Euler Equations and Transversality Conditions

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1 The Necessity of the Transversality Condition at Infinity: A (Very) Special Case

Consider a discrete-time, infinite horizon model that characterizes the optimal consumption of an exhaustible resource (like oil or coal). Let time periods be indexed by $t = 0, 1, 2, \ldots$, and let s_t, $t = 0, 1, 2, \ldots$, denote the stock of the exhaustible resource that remains at the beginning of period t. Let c_t, $t = 0, 1, 2, \ldots$, denote the amount of this resource that is consumed during period t. Since no new units of the resource are ever created, the amount consumed simply subtracts from the available stock according to

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, \ldots$, where the inequality constraint (which will always bind at the optimum) simply recognizes that the resource can be freely disposed of. The optimization problem then involves choosing sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize utility from consuming the resource over the infinite horizon, given by

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where the discount factor lies between zero and one, $0 < \beta < 1$, subject to the constraints $s_t - c_t \geq s_{t+1}$ for all $t = 0, 1, 2, \ldots$, taking as given the level of the initial resource stock $s_0 > 0$.

Strictly speaking, we could also add nonnegativity constraints $c_t \geq 0$ for all $t = 0, 1, 2, \ldots$ and $s_t \geq 0$ for all $t = 1, 2, 3, \ldots$ to the statement of the problem, but the assumption of log utility, which implies that the marginal utility of consumption becomes infinite as the level of consumption approaches zero, also implies that these constraints will never bind at the optimum.

We now know that there are at least three ways of deriving the necessary conditions describing a solution to this problem: using the Kuhn-Tucker theorem and the Lagrangian, using the maximum principle and the Hamiltonian, or using dynamic programming and the Bellman equation.

If we choose to use the Kuhn-Tucker theorem, then we would start by defining the Lagrangian for the problem as

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \tilde{\lambda}_{t+1}(s_t - c_t - s_{t+1}).$$

This definition of the Lagrangian casts the problem in "present value" form, in the sense that $\tilde{\lambda}_t$ measures the present value at $t = 0$ of having an additional unit of the resource available at the end of period t or the beginning of period $t + 1$. Alternatively, we can use the new variable

$$\lambda_{t+1} = \beta^{-t}\tilde{\lambda}_{t+1},$$

to replace $\tilde{\lambda}_t$ with $\beta^t \lambda_t$ and write the Lagrangian in "current value" form as

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \beta^t \lambda_{t+1}(s_t - c_t - s_{t+1}).$$

According to the Kuhn-Tucker theorem, if the sequences $\{c_t^*\}_{t=0}^{\infty}$ and $\{s_t^*\}_{t=1}^{\infty}$ solve the dynamic optimization problem, then there exists a sequence $\{\lambda_t^*\}_{t=1}^{\infty}$ of value for the Lagrange multipliers such that together, these three sequences satisfy:

a) The first-order condition for c_t^*,

$$\frac{\beta^t}{c_t^*} - \beta^t \lambda_{t+1}^* = 0$$

or more simply

$$\frac{1}{c_t^*} = \lambda_{t+1}^* \tag{1}$$

for all $t = 0, 1, 2, \ldots$.

b) The first-order condition for s_t^*,

$$\beta^t \lambda_{t+1}^* - \beta^{t-1}\lambda_t^* = 0$$

or more simply

$$\beta\lambda_{t+1}^* = \lambda_t^* \tag{2}$$

for all $t = 1, 2, 3, \ldots$.

c) The constraint

$$s_t^* - c_t^* \geq s_{t+1}^* \tag{3}$$

d) The nonnegativity condition

$$\lambda_{t+1}^* \geq 0 \tag{4}$$

for all $t = 0, 1, 2, \ldots$.

e) The complementary slackness condition

$$\lambda_{t+1}^*(s_t^* - c_t^* - s_{t+1}^*) = 0 \tag{5}$$

for all $t = 0, 1, 2, \ldots$.

Note that the first-order condition (1) for consumption implies that

$$\lambda_{t+1}^* = \frac{1}{c_t^*} > 0$$

and so, by extension, the complementary slackness condition (5) implies that the constraint (3) will always bind.

Note also that we can use (1) to "solve out" for the Lagrange multipliers in (2), using

$$\lambda_{t+1}^* = \frac{1}{c_t^*}$$

and

$$\lambda_t^* = \frac{1}{c_{t-1}^*}.$$

Hence, the solution to the original dynamic optimization problem can be characterized by finding solutions to a system of two difference equations in the two unknown sequences $\{c_t^*\}_{t=0}^\infty$ and $\{s_t^*\}_{t=1}^\infty$. The first difference equation comes from the first-order conditions and can be written as

$$\beta\lambda_{t+1}^* = \lambda_t^*$$

or

$$\frac{\beta}{c_t^*} = \frac{1}{c_{t-1}^*}$$

or

$$c_t^* = \beta c_{t-1}^*$$

or

$$c_{t+1}^* = \beta c_t^* \tag{6}$$

This optimality condition can be interpreted as one that indicates that it is optimal to equate the marginal rate of intertemporal substitution

$$\frac{\beta/c_t^*}{1/c_{t-1}^*}$$

to the intertemporal price, which is fixed at unity by the technological assumption that the exhaustible resource can be stored across periods without depreciation.

The second difference equation comes from the binding constraint and can be written as

$$s^*_{t+1} = s^*_t - c^*_t. \tag{7}$$

We know that in general, two boundary conditions are needed to pin down a unique solution to this system of two difference equations.

One boundary condition is the initial condition

$$s_0 \text{ given.} \tag{8}$$

In a finite-horizon version of the problem, second boundary condition would be given by the complementary slackness condition on the nonnegativity constraint $s^*_{T+1} \geq 0$ for the terminal value of the stock, which we know from our more general analysis can be written as

$$\beta^T \lambda^*_{T+1} s^*_{T+1} = 0. \tag{9}$$

Moreover, in the finite-horizon version of the problem, we could show that this transversality condition will hold because the nonnegativity constraint on the terminal value of the stock binds at the optimum:

$$s^*_{T+1} = 0.$$

Intuitively, with a finite horizon, if a strictly positive amount of the exhaustible resource remains at the end of period T, then a higher level of utility could be achieved by consuming that positive amount of the resource at one or more periods $t = 0, 1, \dots, T$.

For the infinite-horizon version of the problem, our more general analysis suggests that the relevant terminal, or transversality, condition is given by

$$\lim_{T\to\infty} \beta^T \lambda^*_{T+1} s^*_{T+1} = 0. \tag{10}$$

Notice that the first-order condition (2) implies that $\beta^T \lambda^*_{T+1}$ is going to be constant at the optimum. So in this special case, (10) will hold because

$$\lim_{T\to\infty} s^*_{T+1} = 0.$$

Intuitively, with an infinite horizon, it will never be optimal for the stock of the exhaustible resource to be run all the way down to zero over any finite period of time, since that would entail zero consumption from that point onward. On the other hand, if the stock is not exhausted in the limit, then there is a sense in which the resource is not being consumed "fast enough," in a way that parallels our argument for the finite-horizon case when some amount of the resource remains at the end of the horizon.

This model is simple enough, in fact, that we can prove formally that (10) is a necessary condition for the infinite-horizon case. The proof involves two steps.

Step one is to argue that $\{\beta^t \lambda_{t+1}^* s_{t+1}^*\}_{t=0}^{\infty}$ is a nonincreasing sequence. To show this, note that for all $t = 1, 2, 3, \ldots$, (2), (3), (4), and the nonnegativity of c_t^* imply that

$$\beta^t \lambda_{t+1}^* s_{t+1}^* - \beta^{t-1} \lambda_t^* s_t^* = \beta^t \lambda_{t+1}^* (s_{t+1}^* - s_t^*) \leq \beta^t \lambda_{t+1}^* (s_t^* - c_t^* - s_t^*) = -\beta^t \lambda_{t+1}^* c_t^* \leq 0.$$

Step two is to argue that

$$\inf_t \beta^t \lambda_{t+1}^* s_{t+1}^* = 0.$$

To show this, suppose to the contrary that there exists an $\varepsilon > 0$ such that

$$\beta^t \lambda_{t+1}^* s_{t+1}^* \geq \varepsilon$$

for all $t = 0, 1, 2, \ldots$. Since (1) implies that $\lambda_{t+1}^* > 0$ for all $t = 0, 1, 2, \ldots$, (2) implies that this last condition requires that

$$s_{t+1}^* \geq \gamma$$

for all $t = 0, 1, 2, \ldots$, where γ equals ε divided by the constant, positive value of $\beta^t \lambda_{t+1}^*$ along the optimal path. But, in this case, we can define new sequences $\{c_t^*\}_{t=0}^{\infty}$ and $\{s_t^{**}\}_{t=1}^{\infty}$ with

$$c_0^{**} = c_0^* + \gamma,$$

$$c_t^{**} = c_t^* \text{ for all } t = 1, 2, 3, \ldots,$$

and

$$s_t^{**} = s_t^* - \gamma \text{ for all } t = 1, 2, 3, \ldots$$

that satisfy all of the constraints from the original problem, but yield a higher level of utility, contradicting the assumption that $\{c_t^*\}_{t=0}^{\infty}$ and $\{s_t^*\}_{t=1}^{\infty}$ solve the problem.

Taken together, $\{\beta^t \lambda_{t+1}^* s_{t+1}^*\}_{t=0}^{\infty}$ and $\inf_t \beta^t \lambda_{t+1}^* s_{t+1}^* = 0$ require that (10) hold at the optimum, completing the proof.

The proof turns out to be relative straightforward for this simple problem, but becomes much more difficult to apply in other cases that are only slightly more complicated. Even for the Ramsey model with log utility and Cobb-Douglas production, for instance, this proof does not generalize.

For a much more elaborate proof that does apply to that version of the Ramsey model, see Ivar Ekeland and Jose Alexandre Scheinkman, "Transversality Conditions for Some Infinite Horizon Discrete Time Optimization Problems," *Mathematics of Operations Research*, Vol. 11 (May 1986), pp.216-229.

For the sake of completeness, let's wrap up by completely characterizing the solution to the original dynamic optimization problem.

Once again, that solution must satisfy the difference equations

$$c_{t+1}^* = \beta c_t^* \tag{6}$$

and

$$s_{t+1}^* = s_t^* - c_t^* \tag{7}$$

together with the initial condition

$$s_0 \text{ given} \tag{8}$$

and the transversality condition

$$\lim_{T\to\infty} \beta^T \lambda_{T+1}^* s_{T+1}^* = 0 \tag{10}$$

where the latter requires that

$$\lim_{T\to\infty} s_{T+1}^* = 0.$$

To start, consider (7), which implies

$$s_1^* = s_0 - c_0^*,$$

$$s_2^* = s_1^* - c_1^* = s_0 - c_0^* - c_1^*,$$

$$s_3^* = s_2^* - c_2^* = s_0 - c_0^* - c_1^* - c_2^*$$

or, after repeating T times,

$$s_T^* = s_0 - \sum_{t=0}^{T-1} c_t^*$$

or, after repeating infinitely many times and using the transversality condition

$$0 = s_0 - \sum_{t=0}^{\infty} c_t^*.$$

Rewritten as

$$\sum_{t=0}^{\infty} c_t^* = s_0,$$

this result confirms our intuition about the implications of the transversality condition: it shows that over the infinite horizon, it is optimal to consume the entire resource stock, otherwise, a higher level of utility could be achieved while still satisfying all of the constraints.

Now use (6), which implies that

$$c_t^* = \beta^t c_0^*$$

to pin down the level of the consumption path from

$$s_0 = \sum_{t=0}^{\infty} c_t^* = c_0^* \sum_{t=0}^{\infty} \beta^t = \frac{c_0^*}{1-\beta}.$$

Evidently, it is optimal to start by consuming

$$c_0^* = (1-\beta)s_0$$

at $t = 0$ and then to allow consumption to decrease proportionally according to (6) for all $t = 1, 2, 3, \ldots$.

2 The Sufficiency of the Euler Equation and Transversality Condition

As noted above, proving that the transversality condition is a necessary condition for models even slightly more complex than the exhaustible resource depletion problem turns out to be quite tricky. It is not as difficult, however, to prove that the Euler equation and the transversality conditions are sufficient conditions for discrete-time infinite horizon optimization models, provided that appropriate assumptions are imposed. These assumptions include the requirement that the objective function be concave and satisfy some kind of boundedness condition.

To start with a specific example, consider the Ramsey model of optimal growth, in which a representative consumer or social planner takes the initial capital stock $k_0 > 0$ as given and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and capital stocks $\{k_{t+1}\}_{t=0}^{\infty}$ to maximize the additively time-separable utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{11}$$

subject to the constraints

$$f(k_t) + (1-\delta)k_t \geq c_t + k_{t+1} \tag{12}$$

for all $t = 0, 1, 2, \ldots$. In (11), it is natural to assume that the discount factor lies between zero and one, with $0 < \beta < 1$ and that the single-period utility function is strictly increasing, strictly concave, and continuously differentiable with $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. Likewise, in (12), it makes sense to assume that the depreciation rate also lies between zero and one, with $0 < \delta \leq 1$ and that the production function is strictly increasing, strictly concave, and continuously differentiable with $f(0) = 0$, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$.

Once again, we know that there are at least three ways of deriving optimality conditions for a problem like this: using the Lagrangian, the Hamiltonian, or the Bellman equation. In this particular context, however, it is easiest to observe that, since u is strictly increasing, the constraint in (12) will always bind and can therefore be used to substitute out for consumption in (11). The problem can then be stated more simply as one of choosing $\{k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}), \tag{13}$$

taking $k_0 > 0$ as given, where the single-period "return function"

$$F(k_t, k_{t+1}) = u[f(k_t) + (1-\delta)k_t - k_{t+1}]$$

is strictly increasing in its first argument, strictly decreasing in its second argument, strictly concave, and continuously differentiable.

An older, "classical" approach to solving continuous-time dynamic optimization problems using the "calculus of variations" predates the development of the maximum principle but can still be adapted to the discrete-time case to derive sufficient conditions describing the solution to problems like this, where (13) is maximized by choice of a trajectory for the capital stock subject only to the given initial condition and where the restrictions on u and f guarantee that a nonnegativity constraint on the capital stock, if imposed, would never bind. These sufficient conditions include the "Euler equation," obtained by fixing a value of $t = 0, 1, 2, \ldots$, differentiating the objective function in (13) with respect to the choice of the end-of-period capital stock k_{t+1} for that period t, and setting the derivative equal to zero:

$$F_2(k_t^*, k_{t+1}^*) + \beta F_1(k_{t+1}^*, k_{t+2}^*) = 0 \tag{14}$$

holding $k_t = k_t^*$ and $k_{t+2} = k_{t+2}^*$ fixed at their optimal values. Of course, this Euler equation applies for all $t = 0, 1, 2, \ldots$. The sufficient conditions also include the transversality condition

$$\lim_{T\to\infty} \beta^T F_2(k_T^*, k_{T+1}^*) k_{T+1}^* = 0. \tag{15}$$

In terms of the original utility and production functions, the Euler equation (14) is

$$u'(c_t^*) = \beta[f'(k_{t+1}^*) + 1 - \delta] u'(c_{t+1}^*),$$

which, as we know, can be derived from the Lagrangian, Hamiltonian, or Bellman equation equally well. The transversality condition (15) is

$$\lim_{T\to\infty} \beta^T u'(c_T^*) k_{T+1}^* = 0,$$

which we've also seen before.

To show that the Euler equation and transversality conditions are sufficient conditions, we need to demonstrate that if the sequence $\{k_{t+1}^*\}_{t=0}^\infty$ satisfies (14) and (15), then that same sequence maximizes the objective function in (13). Accordingly, suppose that $\{k_{t+1}^*\}_{t=0}^\infty$ satisfies (14) and (15), let $\{k_{t+1}\}_{t=0}^\infty$ be any other sequence for the capital stock, and let

$$D = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F(k_t^*, k_{t+1}^*) - F(k_t, k_{t+1})], \tag{16}$$

Then result can be established by showing that $D > 0$.

The most difficult technical challenging in proving the result lies in showing that the limits taken in (16) exist in the first place, not just for the optimal sequence $\{k_{t+1}^*\}_{t=0}^\infty$ but also for the arbitrary alternative sequence $\{k_{t+1}\}_{t=0}^\infty$. The easiest way to get this result is to assume that the single-period return function F is bounded both above and below, but this assumption would imply a satiation point for consumption in the optimal growth model. Another approach would be to assume that the single-period utility function u is bounded below and modify (16) by using the limit inferior; Acemoglu (Chapter 6, pp.203-204) takes this approach. Note, however, that the natural log utility function is not bounded below, nor is the more general constant elasticity of intertemporal substitution utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

for values of σ greater than one, which would seem to be the values of greatest empirical relevance! A third approach tries to finesse the issue, by confining the capital stock to a compact set $[\underline{k}, \bar{k}]$ with $0 < \underline{k} < \bar{k} < \infty$; this is an approach that is taken repeatedly in the *Recursive Methods* book by Stokey, Lucas, and Prescott and used by Acemgolu (Chapter 6, pp.215-216) as well. Finally and most satisfactorily, one can "bite the bullet" and work through hard papers like Ekeland and Scheinkman (1986). The point is, there's no easy way to do what we really want, which is to argue that for a wide range of examples in economics, an infinite horizon problem can be viewed as the limit in a sequence of finite horizon problems, taken as the horizon goes to infinity.

Assuming that the limits in (16) do exist, however, the rest of the analysis is fairly straightforward. Since F is strictly concave,

$$F(k_t, k_{t+1}) < F(k_t^*, k_{t+1}^*) + F_1(k_t^*, k_{t+1}^*)(k_t - k_t^*) + F_2(k_t^*, k_{t+1}^*)(k_{t+1} - k_{t+1}^*).$$

Substituting this result into (16) reveals that

$$\begin{aligned} D &= \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F(k_t^*, k_{t+1}^*) - F(k_t, k_{t+1})] \\ &> \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F_1(k_t^*, k_{t+1}^*)(k_t^* - k_t) + F_2(k_t^*, k_{t+1}^*)(k_{t+1}^* - k_{t+1})]. \end{aligned}$$

Next, since $k_0^* = k_0$, the terms in this last sum can be rearranged to obtain

$$\begin{aligned} D &> \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t [F_2(k_t^*, k_{t+1}^*) + \beta F_1(k_{t+1}^*, k_{t+2}^*)](k_{t+1}^* - k_{t+1}) \\ &\quad + \lim_{T\to\infty} \beta^T F_2(k_T^*, k_{T+1}^*) k_{T+1}^* - \lim_{T\to\infty} \beta^T F_2(k_T^*, k_{T+1}^*) k_{T+1} \end{aligned}$$

By (14), all of the terms on the right-hand side of this inequality equal zero except the last, so that

$$D > -\lim_{T\to\infty} \beta^T F_2(k_T^*, k_{T+1}^*) k_{T+1},$$

and since F is strictly decreasing in its second argument, the desired result, $D > 0$, follows.

Thus, (14) and (15) are sufficient conditions for a solution to the problem of choosing $\{k_{t+1}\}_{t=0}^{\infty}$ to maximize the objective function in (13). Before closing, it is worthwhile emphasizing again the assumptions we needed to make in order to obtain this result. First, F must be concave; this assumption is the natural one to make if we expect first-order conditions like (14) to be sufficient conditions for a maximum. Second, each element of the optimal sequence $\{k_{t+1}^*\}_{t=0}^{\infty}$ must be interior to the set of feasible capital stocks; this assumption, too, is natural if we expected first-order conditions like (14) to hold as equalities. Third and perhaps most important, some sort of boundedness assumptions must be imposed to guarantee that the limits (or something like them) taken in (16) exist.