. Substitute this solution for c_t into the first-order condition for u_t and rearrange to obtain

$$\begin{aligned} & (1-\alpha)\beta F k_t^\alpha [(1-u_t)h_t]^{1-\alpha} \left(\frac{u_t}{1-u_t} \right) \\ = & \beta G k_t^\alpha [(1-u_t)h_t]^{1-\alpha} - \beta G (1-\alpha\beta) k_t^\alpha [(1-u_t)h_t]^{1-\alpha}. \end{aligned}$$

Now, use the solutions for F and G to rewrite this last expression more simply as

$$\frac{u_t}{1 - u_t} = \frac{\beta}{1 - \beta},$$

which leads to the solution

$$u_t = \beta.$$

Evidently, it is also optimal to allocate a fixed fraction β of time to human capital accumulation and the remaining fraction $1 - \beta$ to the production of goods and services.

5. Substituting the solutions for c_t and u_t into the binding constraints yields the system of difference equations consisting of

$$h_{t+1} = \beta\gamma h_t$$

and

$$k_{t+1} = \alpha\beta(1 - \beta)^{1-\alpha} k_t^\alpha h_t^{1-\alpha}.$$

6. The first difference equation shown above implies that the stock of human capital expands or contracts at the constant rate $\beta\gamma$. If, in particular, the parameters are such that $\beta\gamma > 1$, then human capital grows over time at a constant rate and, as discussed below, the entire economy experiences sustained economic growth.

7. Using the solutions for c_t and u_t , the Bellman equation can be written as

$$\begin{aligned} E + F \ln(k_t) + G \ln(h_t) &= \ln(1 - \alpha\beta) + \alpha \ln(k_t) + (1 - \alpha) \ln(1 - \beta) \\ &\quad + (1 - \alpha) \ln(h_t) + \beta E + \beta F \ln(\alpha\beta) + \alpha\beta F \ln(k_t) \\ &\quad + (1 - \alpha)\beta F \ln(1 - \beta) + (1 - \alpha)\beta F \ln(h_t) \\ &\quad + \beta G \ln(\gamma) + \beta G \ln(\beta) + \beta G \ln(h_t). \end{aligned}$$

And using the solutions for F and G , this expression can be simplified to read

$$\begin{aligned} E &= \ln(1 - \alpha\beta) + (1 - \alpha) \ln(1 - \beta) + \beta E \\ &\quad + \left(\frac{\alpha\beta}{1 - \alpha\beta} \right) [\ln(\alpha\beta) + (1 - \alpha) \ln(1 - \beta)] \\ &\quad + \left[\frac{\beta(1 - \alpha)}{(1 - \beta)(1 - \alpha\beta)} \right] [\ln(\gamma) + \ln(\beta)], \end{aligned}$$

which can be solved for

$$\begin{aligned} E &= \left(\frac{1}{1 - \beta} \right) [\ln(1 - \alpha\beta) + (1 - \alpha) \ln(1 - \beta)] \\ &\quad + \left[\frac{\alpha\beta}{(1 - \beta)(1 - \alpha\beta)} \right] [\ln(\alpha\beta) + (1 - \alpha) \ln(1 - \beta)] \\ &\quad + \left[\frac{\beta(1 - \alpha)}{(1 - \beta)^2(1 - \alpha\beta)} \right] [\ln(\gamma) + \ln(\beta)], \end{aligned}$$

The results from above can be collected together to form a system of four equations,

$$c_t = (1 - \alpha\beta)(1 - \beta)^{1-\alpha} k_t^\alpha h_t^{1-\alpha},$$

$$u_t = \beta,$$

$$h_{t+1} = \beta\gamma h_t,$$

and

$$k_{t+1} = \alpha\beta(1 - \beta)^{1-\alpha} k_t^\alpha h_t^{1-\alpha},$$

which can be used to construct the sequences $\{c_t\}_{t=0}^\infty$, $\{u_t\}_{t=0}^\infty$, $\{h_t\}_{t=1}^\infty$, and $\{k_t\}_{t=1}^\infty$ that solve this problem for any given initial conditions k_0 and h_0 . This system of difference equations implies that as t grows larger, the growth rates of consumption and physical capital converge to the same constant growth rate $\beta\gamma$ of human capital. An excel spreadsheet for performing these calculations and verifying this convergence result is available through the course webpage at <http://irelandp.com/econ7720.html>.