Solutions to Problem Set 11

ECON 772001 - Math for Economists | Peter Ireland
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1. Natural Resource Depletion

A social planner chooses continuously differentiable functions $c(t)$ and $s(t)$ for $t \in [0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t, \tag{1}$$

subject to $s(0) = s_0$ given and

$$-c(t) \geq \dot{s}(t) \tag{2}$$

for all $t \in [0, \infty)$.

a. The maximized Hamiltonian for this problem can be defined as

$$H(s(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) - \pi(t)c(t).$$

b. According to the maximum principle, the solution to the social planner's problem is characterized by the first-order condition

$$\frac{e^{-\rho t}}{c(t)} - \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_s(s(t), \pi(t); t) = 0$$

and

$$\dot{s}(t) = H_\pi(s(t), \pi(t); t) = -c(t).$$

c. Probably the easiest way to solve the pair of differential equations shown above is to use a guess-and-verify method. Accordingly, guess that those differential equations have solutions for the form

$$\pi(t) = \pi$$

and

$$s(t) = \left(\frac{1}{\pi\rho}\right) e^{-\rho t} + k,$$

where π and k are two constants that remain to be determined. The proposed solution for $\pi(t)$ implies that

$$\dot{\pi}(t) = 0,$$

verifying that solution works to satisfy the first differential equation. The proposed solution for $s(t)$ implies that

$$\dot{s}(t) = -\left(\frac{1}{\pi}\right) e^{-\rho t},$$

which using the first-order condition for $c(t)$ and the solution for $\pi(t)$ just obtained can be rewritten as

$$\dot{s}(t) = -c(t),$$

verifying that the second differential equation holds as well.

d. The general solutions for $\pi(t)$ and $s(t)$ from above can be combined to yield

$$\pi(T)s(T) = \left(\frac{1}{\rho}\right) e^{-\rho T} + \pi k,$$

which reveals that the transversality condition requires

$$\pi k = 0.$$

Meanwhile, the general solution for $s(t)$ coupled with the initial condition requires that

$$s(0) = \left(\frac{1}{\pi\rho}\right) + k = s_0.$$

Taken together, these last two equations pin down values for

$$\pi = \frac{1}{\rho s_0}$$

and

$$k = 0.$$

e. The specific solution

$$\pi(t) = \pi = \frac{1}{\rho s_0}$$

can be combined with the first-order condition for $c(t)$ to obtain the solution

$$c(t) = s_0 \rho e^{-\rho t},$$

which reveals that consumption of the resource is falling over time, a result that echoes the one that you obtained previously for the discrete-time case.

2. Investment with Adjustment Costs

The firm chooses continuously differentiable functions $I(t)$ and $K(t)$ for $t \in [0, \infty)$ to maximize

$$\int_0^\infty e^{-rt}[K(t)^\alpha - I(t) - (\phi/2)I(t)^2]\,\mathrm{d}t,$$

subject to

$$I(t) - \delta K(t) \geq \dot{K}(t). \tag{3}$$

for all $t \in [0, \infty)$, taking the initial capital stock $K(0) > 0$ as given.

a. The maximized Hamiltonian for this problem can be defined as

$$H(K(t),\pi(t);t) = \max_{I(t)} e^{-rt}[K(t)^\alpha - I(t) - (\phi/2)I(t)^2] + \pi(t)[I(t) - \delta K(t)].$$

b. According to the maximum principle, necessary conditions for the values of $I(t)$ and $K(t)$ that solve the firm's infinite-horizon problem include the first-order condition

$$-e^{-rt}[1+\phi I(t)] + \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_K(K(t),\pi(t);t) = -e^{-rt}\alpha K(t)^{\alpha-1} + \pi(t)\delta$$

and

$$\dot{K}(t) = H_\pi(K(t),\pi(t);t) = I(t) - \delta K(t).$$

c. There are many ways to proceed for this part of the problem, but one of them it to start by rearranging the first-order condition to read

$$e^{-rt}[1+\phi I(t)] = \pi(t).$$

Now differentiate both sides of this expression with respect to t to obtain

$$-re^{-rt}[1+\phi I(t)] + e^{-rt}\phi\dot{I}(t) = \dot{\pi}(t)$$

and use the first-order condition and the first differential equation to eliminate all references to $\pi(t)$ and $\dot{\pi}(t)$ on the right-hand side:

$$-re^{-rt}[1+\phi I(t)] + e^{-rt}\phi\dot{I}(t) = -e^{-rt}\alpha K(t)^{\alpha-1} + e^{-rt}[1+\phi I(t)]\delta.$$

Finally, divide through by e^{-rt} and rearrange to obtain

$$\dot{I}(t) = (1/\phi)\{(\delta+r)[1+\phi I(t)] - \alpha K(t)^{\alpha-1}\},$$

which can be combined with the second differential equation,

$$\dot{K}(t) = H_\pi(K(t),\pi(t);t) = I(t) - \delta K(t).$$

to form a two-equation system in the unknown functions $I(t)$ and $K(t)$.

d. In a steady-state, $\dot{I}(t) = \dot{K}(t) = 0$, $I(t) = I^*$, and $K(t) = K^*$. Hence the differential equations just derived require that

$$(\delta+r)(1+\phi I^*) = \alpha(K^*)^{\alpha-1}$$

and

$$I^* = \delta K^*.$$

e. The differential equation for $K(t)$ implies that

$$\dot{K}(t) = 0 \text{ when } I(t) = \delta K(t),$$

$$\dot{K}(t) > 0 \text{ when } I(t) > \delta K(t),$$

and

$$\dot{K}(t) < 0 \text{ when } I(t) < \delta K(t).$$

The differential equation for $I(t)$ implies that

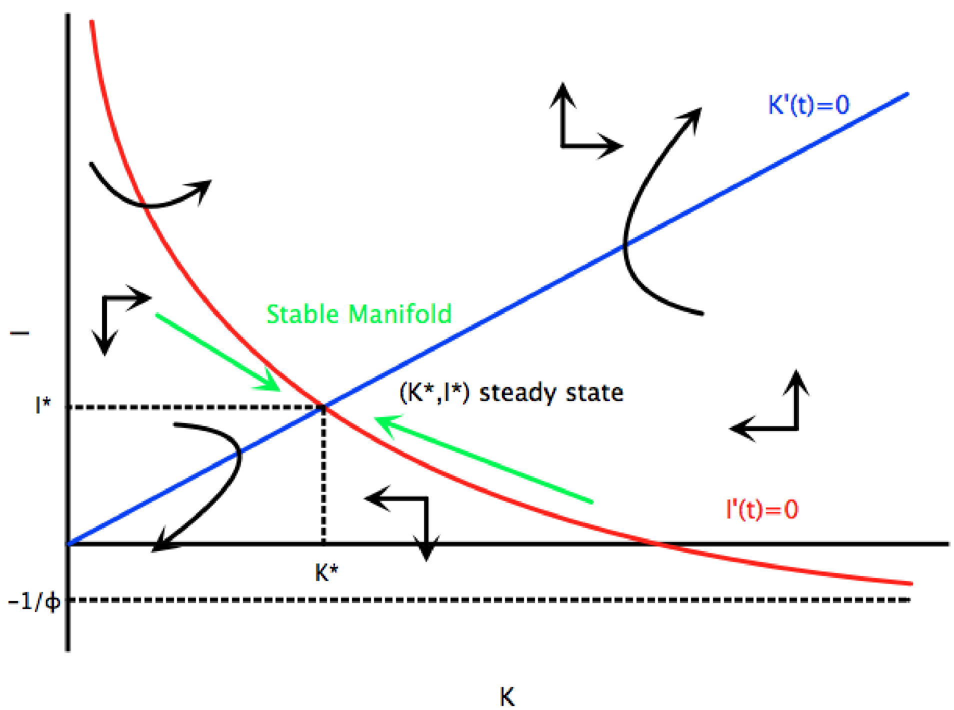
$$\dot{I}(t) = 0 \text{ when } I(t) = \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right],$$

$$\dot{I}(t) > 0 \text{ when } I(t) > \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right],$$

and

$$\dot{I}(t) < 0 \text{ when } I(t) < \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right].$$

The phase diagram shown below illustrates these conditions and also reveals that starting from any value $K(0) > 0$ for the initial capital stock, there is a unique value of investment $I(0)$ such that, starting from $I(0)$ and $K(0)$, the optimally-chosen paths for $I(t)$ and $K(t)$ converge to the steady-state values I^* and K^*.



Strictly speaking, it still remains to show that only those trajectories that follow along the stable manifold are optimal or, equivalently, that all other trajectories are suboptimal.

Intuitively, this argument would establish that if the initial value of investment $I(0)$ is too low, in the sense that it lies below the stable manifold in the phase diagram, investment and the capital stock would converge to zero, which is suboptimal, whereas if the initial value of investment $I(0)$ is too high, in the sense that it lies above the stable manifold, investment and the capital stock would grow so fast that the transversality condition

$$\lim_{T\to\infty} \pi(T)K(T) = 0$$

gets violated. As is often the case in dealing with infinite-horizon dynamic optimization problems, however, actually working through the formal details of these arguments turns out to be surprisingly difficult. For a thorough discussion of both the intuition and the formal results, see Acemoglu's book, section 7.8 (pp.269-274), on "The q-Theory of Investment and Saddle-Path Stability."