Solutions to Problem Set 1

ECON 720001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2020

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1. Profit Maximization

Consider a firm that produces output y with capital k and labor l according to the technology described by

$$k^a l^b \geq y, \tag{1}$$

where $0 < a < 1$, $0 < b < 1$, and $0 < a + b < 1$. The firm sells each unit of output at the price p, rents each unit of capital at the rate r, and hires each unit of labor at the wage w. Hence it chooses y, k, and l to maximize profits

$$py - rk - wl$$

subject to the constraint just shown in (1).

a. The Lagrangian for this problem is

$$L(y, k, l, \lambda) = py - rk - wl + \lambda(k^a l^b - y).$$

b. According to the Kuhn-Tucker theorem, the values y^*, k^*, and l^* that solve the firm's problem, together with the associated value λ^* for the multiplier, must satisfy the first-order conditions

$$L_1(y^*, k^*, l^*, \lambda^*) = p - \lambda^* = 0,$$

$$L_2(y^*, k^*, l^*, \lambda^*) = -r + a\lambda^*(k^*)^{a-1}(l^*)^b = 0,$$

and

$$L_3(y^*, k^*, l^*, \lambda^*) = -w + b\lambda^*(k^*)^a(l^*)^{b-1} = 0,$$

the constraint

$$L_4(y^*, k^*, l^*, \lambda^*) = (k^*)^a(l^*)^b - y^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*[(k^*)^a(l^*)^b - y^*] = 0.$$

c. The first-order condition for y^* reveals that $\lambda^* = p$ and hence $\lambda^* > 0$ whenever $p > 0$. Assuming that this is the case, the complementary slackness condition requires the constraint to hold as an equality. Now the first-order conditions for k^* and l^* can be used together with the binding constraint to solve for y^*, k^*, and l^* in terms of the model's parameters a, b, p, r, and w. While there are a variety of ways of doing this, one approach is to rewrite the first-order condition for k^* as

$$k^* = \left(\frac{a}{r}\right)^{1/(1-a)} p^{1/(1-a)} (l^*)^{b/(1-a)}$$

then substitute this expression into the first-order condition for l^* to find

$$\begin{aligned} l^* &= \left(\frac{b}{w}\right)^{1/(1-b)} p^{1/(1-b)} (k^*)^{a/(1-b)} \\ &= \left(\frac{b}{w}\right)^{1/(1-b)} p^{1/(1-b)} \left(\frac{a}{r}\right)^{a/[(1-a)(1-b)]} p^{a/[(1-a)(1-b)]} (l^*)^{ab/[(1-a)(1-b)]} \end{aligned}$$

and therefore

$$\begin{aligned} (l^*)^{(1-a)(1-b)} &= \left(\frac{b}{w}\right)^{1-a} p^{1-a} \left(\frac{a}{r}\right)^{a} p^{a} (l^*)^{ab} \\ (l^*)^{1-a-b} &= \left(\frac{a}{r}\right)^{a} \left(\frac{b}{w}\right)^{1-a} p \\ l^* &= \left(\frac{a}{r}\right)^{a/(1-a-b)} \left(\frac{b}{w}\right)^{(1-a)/(1-a-b)} p^{1/(1-a-b)}. \end{aligned}$$

Next, substitute this solution for l^* back into the first-order condition for k^* to find

$$\begin{aligned} k^* &= \left(\frac{a}{r}\right)^{1/(1-a)} p^{1/(1-a)} (l^*)^{b/(1-a)} \\ (k^*)^{1-a} &= \left(\frac{a}{r}\right) p (l^*)^{b} \\ (k^*)^{1-a} &= \left(\frac{a}{r}\right) p \left(\frac{a}{r}\right)^{ab/(1-a-b)} \left(\frac{b}{w}\right)^{(1-a)b/(1-a-b)} p^{b/(1-a-b)} \\ (k^*)^{1-a} &= \left(\frac{a}{r}\right)^{(1-a)(1-b)/(1-a-b)} \left(\frac{b}{w}\right)^{(1-a)b/(1-a-b)} p^{(1-a)/(1-a-b)} \\ k^* &= \left(\frac{a}{r}\right)^{(1-b)/(1-a-b)} \left(\frac{b}{w}\right)^{b/(1-a-b)} p^{1/(1-a-b)}. \end{aligned}$$

Finally, substitute the solutions for l^* and k^* into the binding constraint to find

$$\begin{aligned} y &= (k^*)^a (l^*)^b \\ &= \left[\left(\frac{a}{r}\right)^{a(1-b)/(1-a-b)} \left(\frac{b}{w}\right)^{ab/(1-a-b)} p^{a/(1-a-b)}\right] \\ &\quad \times \left[\left(\frac{a}{r}\right)^{ab/(1-a-b)} \left(\frac{b}{w}\right)^{(1-a)b/(1-a-b)} p^{b/(1-a-b)}\right] \\ &= \left(\frac{a}{r}\right)^{a/(1-a-b)} \left(\frac{b}{w}\right)^{b/(1-a-b)} p^{(a+b)/(1-a-b)}. \end{aligned}$$

d. The solutions from above imply that:

   i. The optimal y^*, k^*, and l^* all rise when the output price p rises, holding all other parameters fixed.

   ii. The optimal y^*, k^*, and l^* all fall when the rental rate for capital r rises, holding all other parameters fixed.

   iii. The optimal y^*, k^*, and l^* all fall when the wage rate w rises, holding all other parameters fixed.

   iv. The optimal y^*, k^*, and l^* all remain unchanged when p, r, and w all double at the same time.

2. Utility Maximization

Now consider a consumer who uses his or her income I to purchase c_1 units of good 1 at the price of p_1 per unit and c_2 units of good 2 at the price of p_2 per unit, subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2. \tag{2}$$

Suppose that the consumer has preferences over the two goods described by the utility function

$$U(c_1, c_2) = c_1^a c_2^{1-a}, \tag{3}$$

where $0 < a < 1$.

a. The Lagrangian for the consumer's problem is

$$L(c_1, c_2, \lambda) = c_1^a c_2^{1-a} + \lambda(I - p_1 c_1 - p_2 c_2).$$

b. According to the Kuhn-Tucker theorem, the values c_1^* and c_2^*, that solve the consumer's problem, together with the associated value λ^* for the multiplier, must satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*) = a(c_1^*)^{a-1}(c_2^*)^{1-a} - \lambda^* p_1 = 0$$

and

$$L_2(c_1^*, c_2^*, \lambda^*) = (1-a)(c_1^*)^a(c_2^*)^{-a} - \lambda^* p_2 = 0,$$

the constraint

$$L_3(c_1^*, c_2^*, \lambda^*) = I - p_1 c_1^* - p_2 c_2^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0.$$

c. The first-order conditions for c_1^* and c_2^* reveal that $\lambda^* > 0$ if the prices p_1 and p_2 are both strictly positive, a condition that must hold for the problem to have a well-defined solution in the first place. Assuming that this is the case, the complementary slackness conditions requires the constraint to hold as an equality. Hence, the first-order order conditions can be used together with the binding constraint to solve for c_1^*, c_2^*, and λ^* in terms of the model's parameters I, p_1, p_2, and a. Although there are a number of ways of doing this, one way is to divide the first-order condition for c_1^* by the first-order condition for c_2^* to obtain

$$c_2^* = \left(\frac{1-a}{a}\right)\left(\frac{p_1}{p_2}\right)c_1^*,$$

then substitute this expression into the binding constraint to obtain

$$I = p_1 c_1^* + p_2\left(\frac{1-a}{a}\right)\left(\frac{p_1}{p_2}\right)c_1^* = \left[1+\left(\frac{1-a}{a}\right)\right]p_1 c_1^* = \frac{p_1 c_1^*}{a}$$

or

$$c_1^* = \frac{aI}{p_1}.$$

Next, substitute this solution for c_1^* into the previous expression for c_2^* to find

$$c_2^* = \left(\frac{1-a}{a}\right)\left(\frac{p_1}{p_2}\right)c_1^* = \left(\frac{1-a}{a}\right)\left(\frac{p_1}{p_2}\right)\left(\frac{aI}{p_1}\right)$$

or

$$c_2^* = \frac{(1-a)I}{p_2}.$$

Finally, the first-order condition for c_1^* implies

$$\lambda^* = \frac{a}{p_1}\left(\frac{c_2^*}{c_1^*}\right)^{1-a} = \frac{a}{p_1}\left(\frac{1-a}{a}\right)^{1-a}\left(\frac{p_1}{p_2}\right)^{1-a}$$

and

$$\lambda^* = a^a(1-a)^{1-a}\left(\frac{1}{p_1}\right)^a\left(\frac{1}{p_2}\right)^{1-a}.$$

d. The solutions for c_1^* and c_2^* shown above imply that the consumer spends the fraction a of his or her income on good 1 and the remaining fraction $1-a$ of his or her income on good 2.

3. Utility Maximization (Again)

Redo the four parts of the previous question, but assuming that instead of (3), the consumer's utility is described by

$$U(c_1, c_2) = a\ln(c_1) + (1-a)\ln(c_2),$$

where ln denotes the natural logarithm and where $0 < a < 1$ as before.

a. The Lagrangian for the consumer's problem is now

$$L(c_1, c_2, \lambda) = a\ln(c_1) + (1-a)\ln(c_2) + \lambda(I - p_1c_1 - p_2c_2).$$

b. According to the Kuhn-Tucker theorem, the values c_1^* and c_2^*, that solve the consumer's problem, together with the associated value λ^* for the multiplier, must satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*) = \frac{a}{c_1^*} - \lambda^* p_1 = 0$$

and

$$L_2(c_1^*, c_2^*, \lambda^*) = \frac{1-a}{c_2^*} - \lambda^* p_2 = 0,$$

the constraint

$$L_3(c_1^*, c_2^*, \lambda^*) = I - p_1c_1^* - p_2c_2^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*(I - p_1c_1^* - p_2c_2^*) = 0.$$

c. As before, the first-order conditions for c_1^* and c_2^* reveal that $\lambda^* > 0$ if the prices p_1 and p_2 are both strictly positive, again a condition that must hold if the problem is to have a well-defined solution in the first place. Assuming that this is the case, the complementary slackness conditions requires the constraint to hold as an equality. Hence, the first-order order conditions can be used together with the binding constraint to solve for c_1^*, c_2^*, and λ^* in terms of the model's parameters I, p_1, p_2, and a. In this case, it is probably easiest to rewrite the first-order conditions as

$$c_1^* = \frac{a}{\lambda^* p_1}$$

and

$$c_2^* = \frac{1-a}{\lambda^* p_2}$$

and substitute these expressions into the binding constraint to obtain

$$I = \frac{a}{\lambda^*} + \frac{1-a}{\lambda^*} = \frac{1}{\lambda^*}$$

or

$$\lambda^* = \frac{1}{I}.$$

Now the first-order conditions imply

$$c_1^* = \frac{aI}{p_1},$$

and

$$c_2^* = \frac{(1-a)I}{p_2}.$$

d. The solutions for c_1^* and c_2^* shown above continue to imply that the consumer spends the fraction a of his or her income on good 1 and the remaining fraction $1 - a$ of his or her income on good 2. This is because the two utility functions from this problem and the previous one represent the same underlying preference ordering.