Solutions to Problem Set 12

ECON 772001 - Math for Economists
Boston College, Department of Economics

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1. Linear-Quadratic Dynamic Programming

The problem is to choose sequences $\{z_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ to maximize the objective function

$$\sum_{t=0}^{\infty} \beta^t (Ry_t^2 + Qz_t^2),$$

subject to the constraints y_0 given and

$$Ay_t + Bz_t \geq y_{t+1}$$

for all $t = 0, 1, 2, ...$, where β, R, Q, A, and B are constant, known parameters.

a. The Bellman equation for this problem is

$$v(y_t; t) = \max_{z_t} Ry_t^2 + Qz_t^2 + \beta v(Ay_t + Bz_t; t+1).$$

b. Now guess that the value function takes the quadratic, time-invariant form

$$v(y_t; t) = v(y_t) = Py_t^2,$$

where P is an unknown constant, allowing the Bellman equation to be specialized to read

$$Py_t^2 = \max_{z_t} Ry_t^2 + Qz_t^2 + \beta P(Ay_t + Bz_t)^2.$$

Using this guess, the first-order condition is for z_t is

$$2Qz_t + 2\beta BP(Ay_t + Bz_t) = 0$$

and the envelope condition for y_t is

$$2Py_t = 2Ry_t + 2\beta AP(Ay_t + Bz_t).$$

c. Rewrite the first-order condition as

$$z_t = -\left(\frac{\beta ABP}{Q + \beta B^2 P}\right) y_t$$

and substitute this result into the envelope condition to obtain

$$Py_t = Ry_t + \beta A^2 Py_t - \left(\frac{\beta^2 A^2 B^2 P^2}{Q + \beta B^2 P}\right) y_t$$

or, after combining the last two terms and dividing through by y_t,

$$P = R + \frac{\beta A^2 QP}{Q + \beta B^2 P},$$

which is the Riccati equation.

d. Rewriting the Riccati equation as

$$\beta B^2 P^2 + [(1 - \beta A^2)Q - \beta B^2 R]P - QR = 0$$

makes clear that the quadratic formula can be applied to solve for

$$P = \frac{-[(1 - \beta A^2)Q - \beta B^2 R] \pm \{[(1 - \beta A^2)Q - \beta B^2 R]^2 + 4\beta B^2 QR\}^{1/2}}{2\beta B^2}.$$

Plugging in the specific numerical values $\beta = 0.95$, $R = -0.01$ $Q = -1$, $A = 1.05$, and $B = -1$, the formula provides two possible solutions: $P = 0.0769$ and $P = -0.1368$. The negative value $P = -0.1368$ corresponds to the true solution to the original dynamic optimization problem.

2. Natural Resource Depletion

The social planner chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize a representative consumer's utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

subject to the constraints that the initial stock s_0 is given and that

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, ...$

a. The Bellman equation for this problem is

$$v(s_t; t) = \max_{c_t} \ln(c_t) + \beta v(s_t - c_t; t + 1).$$

b. Now guess that the value function takes the time-invariant form

$$v(s_t; t) = v(s_t) = E + F \ln(s_t),$$

where E and F are constants to be determined. Using this guess, the Bellman equation specializes to

$$E + F \ln(s_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(s_t - c_t),$$

the first-order condition for c_t becomes

$$\frac{1}{c_t} - \frac{\beta F}{s_t - c_t} = 0,$$

and the envelope condition for s_t becomes

$$\frac{F}{s_t} = \frac{\beta F}{s_t - c_t}.$$

c. Rewrite the first-order condition as

$$c_t = \left(\frac{1}{1+\beta F}\right) s_t,$$

and substitute this equation into the envelope condition to obtain

$$\frac{1}{1+\beta F} = 1 - \beta.$$

Solving for F then yields

$$F = \frac{1}{1-\beta}.$$

d. Substitute this solution for F back into the first-order condition yields

$$c_t = (1-\beta)s_t,$$

and substitute this equation back into the binding constraint to obtain

$$s_{t+1} = \beta s_t.$$

These last two equations can be used to construct the optimal sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ starting from any initial value of s_0.

e. Substituting these results back into the Bellman equation yields the solution

$$E = \frac{1}{1-\beta}\left[\ln(1-\beta) + \left(\frac{\beta}{1-\beta}\right)\ln(\beta)\right].$$