Consider a version of the Ramsey model with population growth, CES utility, and labor augmenting technology progress.

In this model, output is produced with $K(t)$ units of capital and $L(t)$ workers according to

$$K(t)^\alpha [A(t)L(t)]^{1-\alpha},$$

where productivity grows at the constant rate g

$$\frac{\dot{A}(t)}{A(t)} = g$$

and population grows at the constant rate n

$$\frac{\dot{L}(t)}{L(t)} = n$$

Capital accumulates according to

$$\dot{K}(t) = K(t)^\alpha [A(t)L(t)]^{1-\alpha} - \delta K(t) - C(t)$$

It will helpful to scale the capital stock and consumption by both the population and the level of technology:

$$k(t) = \frac{K(t)}{A(t)L(t)}$$

and

$$c(t) = \frac{C(t)}{A(t)L(t)}$$

Note then that

$$\begin{aligned} \dot{k}(t) &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)} \left[\frac{\dot{A}(t)L(t) + A(t)\dot{L}(t)}{A(t)L(t)} \right] \\ &= \frac{K(t)^\alpha [A(t)L(t)]^{1-\alpha} - \delta K(t) - C(t)}{A(t)L(t)} - (g+n)k(t) \\ &= k(t)^\alpha - (\delta + g + n)k(t) - c(t) \end{aligned}$$

so that the capital accumulation constraint looks the same as before except with $\delta + g + n$ in place of δ.

Next, consider the general CES utility function

$$\int_0^\infty e^{-\rho t} L(t) \left\{ \frac{[C(t)/L(t)]^{1-\sigma} - 1}{1-\sigma} \right\} dt$$

where $C(t)/L(t)$ is consumption per capita and utility for the entire population is the sum of utilities over $L(t)$ identical consumers. Normalizing the initial population to equal $L(0) = 1$ and using the definition of scaled consumption $c(t)$ an equivalent expression for utility is

$$\int_0^\infty e^{-(\rho-n)t} \left\{ \frac{[c(t)A(t)]^{1-\sigma} - 1}{1-\sigma} \right\} dt$$

We are now ready to state the social planner's problem: Given $k(0)$ and the constant growth rate of $A(t)$, choose functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-(\rho-n)t} \left\{ \frac{[c(t)A(t)]^{1-\sigma} - 1}{1-\sigma} \right\} dt$$

subject to

$$k(t)^\alpha - (\delta + g + n)k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$.

Define the current value Hamiltonian

$$\hat{H}(c(t), k(t), \theta(t); t) = \left\{ \frac{[c(t)A(t)]^{1-\sigma} - 1}{1-\sigma} \right\} + \theta(t)[k(t)^\alpha - (\delta + g + n)k(t) - c(t)].$$

Note that $\hat{H}$ depends on t as well as $c(t)$, $k(t)$, and $\theta(t)$ because of the growing level of productivity.

Similarly, define the maximized current value Hamiltonian

$$H(k(t), \theta(t); t) = \max_{c(t)} \left\{ \frac{[c(t)A(t)]^{1-\sigma} - 1}{1-\sigma} \right\} + \theta(t)[k(t)^\alpha - (\delta + g + n)k(t) - c(t)].$$

The maximum principle implies that the solution to the social planner's problem must satisfy the first-order condition

$$c(t)^{-\sigma} A(t)^{1-\sigma} - \theta(t) = 0$$

for all $t \in [0, \infty)$ and the pair of differential equations

$$\dot{\theta}(t) = (\rho - n)\theta(t) - H_k(k(t), \theta(t); t) = (\rho - n)\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta - g - n]$$

for all $t \in (0, \infty)$ and

$$\dot{k}(t) = H_\theta(k(t), \theta(t); t) = k(t)^\alpha - (\delta + g + n)k(t) - c(t)$$

for all $t \in [0, \infty)$.

As in the simpler case with log utility and no population growth or technological progress, we can use the first-order condition for $c(t)$ to eliminate $\theta(t)$ from this three-equation system of optimality conditions. Note that

$$\theta(t) = c(t)^{-\sigma} A(t)^{1-\sigma}$$

implies

$$\begin{aligned}\dot{\theta}(t) &= -\sigma c(t)^{-\sigma-1}A(t)^{1-\sigma}\dot{c}(t) + (1-\sigma)c(t)^{-\sigma}A(t)^{-\sigma}\dot{A}(t) \\ &= -\sigma\theta(t)\left[\frac{\dot{c}(t)}{c(t)}\right] + (1-\sigma)\theta(t)\left[\frac{\dot{A}(t)}{A(t)}\right] \\ &= -\sigma\theta(t)\left[\frac{\dot{c}(t)}{c(t)}\right] + (1-\sigma)g\theta(t).\end{aligned}$$

Substituting this expression for $\dot{\theta}(t)$ into the differential equation for $\theta(t)$ then yields

$$-\sigma\theta(t)\left[\frac{\dot{c}(t)}{c(t)}\right] + (1-\sigma)g\theta(t) = (\rho-n)\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta - g - n]$$

or

$$\dot{c}(t) = (1/\sigma)[\alpha k(t)^{\alpha-1} - \delta - \sigma g - \rho]c(t).$$

Note that when $\sigma = 1$ so that the general CES utility function reduces to the log case, and when $g = 0$, so there is no technological change, this Euler equation for consumption is the same as the one we derived previously in class.

In general, for this version of the Ramsey model, the solutions for $c(t)$ and $k(t)$ are described by the pair of differential equations

$$\dot{c}(t) = (1/\sigma)[\alpha k(t)^{\alpha-1} - \delta - \sigma g - \rho]c(t).$$

and

$$\dot{k}(t) = k(t)^{\alpha} - (\delta + g + n)k(t) - c(t).$$