**Solutions to Problem Set 13**


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Practice for the Final – Not Collected or Graded

## Human Capital Accumulation and Economic Growth

In this version of the Uzawa-Lucas model, the representative consumer or social planner chooses sequences $\{c_t\}_{t=0}^{\infty}$, $\{u_t\}_{t=0}^{\infty}$, $\{h_t\}_{t=1}^{\infty}$, and $\{k_t\}_{t=1}^{\infty}$, to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

subject to the constraints

$$\gamma u_t h_t \geq h_{t+1} \tag{1}$$

and

$$k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} \geq c_t + k_{t+1} \tag{2}$$

for all $t = 0, 1, 2, \ldots$, taking the initial stocks $h_0$ and $k_0$ as given.

1. The Bellman equation for the problem can be written as

$$v(k_t, h_t; t) = \max_{c_t, u_t} \ln(c_t) + \beta v(k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - c_t, \gamma u_t h_t; t+1)$$

or, using the guess that the value function takes the time-invariant form

$$v(k_t, h_t; t) = v(k_t, h_t) = E + F\ln(k_t) + G\ln(h_t)$$

where $E$, $F$, and $G$ are constants to be determined,

$$\begin{aligned} E + F\ln(k_t) + G\ln(h_t) &= \max_{c_t, u_t} \ln(c_t) + \beta E \\ &\quad + \beta F \ln(k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - c_t) + \beta G \ln(\gamma u_t h_t) \\ &= \max_{c_t, u_t} \ln(c_t) + \beta E + \beta F \ln(k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - c_t) \\ &\quad + \beta G \ln(\gamma) + \beta G \ln(u_t) + \beta G \ln(h_t). \end{aligned}$$

2. Continuing to use the form of the value function conjectured above, the first-order condition for $c_t$ and $u_t$ can be written as

$$\frac{1}{c_t} - \frac{\beta F}{k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - c_t} = 0$$

and

$$-\frac{(1-\alpha)\beta F k_t^{\alpha}(1-u_t)^{-\alpha} h_t^{1-\alpha}}{k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - c_t} + \frac{\beta G}{u_t} = 0,$$

and the envelope conditions for $k_t$ and $h_t$ can be written as

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1}[(1-u_t)h_t]^{1-\alpha}}{k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - c_t}$$

and

$$\frac{G}{h_t} = \frac{(1-\alpha)\beta F k_t^{\alpha}(1-u_t)^{1-\alpha}h_t^{-\alpha}}{k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - c_t} + \frac{\beta G}{h_t}.$$

3. Together with the binding constraints (1) and (2), the four equations from above form a system of six equations in six unknowns: the unknown variables $c_t$, $u_t$, $k_t$, and $h_t$ and the unknown constants $F$ and $G$. Once all these unknowns have been solved for, they can be substituted back into the Bellman equation itself to solve for $E$. As the next step in characterizing the model's solution, rewrite the first-order condition for $c_t$ as

$$c_t = \left(\frac{1}{1+\beta F}\right) k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha}$$

and substitute this result into the envelope condition for $k_t$ to solve for

$$\frac{1}{1+\beta F} = 1 - \alpha\beta$$

or

$$F = \frac{\alpha}{1-\alpha\beta}.$$

Next, substitute these expressions for $c_t$ and $F$ into the envelope condition for $h_t$ and rearrange to obtain

$$\begin{aligned} & (1-\beta)G k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - (1-\beta)G(1-\alpha\beta)k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} \\ = \; & (1-\alpha)\beta\left(\frac{\alpha}{1-\alpha\beta}\right) k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} \end{aligned}$$

or, more simply,

$$G = \frac{1-\alpha}{(1-\beta)(1-\alpha\beta)}.$$

4. The solution for $F$, when substituted back into the first-order condition for $c_t$, implies that

$$c_t = (1-\alpha\beta)k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha}.$$

Evidently, as in the more basic neoclassical growth model with complete depreciation of physical capital, it is optimal to set consumption as a fixed fraction $1-\alpha\beta$ of output. Substitute this solution for $c_t$ into the first-order condition for $u_t$ and rearrange to obtain

$$\begin{aligned} & (1-\alpha)\beta F k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha}\left(\frac{u_t}{1-u_t}\right) \\ = \; & \beta G k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha} - \beta G(1-\alpha\beta)k_t^{\alpha}[(1-u_t)h_t]^{1-\alpha}. \end{aligned}$$

Now, use the solutions for $F$ and $G$ to rewrite this last expression more simply as

$$\frac{u_t}{1-u_t} = \frac{\beta}{1-\beta},$$

which leads to the solution

$$u_t = \beta.$$

Evidently, it is also optimal to allocate a fixed fraction $\beta$ of time to human capital accumulation and the remaining fraction $1-\beta$ to the production of goods and services.

5. Substituting the solutions for $c_t$ and $u_t$ into the binding constraints yields the system of difference equations consisting of

$$h_{t+1} = \beta\gamma h_t$$

and

$$k_{t+1} = \alpha\beta(1-\beta)^{1-\alpha}k_t^{\alpha}h_t^{1-\alpha}.$$

6. The first difference equation shown above implies that the stock of human capital expands or contracts at the constant rate $\beta\gamma$. If, in particular, the parameters are such that $\beta\gamma > 1$, then human capital grows over time at a constant rate and, as discussed below, the entire economy experiences sustained economic growth.

7. Using the solutions for $c_t$ and $u_t$, the Bellman equation can be written as

$$\begin{aligned} E + F\ln(k_t) + G\ln(h_t) \quad = \quad & \ln(1-\alpha\beta) + \alpha\ln(k_t) + (1-\alpha)\ln(1-\beta) \\ & + (1-\alpha)\ln(h_t) + \beta E + \beta F\ln(\alpha\beta) + \alpha\beta F\ln(k_t) \\ & + (1-\alpha)\beta F\ln(1-\beta) + (1-\alpha)\beta F\ln(h_t) \\ & + \beta G\ln(\gamma) + \beta G\ln(\beta) + \beta G\ln(h_t). \end{aligned}$$

And using the solutions for $F$ and $G$, this expression can be simplified to read

$$\begin{aligned} E \quad = \quad & \ln(1-\alpha\beta) + (1-\alpha)\ln(1-\beta) + \beta E \\ & + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)[\ln(\alpha\beta) + (1-\alpha)\ln(1-\beta)] \\ & + \left[\frac{\beta(1-\alpha)}{(1-\beta)(1-\alpha\beta)}\right][\ln(\gamma) + \ln(\beta)], \end{aligned}$$

which can be solved for

$$\begin{aligned} E \quad = \quad & \left(\frac{1}{1-\beta}\right)[\ln(1-\alpha\beta) + (1-\alpha)\ln(1-\beta)] \\ & + \left[\frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)}\right][\ln(\alpha\beta) + (1-\alpha)\ln(1-\beta)] \\ & + \left[\frac{\beta(1-\alpha)}{(1-\beta)^2(1-\alpha\beta)}\right][\ln(\gamma) + \ln(\beta)], \end{aligned}$$

The results from above can be collected together to form a system of four equations,

$$c_t = (1-\alpha\beta)(1-\beta)^{1-\alpha}k_t^\alpha h_t^{1-\alpha},$$

$$u_t = \beta,$$

$$h_{t+1} = \beta\gamma h_t,$$

and

$$k_{t+1} = \alpha\beta(1-\beta)^{1-\alpha}k_t^\alpha h_t^{1-\alpha},$$

which can be used to construct the sequences $\{c_t\}_{t=0}^\infty$, $\{u_t\}_{t=0}^\infty$, $\{h_t\}_{t=1}^\infty$, and $\{k_t\}_{t=1}^\infty$ that solve this problem for any given initial conditions $k_0$ and $h_0$. This system of difference equations implies that at $t$ grows larger, the growth rates of consumption and physical capital converge to the same constant growth rate $\beta\gamma$ of human capital. An excel spreadsheet for performing these calculations and verifying this convergence result is available through the course webpage at `http://www2.bc.edu/peter-ireland/econ7720.html`.