Solutions to Problem Set 10

ECON 772001 - Math for Economists — Peter Ireland
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Practice for the Midterm – Not Collected or Graded

Consider an economy populated by a large number of identical consumers, each of whom takes s_0 as given, and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the budget constraint

$$\frac{d_t s_t - c_t}{p_t} \geq s_{t+1} - s_t$$

for all $t = 0, 1, 2, ...$

1. The Kuhn-Tucker Formulation

With the Lagrangian for this problem written as

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \pi_{t+1} \left(s_t + \frac{d_t s_t - c_t}{p_t} - s_{t+1} \right)$$

the first-order condition for c_t is

$$\beta^t u'(c_t) - \frac{\pi_{t+1}}{p_t} = 0$$

and the first-order condition for s_t is

$$\pi_{t+1} + \pi_{t+1} \left(\frac{d_t}{p_t} \right) - \pi_t = 0.$$

The first of these two conditions must hold for all $t = 0, 1, 2, ...$ and the second must hold for all $t = 1, 2, 3, ...$ Together with the binding constraint

$$\frac{d_t s_t - c_t}{p_t} = s_{t+1} - s_t$$

for all $t = 0, 1, 2, ...,$ these conditions form a system of three equations in the three unknowns c_t, s_t, and π_{t+1}.

2. The Maximum Principle

The maximized Hamiltonian for the consumer's problem can be defined as

$$H(s_t, \pi_{t+1}; t) = \max_{c_t} \beta^t u(c_t) + \pi_{t+1} \left(\frac{d_t s_t - c_t}{p_t} \right).$$

Notably, the problem on the right-hand side of this definition is an unconstrained problem. Hence, the envelope theorem implies that

$$H_s(s_t, \pi_{t+1}; t) = \pi_{t+1} \left(\frac{d_t}{p_t} \right)$$

and

$$H_\pi(s_t, \pi_{t+1}; t) = \frac{d_t s_t - c_t}{p_t},$$

where c_t satisfies the first-order condition

$$\beta^t u'(c_t) - \frac{\pi_{t+1}}{p_t} = 0.$$

The maximum principle applied to this problem simply observes that the same three-equation system that, according to the Kuhn-Tucker theorem, must characterize the consumer's optimal choices can be derived using the first-order condition for c_t just stated together with the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_s(s_t, \pi_{t+1}; t) = -\pi_{t+1} \left(\frac{d_t}{p_t} \right)$$

and

$$s_{t+1} - s_t = H_\pi(s_t, \pi_{t+1}; t) = \frac{d_t s_t - c_t}{p_t}.$$

3. An Economic Interpretation of the Results

A problem with the optimality conditions, regardless of whether they are derived using the Kuhn-Tucker theorem or with the help of the maximum principle, is that they make reference to a variable, the Lagrange multiplier on the budget constraint, that lacks an immediate economic interpretation. To sidestep this problem, note that since the first-order condition for c_t must hold for all $t = 0, 1, 2, ...$, it implies that

$$\pi_{t+1} = \beta^t u'(c_t) p_t$$

and

$$\pi_t = \beta^{t-1} u'(c_{t-1}) p_{t-1}.$$

Using these two equations to eliminate reference to the multipliers in the first-order condition for s_t yields

$$\beta^t u'(c_t)(p_t + d_t) = \beta^{t-1} u'(c_{t-1}) p_{t-1}.$$

And using the definition

$$R_t^s = \frac{d_t + p_t}{p_{t-1}}$$

on the stock allows this last expression to be rewritten even more compactly as

$$\frac{\beta u'(c_t)}{u'(c_{t-1})} = \frac{1}{R_t^s}.$$

The expression on the left-hand-side of this last equation measures the consumer's intertemporal marginal rate of substitution; by relating this marginal rate of substitution to the expression involving the asset return on the right-hand-side, this equation resembles the more familiar static condition relating an optimizing consumer's marginal rate of substitution between two goods consumed at a given point in time to the relative price of those same two goods.

4. An Equilibrium Asset-Pricing Formula

To derive the equilibrium asset-pricing formula for the share price, it is easiest to start by going back to the original statement of the first-order condition for s_t, which can be rewritten as

$$\pi_t = \pi_{t+1} + \pi_{t+1}\left(\frac{d_t}{p_t}\right).$$

Since this condition must hold for all periods $t = 0, 1, 2, ...$, it also implies that

$$\pi_{t+1} = \pi_{t+2} + \pi_{t+2}\left(\frac{d_{t+1}}{p_{t+1}}\right).$$

Substituting this last expression into the one just before yields

$$\pi_t = \pi_{t+2} + \pi_{t+2}\left(\frac{d_{t+1}}{p_{t+1}}\right) + \pi_{t+1}\left(\frac{d_t}{p_t}\right)$$

and repeating these steps an infinite number of times while also invoking the transversality condition

$$\lim_{T\to\infty} \pi_{T+1} = 0$$

leads to

$$\pi_t = \sum_{j=1}^{\infty} \pi_{t+j}\left(\frac{d_{t+j-1}}{p_{t+j-1}}\right).$$

Next, use the first-order condition for c_t, which implies that

$$\pi_{t+j} = \beta^{t+j-1} u'(c_{t+j-1}) p_{t+j-1}$$

for all $j = 0, 1, 2, ...$ to rewrite the formula as

$$\beta^{t-1} u'(c_{t-1}) p_{t-1} = \sum_{j=1}^{\infty} \beta^{t+j-1} u'(c_{t+j-1}) d_{t+j-1}.$$

Finally, divide both sides by $\beta^{t-1}u'(c_{t-1})$, substitute in the market-clearing conditions

$$c_{t+j-1} = d_{t+j-1}$$

for all $j = 0, 1, 2, ...$, and roll the expression forward one period to obtain the desired result that

$$p_t = \sum_{j=1}^{\infty} \left[\frac{\beta^j u'(d_{t+j})}{u'(d_t)} \right] d_{t+j}$$

must hold in equilibrium, indicating that the share price equals the present discounted value of future dividends, where the discounting is done with reference to the consumer's intertemporal marginal rate of substitution.

5. An Interesting Special Case

When the representative consumer's single-period utility takes the (natural) logarithmic form

$$u(c_t) = \ln(c_t),$$

the asset-pricing formula from above collapses to

$$p_t = d_t \sum_{j=1}^{\infty} \beta^j,$$

implying that in this special case, the dividend yield d_t/p_t will remain constant over time even as the dividends d_t and the stock price p_t themselves fluctuate.