The Kuhn-Tucker and Envelope Theorems

Peter Ireland*
ECON 772001 - Math for Economists
Boston College, Department of Economics

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The Kuhn-Tucker and envelope theorems can be used to characterize the solution to a wide range of constrained optimization problems: static or dynamic, and under perfect foresight or featuring randomness and uncertainty. In addition, these same two results provide foundations for the work on the maximum principle and dynamic programming that we will do later on. For both of these reasons, the Kuhn-Tucker and envelope theorems provide the starting point for our analysis. Let's consider each in turn, first in fairly general or abstract settings and then applied to some economic examples.

1 The Kuhn-Tucker Theorem

References:

Dixit, Chapters 2 and 3.

Simon-Blume, Chapters 18 and 19.

Acemoglu, Appendix A.

Consider a simple constrained optimization problem:

$x \in \mathbf{R}$ choice variable

$F : \mathbf{R} \rightarrow \mathbf{R}$ objective function, continuously differentiable

$c \geq G(x)$ constraint, with $c \in \mathbf{R}$ and $G : \mathbf{R} \rightarrow \mathbf{R}$, also continuously differentiable.

The problem can be stated as:

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

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This problem is "simple" because it is static and contains no random or stochastic elements that would force decisions to be made under uncertainty. This problem is also "simple" because it has a single choice variable and a single constraint. All these simplifications will make our statement and proof of the Kuhn-Tucker theorem as clean and intuitive as possible. But the results can be generalized along all of these dimensions and, throughout the semester, we will work through examples that do so.

Probably the easiest way to solve this problem is via the method of Lagrange multipliers. The mathematical foundations that allow for the application of this method are given to us by Lagrange's Theorem or, in its most general form, the Kuhn-Tucker Theorem.

To prove this theorem, begin by defining the Lagrangian:

$$L(x, \lambda) = F(x) + \lambda[c - G(x)]$$

for any $x \in \mathbf{R}$ and $\lambda \in \mathbf{R}$.

Theorem (Kuhn-Tucker) Suppose that x^* maximizes $F(x)$ subject to $c \geq G(x)$, where F and G are both continuously differentiable, and suppose that $G'(x^*) \neq 0$. Then there exists a value λ^* of λ such that x^* and λ^* satisfy the following four conditions:

$$L_1(x^*, \lambda^*) = F'(x^*) - \lambda^* G'(x^*) = 0, \tag{1}$$

$$L_2(x^*, \lambda^*) = c - G(x^*) \geq 0, \tag{2}$$

$$\lambda^* \geq 0, \tag{3}$$

and

$$\lambda^*[c - G(x^*)] = 0. \tag{4}$$

Proof Consider two possible cases, depending on whether or not the constraint is binding at x^*.

Case 1: Nonbinding Constraint.

If $c > G(x^*)$, then let $\lambda^* = 0$. Clearly, (2)-(4) are satisfied, so it only remains to show that (1) must hold. With $\lambda^* = 0$, (1) holds if and only if

$$F'(x^*) = 0. \tag{5}$$

We can show that (5) must hold using a proof by contradiction. Suppose that instead of (5), it turns out that

$$F'(x^*) < 0.$$

Then, by the continuity of F and G, there must exist an $\varepsilon > 0$ such that

$$F(x^* - \varepsilon) > F(x^*) \text{ and } c > G(x^* - \varepsilon).$$

But this result contradicts the assumption that x^* maximizes $F(x)$ subject to $c \geq G(x)$. Similarly, if it turns out that

$$F'(x^*) > 0,$$

then by the continuity of F and G there must exist an $\varepsilon > 0$ such that

$$F(x^* + \varepsilon) > F(x^*) \text{ and } c > G(x^* + \varepsilon),$$

But, again, this result contradicts the assumption that x^* maximizes $F(x)$ subject to $c \geq G(x)$. This establishes that (5) must hold, completing the proof for case 1.

Case 2: Binding Constraint.

If $c = G(x^*)$, then let $\lambda^* = F'(x^*)/G'(x^*)$. This is possible, given the assumption that $G'(x^*) \neq 0$. Clearly, (1), (2), and (4) are satisfied, so it only remains to show that (3) must hold. With $\lambda^* = F'(x^*)/G'(x^*)$, (3) holds if and only if

$$F'(x^*)/G'(x^*) \geq 0. \tag{6}$$

We can show that (6) must hold using a proof by contradiction. Suppose that instead of (6), it turns out that

$$F'(x^*)/G'(x^*) < 0.$$

One way that this can happen is if $F'(x^*) > 0$ and $G'(x^*) < 0$. But if these conditions hold, then the continuity of F and G implies the existence of an $\varepsilon > 0$ such that

$$F(x^* + \varepsilon) > F(x^*) \text{ and } c = G(x^*) > G(x^* + \varepsilon),$$

which contradicts the assumption that x^* maximizes $F(x)$ subject to $c \geq G(x)$. And if, instead, $F'(x^*)/G'(x^*) < 0$ because $F'(x^*) < 0$ and $G'(x^*) > 0$, then the continuity of F and G implies the existence of an $\varepsilon > 0$ such that

$$F(x^* - \varepsilon) > F(x^*) \text{ and } c = G(x^*) > G(x^* - \varepsilon),$$

which again contradicts the assumption that x^* maximizes $F(x)$ subject to $c \geq G(x)$. This establishes that (6) must hold, completing the proof for case 2.

Notes:

a) The theorem can be extended to handle cases with more than one choice variable and more than one constraint: see Dixit, Simon-Blume, Acemoglu, or section 4.1 of the notes below.

b) Equations (1)-(4) are necessary conditions: If x^* is a solution to the optimization problem, then there exists a λ^* such that (1)-(4) must hold. But (1)-(4) are not sufficient conditions: if x^* and λ^* satisfy (1)-(4), it does not follow automatically that x^* is a solution to the optimization problem.

As an example that illustrates point (b), consider the problem:

$$\max_x e^x - \frac{2}{1+x^2} \text{ subject to } 1 \geq x.$$

With the Lagrangian defined as

$$L(x,\lambda) = e^x - \frac{2}{1+x^2} + \lambda(1-x),$$

the Kuhn-Tucker conditions are

$$L_1(x^*,\lambda^*) = e^{x^*} + \frac{4x^*}{[1+(x^*)^2]^2} - \lambda^* = 0,$$

$$L_2(x^*,\lambda^*) = 1 - x^* \geq 0,$$

$$\lambda^* \geq 0,$$

and

$$\lambda^*(1-x^*) = 0.$$

These conditions are satisfied when $x^* = 1$ and $\lambda^* = e + 1$, which corresponds to the solution to the problem, but they are also satisfied when $x^* = -0.2205$ and $\lambda^* = 0$, which corresponds instead to the solution to the *minimization* problem

$$\min_x e^x - \frac{2}{1+x^2} \text{ subject to } 1 \geq x.$$

But despite point (b) listed above, the Kuhn-Tucker theorem is extremely useful in practice.

Suppose that we are looking for the solution x^* to the constrained optimization problem

$$\max_x F(x) \text{ subject to } c \geq G(x).$$

The theorem tells us that if we form the Lagrangian

$$L(x,\lambda) = F(x) + \lambda[c - G(x)],$$

then x^* and the associated λ^* must satisfy the first-order condition (FOC) obtained by differentiating L by x and setting the result equal to zero:

$$L_1(x^*,\lambda^*) = F'(x^*) - \lambda^* G'(x^*) = 0, \tag{1}$$

In addition, we know that x^* must satisfy the constraint:

$$c \geq G(x^*). \tag{2}$$

We know that the Lagrange multiplier λ^* must be nonnegative:

$$\lambda^* \geq 0. \tag{3}$$

And finally, we know that the complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0, \tag{4}$$

must hold: If $\lambda^* > 0$, then the constraint must bind; if the constraint does not bind, then $\lambda^* = 0$.

In searching for the value of x that solves the constrained optimization problem, we only need to consider values of x^* that satisfy (1)-(4). In the example from above, for instance, it is straightforward to compare the values of the objective function when $x = 1$ and $x = -0.2205$ to conclude that the solution to the optimization problem is $x^* = 1$.

Two pieces of terminology:

a) The extra assumption that $G'(x^*) \neq 0$ is needed to guarantee the existence of a multiplier λ^* satisfying (1)-(4). This extra assumption is called the constraint qualification, and almost always holds in practice.

b) Note that (1) is a FOC for x, while (2) is like a FOC for λ. In many economic applications, where $F(x)$ is concave and $G(x)$ is convex, x^* maximizes $L(x, \lambda)$, while λ^* minimizes $L(x, \lambda)$. For this reason, (x^*, λ^*) is typically a saddle-point of $L(x, \lambda)$.

To see that the saddle-point property characterizes the solution to "concave programming" problems, consider

$$\max_x F(x) \text{ subject to } c \geq G(x),$$

where F is concave, G is convex, and the constraint qualification holds. Under these assumptions, the Kuhn-Tucker conditions (1)-(4) are both necessary and sufficient for the value x^* of x that solves the problem and the associated value λ^* of the Lagrange multiplier (see Simon and Blume's Theorem 21.22, pp.532-533).

Consider first the Lagrangian, viewed as a function of x with λ fixed at λ^*:

$$L(x, \lambda^*) = F(x) + \lambda^*[c - G(x)].$$

Since F is concave, G is convex, and $\lambda^* > 0$, $L(x, \lambda^*)$ is a concave function of x, and is therefore maximized when

$$L(x^*, \lambda^*) = F'(x^*) - \lambda^* G'(x^*)$$

holds. But this first-order condition for maximizing L with respect to x is the same as the first-order condition (1) for maximizing $F(x)$ with respect to x subject to the constraint $c \geq G(x)$.

Now consider the Lagrangian, viewed as a function of λ with x fixed at x^*:

$$L(x^*, \lambda) = F(x^*) + \lambda[c - G(x^*)].$$

When $c > G(x^*)$, $L(x^*, \lambda)$ is minimized subject to $\lambda > 0$ with $\lambda^* = 0$. And when $c = G(x^*)$, $L(x^*, \lambda)$ is minimized subject to $\lambda > 0$ with $\lambda^* = F'(x^*)/G'(x^*)$ so long as the constraint qualification holds.

Thus, point (b) gives us some intuition about how the Kuhn-Tucker theorem works. It uses the Lagrangian to turn a constrained optimization problem into an unconstrained problem, where the solution for x^* is a critical point of $L(x, \lambda^*)$ rather than a critical point of $F(x)$. And when the extra assumptions made in concave programming are adopted, x^* is not only a critical point of $L(x, \lambda^*)$, it also maximizes $L(x, \lambda^*)$.

As an example illustrating the saddle-point property, consider the problem

$$\max_x \ln(x) \text{ subject to } 1 \geq x.$$

Since the logarithmic objective function is strictly increasing, we know that $x^* = 1$ is the solution. Forming the Lagrangian

$$L(x, \lambda) = \ln(x) + \lambda(1 - x)$$

and taking the first-order condition

$$L_1(x^*, \lambda^*) = \frac{1}{x^*} - \lambda^* = 0$$

reveals that the associated value of λ is $\lambda^* = 1$. Substituting this value for λ back into the Lagrangian

$$L(x, \lambda^*) = \ln(x) + 1 - x,$$

we can see that $L(x, \lambda^*)$ is maximized with $x^* = 1$.

Note, however, that in the general case where $F(x)$ need not be concave, x^* will always be a critical point of the Lagrangian – that is, it will satisfy the first-order condition (1) – but x^* need not maximize $L(x, \lambda)$. An example of how this can happen is given by Dixit's example 7.2 (p.103). Consider the problem

$$\max_x e^x \text{ subject to } 1 \geq x.$$

Since the exponential objective function is strictly increasing, we know again that this problem is solved with $x^* = 1$. Forming the Lagrangian and taking the first-order condition

$$L_1(x^*, \lambda^*) = e^{x^*} - \lambda^* = 0,$$

shows that the associated value of λ^* is e. But $\lambda^* = e$ implies that

$$L(x, \lambda^*) = e^x + \lambda^*(1 - x) = e^x + e(1 - x) = e^x - ex + e,$$

and since $e^x - ex$ grows without bound as either $x \to \infty$ or $x \to -\infty$, $x^* = 1$ does not maximize the Lagrangian given λ^*. Hence, (x^*, λ^*) is not a saddle-point of L, since the objective function from the original problem is convex, not concave.

One final note:

Our general constraint, $c \geq G(x)$, nests as a special case the nonnegativity constraint $x \geq 0$, obtained by setting $c = 0$ and $G(x) = -x$.

So nonnegativity constraints can be introduced into the Lagrangian in the same way as all other constraints. If we consider, for example, the extended problem

$$\max_x F(x) \text{ subject to } c \geq G(x) \text{ and } x \geq 0,$$

then we can introduce a second multiplier μ, form the Lagrangian as

$$L(x, \lambda, \mu) = F(x) + \lambda[c - G(x)] + \mu x,$$

and write the first order condition for the optimal x^* as

$$L_1(x^*, \lambda^*, \mu^*) = F'(x^*) - \lambda^* G'(x^*) + \mu^* = 0. \tag{1'}$$

In addition, analogs to our earlier conditions (2)-(4) must also hold for the second constraint: $x^* \geq 0$, $\mu^* \geq 0$, and $\mu^* x^* = 0$.

Kuhn and Tucker's original statement of the theorem, however, does not incorporate nonnegativity constraints into the Lagrangian. Instead, even with the additional nonnegativity constraint $x \geq 0$, they continue to define the Lagrangian as

$$L(x, \lambda) = F(x) + \lambda[c - G(x)].$$

If this case, the first order condition for x^* must be modified to read

$$L_1(x^*, \lambda^*) = F'(x^*) - \lambda^* G'(x^*) \leq 0, \text{with equality if } x^* > 0. \tag{1''}$$

Of course, in (1′), $\mu^* \geq 0$ in general and $\mu^* = 0$ if $x^* > 0$. So a close inspection reveals that these two approaches to handling nonnegativity constraints lead in the end to the same results.

2 The Envelope Theorem

References:

Dixit, Chapter 5.

Simon-Blume, Chapter 19.

Acemoglu, Appendix A.

In our discussion of the Kuhn-Tucker theorem, we considered an optimization problem of the form

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Now, let's generalize the problem by allowing the functions F and G to depend on a parameter $\theta \in \mathbf{R}$. The problem can now be stated as

$$\max_x F(x, \theta) \text{ subject to } c \geq G(x, \theta)$$

For this problem, define the maximum value function $V : \mathbf{R} \to \mathbf{R}$ as

$$V(\theta) = \max_x F(x, \theta) \text{ subject to } c \geq G(x, \theta)$$

Note that evaluating V requires a two-step procedure:

First, given θ, find the value of x^* that solves the constrained optimization problem.

Second, substitute this value of x^*, together with the given value of θ, into the objective function to obtain

$$V(\theta) = F(x^*, \theta)$$

Now suppose that we want to investigate the properties of this function V. Suppose, in particular, that we want to take the derivative of V with respect to its argument θ.

As the first step in evaluating $V'(\theta)$, consider solving the constrained optimization problem for any given value of θ by setting up the Lagrangian

$$L(x, \lambda) = F(x, \theta) + \lambda[c - G(x, \theta)]$$

We know from the Kuhn-Tucker theorem that the solution x^* to the optimization problem and the associated value of the multiplier λ^* must satisfy the complementary slackness condition:

$$\lambda^*[c - G(x^*, \theta)] = 0$$

Use this last result to rewrite the expression for V as

$$V(\theta) = F(x^*, \theta) = F(x^*, \theta) + \lambda^*[c - G(x^*, \theta)]$$

So suppose that we tried to calculate $V'(\theta)$ simply by differentiating both sides of this equation with respect to θ:

$$V'(\theta) = F_2(x^*, \theta) - \lambda^* G_2(x^*, \theta).$$

But, in principle, this formula may not be correct. The reason is that x^* and λ^* will themselves depend on the parameter θ, and we must take this dependence into account when differentiating V with respect to θ.

However, the envelope theorem tells us that our formula for $V'(\theta)$ is, in fact, correct. That is, the envelope theorem tells us that we can ignore the dependence of x^* and λ^* on θ in calculating $V'(\theta)$.

To see why, for any θ, let $x^*(\theta)$ denote the solution to the problem: max $F(x, \theta)$ subject to $c \geq G(x, \theta)$, and let $\lambda^*(\theta)$ be the associated Lagrange multiplier.

Theorem (Envelope) Let F and G be continuously differentiable functions of x and θ. For any given θ, let $x^*(\theta)$ maximize $F(x,\theta)$ subject to $c \geq G(x,\theta)$, and let $\lambda^*(\theta)$ be the associated value of the Lagrange multiplier. Suppose, further, that $x^*(\theta)$ and $\lambda^*(\theta)$ are also continuously differentiable functions, and that the constraint qualification $G_1[x^*(\theta),\theta] \neq 0$ holds for all values of θ. Then the maximum value function defined by

$$V(\theta) = \max_x F(x,\theta) \text{ subject to } c \geq G(x,\theta)$$

satisfies

$$V'(\theta) = F_2[x^*(\theta),\theta] - \lambda^*(\theta)G_2[x^*(\theta),\theta]. \tag{7}$$

Proof The Kuhn-Tucker theorem tells us that for any given value of θ, $x^*(\theta)$ and $\lambda^*(\theta)$ must satisfy

$$L_1[x^*(\theta),\lambda^*(\theta)] = F_1[x^*(\theta),\theta] - \lambda^*(\theta)G_1[x^*(\theta),\theta] = 0, \tag{1}$$

and

$$\lambda^*(\theta)\{c - G[x^*(\theta),\theta]\} = 0. \tag{4}$$

In light of (4),

$$V(\theta) = F[x^*(\theta),\theta] = F[x^*(\theta),\theta] + \lambda^*(\theta)\{c - G[x^*(\theta),\theta]\}$$

Differentiating both sides of this expression with respect to θ yields

$$\begin{aligned} V'(\theta) \quad &= \quad F_1[x^*(\theta),\theta]x^{*\prime}(\theta) + F_2[x^*(\theta),\theta] \\ &\quad +\lambda^{*\prime}(\theta)\{c - G[x^*(\theta),\theta]\} - \lambda^*(\theta)G_1[x^*(\theta),\theta]x^{*\prime}(\theta) - \lambda^*(\theta)G_2[x^*(\theta),\theta] \end{aligned}$$

which shows that, in principle, we must take the dependence of x^* and λ^* on θ into account when calculating $V'(\theta)$.

Note, however, that

$$\begin{aligned} V'(\theta) \quad &= \quad \{F_1[x^*(\theta),\theta] - \lambda^*(\theta)G_1[x^*(\theta),\theta]\}x^{*\prime}(\theta) \\ &\quad +F_2[x^*(\theta),\theta] + \lambda^{*\prime}(\theta)\{c - G[x^*(\theta),\theta]\} - \lambda^*(\theta)G_2[x^*(\theta),\theta], \end{aligned}$$

which by (1) reduces to

$$V'(\theta) = F_2[x^*(\theta),\theta] + \lambda^{*\prime}(\theta)\{c - G[x^*(\theta),\theta]\} - \lambda^*(\theta)G_2[x^*(\theta),\theta]$$

Thus, it only remains to show that

$$\lambda^{*\prime}(\theta)\{c - G[x^*(\theta),\theta]\} = 0 \tag{8}$$

Clearly, (8) holds for any θ such that the constraint is binding.

For θ such that the constraint is not binding, (4) implies that $\lambda^*(\theta)$ must equal zero. Furthermore, by the continuity of G and x^*, if the constraint does not bind at θ, there exists an $\varepsilon^* > 0$ such that the constraint does not bind for all $\theta + \varepsilon$ with $\varepsilon^* > |\varepsilon|$. Hence, (4) also implies that $\lambda^*(\theta + \varepsilon) = 0$ for all $\varepsilon^* > |\varepsilon|$. Using the definition of the derivative

$$\lambda^{*\prime}(\theta) = \lim_{\varepsilon \to 0} \frac{\lambda^*(\theta + \varepsilon) - \lambda^*(\theta)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{0}{\varepsilon} = 0,$$

it once again becomes apparent that (8) must hold.

Thus,

$$V'(\theta) = F_2[x^*(\theta), \theta] - \lambda^*(\theta) G_2[x^*(\theta), \theta]$$

as claimed in the theorem.

Once again, this theorem is useful because it tells us that we can ignore the dependence of x^* and λ^* on θ in calculating $V'(\theta)$.

And once again, the theorem can be extended to apply in more general settings: see Dixit, Simon-Blume, Acemoglu, or section 4.2 of the notes below.

It is worth noting that the assumptions required by the envelope theorem are more restrictive than those required by the Kuhn-Tucker theorem. For example, the statement of the Kuhn-Tucker theorem makes reference to the value x^* of x that solves the constrained optimization problem, thereby assuming implicitly that a solution to the problem exists. But the envelope theorem requires that x^* be depicted as a function of θ, implying that the solution must not only exist but be unique as well. Further, the solution must vary smoothly with θ, so that x^* and λ^* can be written as continuously differentiable functions of the parameter.

Although most statements of the envelope theorem assume directly that $x^*(\theta)$ and $\lambda^*(\theta)$ are continuously differentiable, it is also possible to impose restrictions on $F(x, \theta)$ and $G(x, \theta)$ that guarantee this. Assume, for example, that the constraint always binds at the optimum, so that by the Kuhn-Tucker theorem, $x^*(\theta)$ and $\lambda^*(\theta)$ must satisfy

$$L_1[x^*(\theta), \lambda^*(\theta)] = F_1[x^*(\theta), \theta] - \lambda^*(\theta) G_1[x^*(\theta), \theta] = 0$$

and

$$L_2[x^*(\theta), \lambda^*(\theta)] = \lambda^*(\theta)\{c - G[x^*(\theta), \theta]\} = 0.$$

If F and G are two times continuously differentiable in x and θ, and if the matrix of second derivatives of the Lagrangian

$$\begin{bmatrix} L_{11}[x^*(\theta), \lambda^*(\theta)] & L_{12}[x^*(\theta), \lambda^*(\theta)] \\ L_{21}[x^*(\theta), \lambda^*(\theta)] & L_{22}[x^*(\theta), \lambda^*(\theta)] \end{bmatrix}$$

is nonsingular, then the implicit function theorem (Simon and Blume, Theorem 15.7, pp.355-356) will imply that $x^*(\theta)$ and $\lambda^*(\theta)$ exist and are continuously differentiable.

But what is the intuition for why the envelope theorem holds? To obtain some intuition, begin by considering the simpler, unconstrained optimization problem:

$$\max_x F(x, \theta),$$

where x is the choice variable and θ is the parameter.

Associated with this unconstrained problem, define the maximum value function in the same way as before:

$$V(\theta) = \max_x F(x, \theta).$$

To evaluate V for any given value of θ, use the same two-step procedure as before. First, find the value $x^*(\theta)$ that solves the unconstrained maximization problem for that value of θ. Second, substitute that value of x back into the objective function to obtain

$$V(\theta) = F[x^*(\theta), \theta].$$

Now differentiate both sides of this expression through by θ, carefully taking the dependence of x^* on θ into account:

$$V'(\theta) = F_1[x^*(\theta), \theta]x^{*\prime}(\theta) + F_2[x^*(\theta), \theta].$$

But, if $x^*(\theta)$ is the value of x that maximizes F given θ, we know that $x^*(\theta)$ must be a critical point of F:

$$F_1[x^*(\theta), \theta] = 0.$$

Hence, for the unconstrained problem, the envelope theorem implies that

$$V'(\theta) = F_2[x^*(\theta), \theta],$$

so that, again, we can ignore the dependence of x^* on θ in differentiating the maximum value function. And this result holds not because x^* fails to depend on θ: to the contrary, in fact, x^* will typically depend on θ through the function $x^*(\theta)$. Instead, the result holds because since x^* is chosen optimally, $x^*(\theta)$ is a critical point of F given θ.

Now return to the constrained optimization problem

$$\max_x F(x, \theta) \text{ subject to } c \geq G(x, \theta)$$

and define the maximum value function as before:

$$V(\theta) = \max_x F(x, \theta) \text{ subject to } c \geq G(x, \theta).$$

The envelope theorem for this constrained problem tells us that we can also ignore the dependence of x^* on θ when differentiating V with respect to θ, but only if we start by adding the complementary slackness condition to the maximized objective function to first obtain

$$V(\theta) = F[x^*(\theta), \theta] + \lambda^*(\theta)\{c - G[x^*(\theta), \theta]\}.$$

In taking this first step, we are actually evaluating the entire Lagrangian at the optimum, instead of just the objective function. We need to take this first step because for the constrained problem, the Kuhn-Tucker condition (1) tells us that $x^*(\theta)$ is a critical point, not of the objective function by itself, but of the entire Lagrangian formed by adding the product of the multiplier and the constraint to the objective function.

And what gives the envelope theorem its name? The "envelope" theorem refers to a geometrical presentation of the same result that we've just worked through.

To see where that geometrical interpretation comes from, consider again the simpler, unconstrained optimization problem:

$$\max_x F(x, \theta),$$

where x is the choice variable and θ is a parameter.

Following along with our previous notation, let $x^*(\theta)$ denote the solution to this problem for any given value of θ, so that the function $x^*(\theta)$ tells us how the optimal choice of x depends on the parameter θ.

Also, continue to define the maximum value function V in the same way as before:

$$V(\theta) = \max_x F(x, \theta).$$

Now let θ_1 denote a particular value of θ, and let x_1 denote the optimal value of x associated with this particular value θ_1. That is, let

$$x_1 = x^*(\theta_1).$$

After substituting this value of x_1 into the function F, we can think about how $F(x_1, \theta)$ varies as θ varies—that is, we can think about $F(x_1, \theta)$ as a function of θ, holding x_1 fixed.

In the same way, let θ_2 denote another particular value of θ, with $\theta_2 > \theta_1$ let's say. And following the same steps as above, let x_2 denote the optimal value of x associated with this particular value θ_2, so that

$$x_2 = x^*(\theta_2).$$

Once again, we can hold x_2 fixed and consider $F(x_2, \theta)$ as a function of θ.

The geometrical presentation of the envelope theorem can be derived by thinking about the properties of these three functions of θ: $V(\theta)$, $F(x_1, \theta)$, and $F(x_2, \theta)$.

One thing that we know about these three functions is that for $\theta = \theta_1$:

$$V(\theta_1) = F(x_1, \theta_1) > F(x_2, \theta_1),$$

where the first equality and the second inequality both follow from the fact that, by definition, x_1 maximizes $F(x, \theta_1)$ by choice of x.

Another thing that we know about these three functions is that for $\theta = \theta_2$:

$$V(\theta_2) = F(x_2, \theta_2) > F(x_1, \theta_2),$$

because again, by definition, x_2 maximizes $F(x, \theta_2)$ by choice of x.

On a graph, these relationships imply that:

At θ_1, $V(\theta)$ coincides with $F(x_1, \theta)$, which lies above $F(x_2, \theta)$.

At θ_2, $V(\theta)$ coincides with $F(x_2, \theta)$, which lies above $F(x_1, \theta)$.

And we could find more and more values of V by repeating this procedure for more and more specific values of θ_i, $i = 1, 2, 3, \ldots$.

In other words:

$V(\theta)$ traces out the "upper envelope" of the collection of functions $F(x_i, \theta)$, formed by holding $x_i = x^*(\theta_i)$ fixed and varying θ.

Moreover, $V(\theta)$ is tangent to each individual function $F(x_i, \theta)$ at the value θ_i of θ for which x_i is optimal, or equivalently:

$$V'(\theta) = F_2[x^*(\theta), \theta],$$

which is the same analytical result that we derived earlier for the unconstrained optimization problem.

If, for example,

$$F(x, \theta) = -(x - \theta)^2 + \theta^2 = -x^2 + 2x\theta,$$

then

$$V(\theta) = \max_x -(x - \theta)^2 + \theta^2 = \theta^2,$$

since, in this case, $x^*(\theta) = \theta$ for all values of θ.

The figure below sets $\theta_1 = 2$ and $\theta_2 = 7$; hence $x_1 = 2$ and $x_2 = 7$, then plots

$$F(x_1, \theta) = -4 + 4\theta,$$

$$F(x_2, \theta) = -49 + 14\theta,$$

and

$$V(\theta) = \theta^2$$

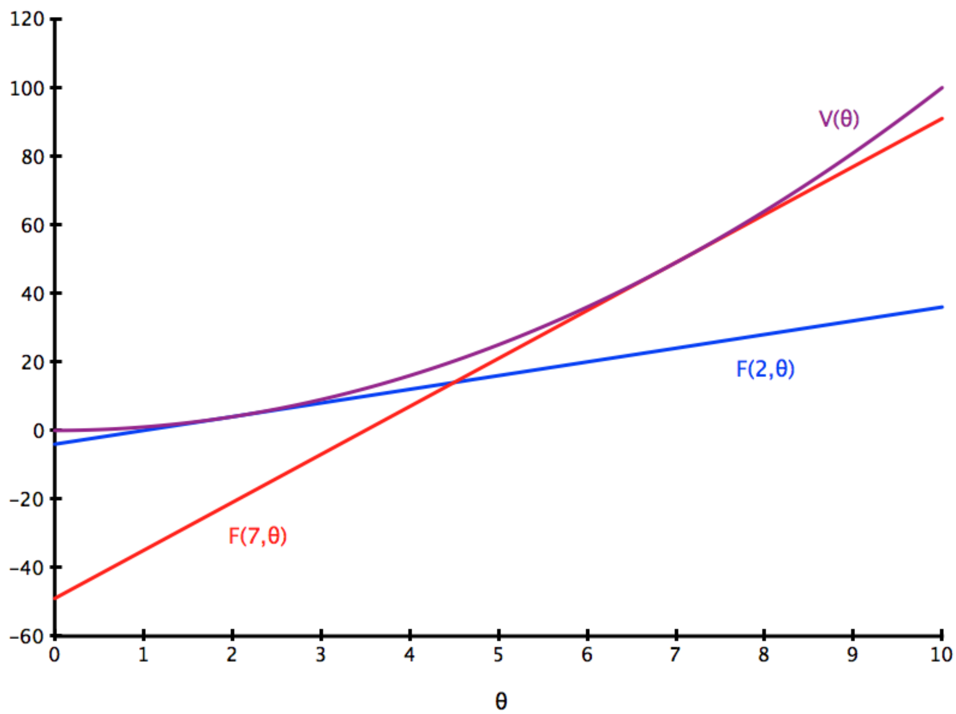
to show how

$$V(\theta_1) = F(x_1, \theta_1) > F(x_2, \theta_1) \text{ at } \theta_1 = 2,$$

and

$$V(\theta_2) = F(x_2, \theta_2) > F(x_1, \theta_2) \text{ at } \theta_2 = 7,$$

and how, more generally, $V(\theta)$ traces out the upper envelope of the family of functions $F(x_i, \theta)$, where each x_i maximizes $F(x, \theta)$ for some value θ_i of θ.



To generalize these arguments so that they apply to the constrained optimization problem

$$\max_x F(x,\theta) \text{ subject to } c \geq G(x,\theta),$$

simply use the fact that in many cases (as when F is concave and G is convex) the value $x^*(\theta)$ that solves the constrained optimization problem for any given value of θ also maximizes the Lagrangian function

$$L(x,\lambda,\theta) = F(x,\theta) + \lambda[c - G(x,\theta)],$$

so that

$$\begin{aligned} V(\theta) &= \max_x F(x,\theta) \text{ subject to } c \geq G(x,\theta) \\ &= \max_x L(x,\lambda,\theta) \end{aligned}$$

Now just replace the function F with the function L in working through the arguments from above to conclude that

$$V'(\theta) = L_3[x^*(\theta),\lambda^*(\theta),\theta] = F_2[x^*(\theta),\theta] - \lambda^*(\theta)G_2[x^*(\theta),\theta],$$

which is again the same result that we derived before for the constrained optimization problem.

Note that in the figure, the maximum value function $V(\theta)$ is convex, with a strictly positive second derivative, whereas the functions $F(x_1,\theta)$ and $F(x_2,\theta)$ are both linear. In general, it can be shown that there is a sense in which the maximum value function $V(\theta)$ will always be "more convex" or "less concave" than each member of the family of functions $F(x_i,\theta)$.

To see this, consider the two functions $V(\theta)$ and $F(x_1, \theta)$ as they were defined for the general unconstrained problem

$$\max_x F(x, \theta).$$

Second order Taylor approximations imply that for values of θ sufficiently close to θ_1:

$$V(\theta) \approx V(\theta_1) + V'(\theta_1)(\theta - \theta_1) + (1/2)V''(\theta_1)(\theta - \theta_1)^2$$

and

$$F(x_1, \theta) \approx F(x_1, \theta_1) + F_2(x_1, \theta_1)(\theta - \theta_1) + (1/2)F_{22}(x_1, \theta_1)(\theta - \theta_1)^2.$$

However,

$$V(\theta_1) = F(x_1, \theta_1),$$

since $x_1 = x^*(\theta_1)$, and

$$V'(\theta_1) = F_2(x_1, \theta_1),$$

by the envelope theorem itself. Since the definition of V implies that

$$V(\theta) \geq F(x_1, \theta)$$

for all values of θ, the Taylor approximations imply

$$V''(\theta_1) \geq F_{22}(x_1, \theta_1),$$

so that, more specifically, the second derivative of the maximum value function will always be larger than the second derivatives of the functions $F(x_i, \theta)$.

Finally, to begin getting a feel for the usefulness of the envelope theorem, consider a preliminary economic example. Suppose that a firm hires n workers, paying each the competitive real wage w (real wage, so that we don't have to consider separately the price of output), in order to produce y units of output according to the production function

$$n^\alpha \geq y,$$

where α lies between zero and one: $0 < \alpha < 1$.

For simplicity, let's depict the firm as solving the unconstrained optimization problem

$$\max_n n^\alpha - wn,$$

with first-order condition

$$\alpha(n^*)^{\alpha-1} - w = 0$$

that leads to the labor demand curve

$$n^*(w) = \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} w^{\frac{1}{\alpha-1}}.$$

Next, define the maximum value function

$$V(w) = \max_n n^\alpha - wn,$$

so that

$$V(w) = [n^*(w)]^\alpha - wn^*(w).$$

The envelope theorem immediately implies that

$$V'(w) = -n^*(w) = -\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} w^{\frac{1}{\alpha-1}}.$$

Suppose instead we substitute our earlier solution for $n^*(w)$ into the expression for $V(w)$:

$$\begin{aligned}
V(w) &= [n^*(w)]^\alpha - wn^*(w) \\
&= \left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} w^{\frac{1}{\alpha-1}}\right]^\alpha - w\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} w^{\frac{1}{\alpha-1}} \\
&= \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} w^{\frac{\alpha}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} w^{\frac{\alpha}{\alpha-1}} \\
&= \left[\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\right] w^{\frac{\alpha}{\alpha-1}}.
\end{aligned}$$

Now differentiate with respect to w and simplify to get

$$\begin{aligned}
V'(w) &= \left(\frac{\alpha}{\alpha-1}\right)\left[\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\right] w^{\frac{1}{\alpha-1}} \\
&= \left(\frac{\alpha}{\alpha-1}\right)\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{1}{\alpha}-1\right) w^{\frac{1}{\alpha-1}} \\
&= -\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} w^{\frac{1}{\alpha-1}},
\end{aligned}$$

exactly as we found, much more quickly, using the envelope theorem.

In terms of its economics, this example shows that when the firm faces a higher wage, there will be two effects on its profits.

The direct effect: it must pay each of its workers a higher wage, so profits fall by $-n^*(w)$.

The indirect effect: it can try to mitigate the direct effect by hiring fewer workers.

The envelope theorem says that since the firm has already chosen n^* optimally, the first-order effect on profits of adjusting this decision in response to an arbitrarily small change in the wage is zero. Only the direct effect remains: $V'(w) = -n^*(w)$.

3 Three Examples

3.1 Utility Maximization

A consumer has a utility function defined over consumption of two goods: $U(c_1, c_2)$

Prices: p_1 and p_2

Income: I

Budget constraint: $I \geq p_1c_1 + p_2c_2 = G(c_1, c_2)$

The consumer's problem is:

$$\max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2$$

The Kuhn-Tucker theorem tells us that if we set up the Lagrangian:

$$L(c_1, c_2, \lambda) = U(c_1, c_2) + \lambda(I - p_1c_1 - p_2c_2)$$

Then the optimal consumptions c_1^* and c_2^* and the associated multiplier λ^* must satisfy the FOC:

$$L_1(c_1^*, c_2^*, \lambda^*) = U_1(c_1^*, c_2^*) - \lambda^* p_1 = 0$$

and

$$L_2(c_1^*, c_2^*, \lambda^*) = U_2(c_1^*, c_2^*) - \lambda^* p_2 = 0$$

Move the terms with minus signs to the other side, and divide the first of these FOC by the second to obtain

$$\frac{U_1(c_1^*, c_2^*)}{U_2(c_1^*, c_2^*)} = \frac{p_1}{p_2},$$

which is just the familiar condition that says that the optimizing consumer should set the slope of his or her indifference curve, the marginal rate of substitution, equal to the slope of his or her budget constraint, the ratio of prices.

Now consider I as one of the model's parameters, and let the functions $c_1^*(I)$, $c_2^*(I)$, and $\lambda^*(I)$ describe how the optimal choices c_1^* and c_2^* and the associated value λ^* of the multiplier depend on I.

In addition, define the maximum value function as

$$V(I) = \max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2$$

The Kuhn-Tucker theorem tells us that

$$\lambda^*(I)[I - p_1c_1^*(I) - p_2c_2^*(I)] = 0$$

and hence

$$V(I) = U[c_1^*(I), c_2^*(I)] = U[c_1^*(I), c_2^*(I)] + \lambda^*(I)[I - p_1c_1^*(I) - p_2c_2^*(I)].$$

The envelope theorem tells us that we can ignore the dependence of c_1^* and c_2^* on I in calculating

$$V'(I) = \lambda^*(I),$$

which gives us an interpretation of the multiplier λ^* as the marginal utility of income.

3.2 Cost Minimization

The Kuhn-Tucker and envelope conditions can also be used to study constrained minimization problems.

Consider a firm that produces output y using capital k and labor l, according to the technology described by

$$f(k, l) \geq y.$$

r = rental rate for capital

w = wage rate

Suppose that the firm takes its output y as given, and chooses inputs k and l to minimize costs. Then the firm solves

$$\min_{k,l} rk + wl \text{ subject to } f(k, l) \geq y$$

If we set up the Lagrangian as

$$L(k, l, \lambda) = rk + wl - \lambda[f(k, l) - y],$$

where the term involving the multiplier λ is subtracted rather than added in the case of a minimization problem, the Kuhn-Tucker conditions (1)-(4) continue to apply, exactly as before.

Thus, according to the Kuhn-Tucker theorem, the optimal choices k^* and l^* and the associated multiplier λ^* must satisfy the FOC:

$$L_1(k^*, l^*, \lambda^*) = r - \lambda^* f_1(k^*, l^*) = 0 \tag{9}$$

and

$$L_2(k^*, l^*, \lambda^*) = w - \lambda^* f_2(k^*, l^*) = 0 \tag{10}$$

Move the terms with minus signs over to the other side, and divide the first FOC by the second to obtain

$$\frac{f_1(k^*, l^*)}{f_2(k^*, l^*)} = \frac{r}{w},$$

which is another familiar condition that says that the optimizing firm chooses factor inputs so that the marginal rate of substitution between inputs in production equals the ratio of factor prices.

Now suppose that the constraint binds, as it usually will:

$$y = f(k^*, l^*) \tag{11}$$

Then (9)-(11) represent 3 equations that determine the three unknowns k^*, l^*, and λ^* as functions of the model's parameters r, w, and y. In particular, we can think of the functions

$$k^* = k^*(r, w, y)$$

and

$$l^* = l^*(r, w, y)$$

as demand curves for capital and labor: strictly speaking, they are conditional (on y) factor demand functions.

Now define the minimum cost function as

$$\begin{aligned} C(r, w, y) &= \min_{k,l} rk + wl \text{ subject to } f(k, l) \geq y \\ &= rk^*(r, w, y) + wl^*(r, w, y) \\ &= rk^*(r, w, y) + wl^*(r, w, y) - \lambda^*(r, w, y)\{f[k^*(r, w, y), l^*(r, w, y)] - y\} \end{aligned}$$

The envelope theorem tells us that in calculating the derivatives of the cost function, we can ignore the dependence of k^*, l^*, and λ^* on r, w, and y.

Hence:

$$C_1(r, w, y) = k^*(r, w, y),$$

$$C_2(r, w, y) = l^*(r, w, y),$$

and

$$C_3(r, w, y) = \lambda^*(r, w, y).$$

The first two of these equations are statements of Shephard's lemma; they tell us that the derivatives of the cost function with respect to factor prices coincide with the conditional factor demand curves. The third equation gives us an interpretation of the multiplier λ^* as a measure of the marginal cost of increasing output.

Thus, our two examples illustrate how we can apply the Kuhn-Tucker and envelope theorems in specific economic problems.

The two examples also show how, in the context of specific economic problems, it is often possible to attach an economic interpretation to the multiplier λ^*.

3.3 Le Chatelier's Principle

For the next example, extend the previous cost minimization problem by introducing a third input:

$m =$ materials input

$q =$ price of materials

Define the minimum cost function

$$C(r, w, q, y) = \min_{k,l,m} rk + wl + qm \text{ subject to } f(k, l, m) \geq y.$$

Then Shephard's lemma implies

$$C_1(r, w, q, y) = k^*(r, w, q, y),$$

$$C_2(r, w, q, y) = l^*(r, w, q, y),$$

and

$$C_3(r, w, q, y) = m^*(r, w, q, y),$$

where $k^*(r, w, q, y)$, $l^*(r, w, q, y)$, and $m^*(r, w, q, y)$ are the conditional factor demand curves and the envelope theorem also implies that

$$C_4(r, w, q, y) = \lambda^*(r, w, q, y),$$

so that the Lagrange multiplier $\lambda^*(r, w, q, y)$ is again a measure of marginal cost.

Next, let's use Shephard's lemma to deduce another property of the conditional factor demand curves. Since

$$C_1(r, w, q, y) = k^*(r, w, q, y),$$

it follows that

$$C_{12}(r, w, q, y) = k_2^*(r, w, q, y).$$

And since

$$C_2(r, w, q, y) = l^*(r, w, q, y),$$

it follows that

$$C_{21}(r, w, q, y) = l_1^*(r, w, q, y).$$

But, since the symmetry of the second partial derivatives of C implies

$$C_{12}(r, w, q, y) = C_{21}(r, w, q, y),$$

Shephard's lemma can be viewed as having, as a corollary, the "reciprocity" condition for conditional factor demand curves:

$$k_2^*(r, w, q, y) = l_1^*(r, w, q, y),$$

or, equivalently,

$$\frac{\partial k^*(r, w, q, y)}{\partial w} = \frac{\partial l^*(r, w, q, y)}{\partial r}.$$

Now consider a "short-run" version of the firm's problem, treating the capital stock k as fixed at some pre-determined level $\bar{k}$:

$$\min_{l,m} r\bar{k} + wl + qm \text{ subject to } f(\bar{k}, l, m) \geq y.$$

With the Lagrangian for the short-run problem defined as

$$L(l, m, \mu) = r\bar{k} + wl + qm - \mu[f(\bar{k}, l, m) - y],$$

the first-order conditions are

$$w - \mu^s f_2(\bar{k}, l^s, m^s) = 0$$

and

$$q - \mu^s f_3(\bar{k}, l^s, m^s) = 0$$

and the binding constraint is

$$f(\bar{k}, l^s, m^s) - y = 0.$$

Interestingly, none of these optimality conditions depends on the rental rate r for capital. Hence, we can solve for short-run conditional factor demand curves of the form

$$l^s = l^s(w, q, y, \bar{k})$$

and

$$m^s = m^s(w, q, y, \bar{k}),$$

and use another application of the envelope theorem, this time to the short-run problem, to interpret

$$\mu^s = \mu^s(w, q, y, \bar{k})$$

as a measure of "short-run marginal cost."

Le Chatelier's principle says that labor demand should be more responsive to a change in the wage rate w in the long run than in the short run since, intuitively, in the long run the firm has the chance to substitute capital for labor in response to that change in the wage.

To show the sense in which this conjecture proves true, suppose that the fixed, short-run value of $\bar{k}$ just happens to equal the optimal value k^* that would have been chosen anyway in the long run:

$$\bar{k} = k^*(r, w, q, y).$$

In this case, a comparison of the Kuhn-Tucker conditions for the short-run problem with those for the long-run problem confirms that the firm will choose a value of l^s in the short run equal to the long-run value l^*:

$$l^*(r, w, q, y) = l^s[w, q, y, k^*(r, w, q, y)].$$

Differentiate both sides of this expression with respect to w to obtain

$$l_2^*(r,w,q,y) = l_1^s[w,q,y,k^*(r,w,q,y)] + l_4^s[w,q,y,k^*(r,w,y)]k_2^*(r,w,q,y).$$

The term on the left-hand side of this expression measures the long-run responsiveness of labor demand to a change in the wage; the first term on the right-hand side of this expression measures the short-run responsiveness of labor demand to a change in the wage. The second-order conditions for the long-run and short-run problems will generally imply that under "typical" conditions, both of these terms are negative: when the wage goes up, labor demand falls. But, is the long-run response "more negative?" This depends on the sign of the second term on the right-hand side, which in turn depends on l_4^s, measuring the response of short-run labor demand to a change in $\bar{k}$, and k_2^*, measuring the response of long-run capital demand to a change in w. The signs of both terms would seem to be ambiguous, dependent in some loose sense on whether, in the long run, the capital and labor inputs are "complements" or "substitutes."

Yet, as it turns out, something more definite can be said. Return once more to

$$l^*(r,w,q,y) = l^s[w,q,y,k^*(r,w,y)],$$

but this time differentiate both sides with respect to r to obtain

$$l_1^*(r,w,q,y) = l_4^s[w,q,y,k^*(r,w,y)]k_1^*(r,w,q,y).$$

Rearrange this expression so that it reads

$$l_4^s[w,q,y,k^*(r,w,q,y)] = \frac{l_1^*(r,w,q,y)}{k_1^*(r,w,q,y)},$$

and substitute this result into the previous one

$$l_2^*(r,w,q,y) = l_1^s[w,q,y,k^*(r,w,q,y)] + l_4^s[w,q,y,k^*(r,w,q,y)]k_2^*(r,w,q,y).$$

to get

$$l_2^*(r,w,q,y) = l_1^s[w,q,y,k^*(r,w,q,y)] + \frac{l_1^*(r,w,q,y)k_2^*(r,w,q,y)}{k_1^*(r,w,q,y)}.$$

Now it is clear that the second term must be negative. This is because the expression in the numerator is, by the reciprocity condition implied by Shephard's lemma, equal to the perfect square $[l_1^*(r,w,q,y)]^2$; meanwhile, the term in the denominator will be negative, as implied by the second-order conditions to the long-run problem: the optimal capital stock falls when the rental rate for capital rises.

Thus, it must be that

$$l_2^*(r,w,q,y) \leq l_1^s[w,q,y,k^*(r,w,y)]$$

and, since both of these terms are negative,

$$\left|\frac{\partial l^*(r,w,q,y)}{\partial w}\right| \geq \left|\frac{\partial l^s(w,q,y,k^*)}{\partial w}\right|,$$

which is the relationship we were after: labor demand is more responsive to a change in the wage in the long run than in the short run. Note, however, that this is a result that only holds locally, when $\bar{k}$ is sufficiently close to $k^*(r,w,q,y)$.

4 Generalizing the Basic Results

4.1 The Kuhn-Tucker Theorem

Our "simple" version of the Kuhn-Tucker theorem applies to a problem with only one choice variable and one constraint.

Section 19.6 of Simon and Blume's book develops a proof for the more general case, with n choice variables and m constraints. Their proof makes repeated, clever use of the implicit function theorem, which makes the arguments surprisingly short but also works to obscure some of the intuition provided by the analysis of the simplest case.

Nevertheless, having gained the intuition the intuition from working through the simple case, it is useful to see how the result extends.

Simon and Blume (Chapter 15) and Acemoglu (Appendix A) both present fairly general statements of the implicit function theorem. The special case or application of their results that we will need works as follows.

Consider a system of n equations, involving n "endogenous" variables $y_1, y_2, \ldots, y_n$ and n "exogenous" variables $c_1, c_2, \ldots c_n$:

$$\begin{aligned} H_1(y_1, y_2, \ldots, y_n) &= c_1, \\ H_2(y_1, y_2, \ldots, y_n) &= c_2, \\ &\vdots \\ H_m(y_1, y_2, \ldots, y_n) &= c_n. \end{aligned}$$

Now suppose that for a specific set of values $c_1^*, c_2^*, \ldots c_n^*$ for the exogenous variables, all the equations in the system are satisfied with the endogenous variables set equal to $y_1^*, y_2^*, \ldots, y_n^*$, so that

$$\begin{aligned} H_1(y_1^*, y_2^*, \ldots, y_n^*) &= c_1^*, \\ H_2(y_1^*, y_2^*, \ldots, y_n^*) &= c_2^*, \\ &\vdots \\ H_n(y_1^*, y_2^*, \ldots, y_n^*) &= c_n^*. \end{aligned}$$

Assume that each function H_i, $i = 1, \ldots, n$, is continuously differentiable and that the $n \times n$ matrix of derivatives

$$\begin{bmatrix} \partial H_1/\partial y_1 & \cdots & \partial H_1/\partial y_n \\ \partial H_2/\partial y_1 & \cdots & \partial H_2/\partial y_n \\ \vdots & \ddots & \vdots \\ \partial H_n/\partial y_1 & \cdots & \partial H_n/\partial y_n \end{bmatrix}$$

is nonsingular at $y_1^*, y_2^*, \ldots, y_n^*$.

Then there exist continuously differentiable functions

$$\begin{aligned} &y_1(c_1, c_2, \ldots, c_n), \\ &y_2(c_1, c_2, \ldots, c_n), \\ &\quad\vdots \\ &y_n(c_1, c_2 \ldots, c_n), \end{aligned}$$

defined in an open subset C of $\mathbf{R}^n$ containing $(c_1^*, c_2^*, \ldots, c_n^*)$, such that

$$\begin{aligned} H_1(y_1(c_1, c_2, \ldots, c_n), y_2(c_1, c_2, \ldots, c_n), \ldots, y_n(c_1, c_2, \ldots, c_n)) &= c_1, \\ H_2(y_1(c_1, c_2, \ldots, c_n), y_2(c_1, c_2, \ldots, c_n), \ldots, y_n(c_1, c_2, \ldots, c_n)) &= c_2, \\ &\vdots \\ H_n(y_1(c_1, c_2, \ldots, c_n), y_2(c_1, c_2, \ldots, c_n), \ldots, y_n(c_1, c_2, \ldots, c_n)) &= c_n. \end{aligned}$$

for all $(c_1, c_2, \ldots, c_n) \in C$.

With this result in hand, consider the following generalized version of the Kuhn-Tucker theorem we proved earlier. Let there be n choice variables, $x_1, x_2, \ldots, x_n$. The objective function $F : \mathbf{R}^n \to \mathbf{R}$ is continuously differentiable, as are the m functions $G_j : \mathbf{R}^n \to \mathbf{R}$, $j = 1, 2, \ldots, m$ that enter into the constraints

$$c_j \geq G_j(x_1, x_2, \ldots, x_n),$$

where $c_j \in \mathbf{R}$ for all $j = 1, 2, \ldots, m$.

The problem can be stated as:

$$\max_{x_1, x_2, \ldots, x_n} F(x_1, x_2, \ldots, x_n) \text{ subject to } c_j \geq G_j(x_1, x_2, \ldots, x_n) \text{ for all } j = 1, 2, \ldots, m.$$

Note that, typically, $m \leq n$ will have to hold so that there is a set of values for the choice variables that satisfy all of the constraints.

To define the Lagrangian, introduce the multipliers λ_j, $j = 1, 2, \ldots, m$, one for each constraint. Then

$$L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m) = F(x_1, x_2, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j [c_j - G_j(x_1, x_2, \ldots, x_n)].$$

Theorem (Kuhn-Tucker) Suppose that $x_1^*, x_2^*, \ldots, x_n^*$ maximize $F(x_1, x_2, \ldots, x_n)$ subject to $c_j \geq G_j(x_1, x_2, \ldots, x_n)$ for all $j = 1, 2, \ldots, m$, where F and the G_j's are all continuously differentiable. Suppose (without loss of generality) that the first $\bar{m} \leq m$ constraints bind at the optimum and that the remaining $m - \bar{m} \geq 0$ constraints are nonbinding, and assume that the $\bar{m} \times n$ matrix of derivatives

$$\begin{bmatrix} G_{1,1}(x_1^*, x_2^*, \ldots, x_n^*) & \ldots & G_{1,n}(x_1^*, x_2^*, \ldots, x_n^*) \\ G_{2,1}(x_1^*, x_2^*, \ldots, x_n^*) & \ldots & G_{2,n}(x_1^*, x_2^*, \ldots, x_n^*) \\ \vdots & \ddots & \vdots \\ G_{\bar{m},1}(x_1^*, x_2^*, \ldots, x_n^*) & \ldots & G_{\bar{m},n}(x_1^*, x_2^*, \ldots, x_n^*) \end{bmatrix}, \tag{12}$$

where $G_{j,i} = \partial G_j / \partial x_i$, has rank $\bar{m}$. Then there exist values $\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*$ that, together with $x_1^*, x_2^*, \dots, x_n^*$, satisfy:

$$L_i(x_1^*, x_2^*, \dots, x_n^*, \lambda_1^*, \lambda_2^*, \dots, \lambda_n^*) = F_i(x_1^*, x_2^*, \dots, x_n^*) - \sum_{j=1}^{m} \lambda_j^* G_{j,i}(x_1^*, x_2^*, \dots, x_n^*) = 0 \tag{13}$$

for $i = 1, 2, \dots, n$,

$$L_{n+j}(x_1^*, x_2^*, \dots, x_n^*, \lambda_1^*, \lambda_2^*, \dots, \lambda_n^*) = c_j - G_j(x_1^*, x_2^*, \dots, x_n^*) \geq 0, \tag{14}$$

for $j = 1, 2, \dots, m$,

$$\lambda_j^* \geq 0, \tag{15}$$

for $j = 1, 2, \dots, m$, and

$$\lambda_j^*[c_j - G_j(x_1^*, x_2^*, \dots, x_n^*)] = 0, \tag{16}$$

for $j = 1, 2, \dots, m$.

Proof To begin, set the multipliers $\lambda_{\bar{m}+1}^*, \lambda_{\bar{m}+2}^*, \dots, \lambda_m^*$ associated with the nonbinding contraints equal to zero. Since each of the functions G_j, $j = \bar{m}+1, \bar{m}+2, \dots, m$, is continuously differentiable, sufficiently small adjustments in the choice variables can be made without violating these $m - \bar{m}$ constraints or causing any of them to become binding.

Next, note that the $\bar{m}+1 \times n$ matrix

$$\begin{bmatrix} F_1(x_1^*, x_2^*, \dots, x_n^*) & \dots & F_n(x_1^*, x_2^*, \dots, x_n^*) \\ G_{1,1}(x_1^*, x_2^*, \dots, x_n^*) & \dots & G_{1,n}(x_1^*, x_2^*, \dots, x_n^*) \\ G_{2,1}(x_1^*, x_2^*, \dots, x_n^*) & \dots & G_{2,n}(x_1^*, x_2^*, \dots, x_n^*) \\ \vdots & \ddots & \vdots \\ G_{\bar{m},1}(x_1^*, x_2^*, \dots, x_n^*) & \dots & G_{\bar{m},n}(x_1^*, x_2^*, \dots, x_n^*) \end{bmatrix}. \tag{17}$$

must have rank $\bar{m} < \bar{m}+1$. To see why, consider the system of equations

$$\begin{aligned} F(x_1, x_2, \dots, x_n) &= y^* \\ G_1(x_1, x_2, \dots, x_n) &= c_1 \\ G_2(x_1, x_2, \dots, x_n) &= c_2 \\ &\vdots \\ G_{\bar{m}}(x_1, x_2, \dots, x_n) &= c_{\bar{m}}. \end{aligned}$$

With y^* set equal to the maximized value of the objective function,

$$y^* = F(x_1^*, x_2^*, \dots, x_n^*),$$

each of these $\bar{m}+1$ equations holds when the functions are evaluated at $x_1^*, x_2^*, \dots, x_n^*$. In this case, the implicit function theorem implies that it should be possible to adjust the values of $\bar{m}+1$ of the choice variables so to find a new set of values $x_1^{**}, x_2^{**}, \dots, x_n^{**}$ such that

$$\begin{aligned} F(x_1^{**}, x_2^{**}, \dots, x_n^{**}) &= y^* + \varepsilon \\ G_1(x_1^{**}, x_2^{**}, \dots, x_n^{**}) &= c_1 \\ G_2(x_1^{**}, x_2^{**}, \dots, x_n^{**}) &= c_2 \\ &\vdots \\ G_{\bar{m}}(x_1^{**}, x_2^{**}, \dots, x_n^{**}) &= c_{\bar{m}}. \end{aligned}$$

for a strictly positive but sufficiently small value of ε. But this contradicts the assumption that $x_1^*, x_2^*, \dots, x_n^*$ solves the constrained optimization problem.

Since the matrix in (17) has rank $\bar{m} < \bar{m}+1$, its $\bar{m}+1$ rows must be linearly dependent. Hence, there exist scalars $\alpha_0, \alpha_1, \dots \alpha_{\bar{m}}$, at least one of which is nonzero, such that

$$\begin{aligned} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} &= \alpha_0 \begin{bmatrix} F_1(x_1^*, x_2^*, \dots, x_n^*) \\ \vdots \\ F_n(x_1^*, x_2^*, \dots, x_n^*) \end{bmatrix} \\ &+ \alpha_1 \begin{bmatrix} G_{1,1}(x_1^*, x_2^*, \dots, x_n^*) \\ \vdots \\ G_{1,n}(x_1^*, x_2^*, \dots, x_n^*) \end{bmatrix} + \dots + \alpha_{\bar{m}} \begin{bmatrix} G_{\bar{m},1}(x_1^*, x_2^*, \dots, x_n^*) \\ \vdots \\ G_{\bar{m},n}(x_1^*, x_2^*, \dots, x_n^*) \end{bmatrix}. \end{aligned} \tag{18}$$

Moreover, in (18), $\alpha_0 \neq 0$, since otherwise, the matrix in (12) would have rank less than $\bar{m}$.

Thus, for $j = 1, 2, \dots, \bar{m}$, set $\lambda_j^* = -\alpha_j/\alpha_0$. With these settings for $\lambda_1^*, \lambda_2^*, \dots, \lambda_{\bar{m}}^*$, plus the settings $\lambda_{\bar{m}+1}^* = \lambda_{\bar{m}+2}^* = \lambda_m^* = 0$ chosen earlier, (18) implies that (13) must hold for all $i = 1, 2, \dots, n$. Clearly, (14) and (16) are satisfied for all $j = 1, 2, \dots, m$, and (15) holds for all $j = \bar{m}+1, \bar{m}+2, \dots, m$. So it only remains to show that (15) holds for $j = 1, 2, \dots, \bar{m}$.

To see that these last conditions must hold, consider the system of equations

$$\begin{aligned} G_1(x_1, x_2, \dots, x_n) &= c_1 - \delta \\ G_2(x_1, x_2, \dots, x_n) &= c_2 \\ &\vdots \\ G_{\bar{m}}(x_1, x_2, \dots, x_n) &= c_{\bar{m}}, \end{aligned} \tag{19}$$

where $\delta \geq 0$. These equations hold, with $\delta = 0$, at $x_1^*, x_2^*, \dots, x_n^*$. And since the matrix in (12) has rank $\bar{m}$, the implicit function theorem implies that there are functions $x_1(\delta), x_2(\delta), \dots, x_n(\delta)$ such that the same equations hold for all sufficiently small values of δ.

Since $c_1 - \delta \leq c_1$, the choices $x_1(\delta), x_2(\delta), \ldots, x_n(\delta)$ satisfy all of the constraints from the original optimization problem. And since, by assumption, $x_1(0) = x_1^*, x_2(0) = x_2^*, \ldots, x_n(0) = x_n^*$ maximizes the objective function subject to the constraints, it must be that

$$\left.\frac{dF(x_1(\delta), x_2(\delta), \ldots, x_n(\delta))}{d\delta}\right|_{\delta=0} = \sum_{i=1}^{n} F_i(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0) \leq 0. \tag{20}$$

In addition, the equations in (19) implicitly defining $x_1(\delta), x_2(\delta), \ldots, x_n(\delta)$ imply

$$\left.\frac{dG_1(x_1(\delta), x_2(\delta), \ldots, x_n(\delta))}{d\delta}\right|_{\delta=0} = \sum_{i=1}^{n} G_{1,i}(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0) = -1 \tag{21}$$

and

$$\left.\frac{dG_j(x_1(\delta), x_2(\delta), \ldots, x_n(\delta))}{d\delta}\right|_{\delta=0} = \sum_{i=1}^{n} G_{j,i}(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0) = 0 \tag{22}$$

for $j = 2, 3, \ldots, \bar{m}$.

Putting all these results together, (13) implies

$$0 = F_i(x_1^*, x_2^*, \ldots, x_n^*) - \sum_{j=1}^{m} \lambda_j^* G_{j,i}(x_1^*, x_2^*, \ldots, x_n^*).$$

for all $i = 1, 2, \ldots, n$. Multiplying each of these equations by $x_i'(0)$ and summing over all i yields

$$0 = \sum_{i=1}^{n} F_i(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0) - \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_j^* G_{j,i}(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0),$$

or

$$0 = \sum_{i=1}^{n} F_i(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0) - \sum_{j=1}^{m} \lambda_j^* \left[\sum_{i=1}^{n} G_{j,i}(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0)\right],$$

or, since $\lambda_j^* = 0$ for $j = \bar{m}+1, \bar{m}+2, \ldots, m$,

$$0 = \sum_{i=1}^{n} F_i(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0) - \sum_{j=1}^{\bar{m}} \lambda_j^* \left[\sum_{i=1}^{n} G_{j,i}(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0)\right].$$

In light of (21) and (22), this last equation simplifies to

$$0 = \sum_{i=1}^{n} F_i(x_1^*, x_2^*, \ldots, x_n^*)x_i'(0) + \lambda_1^*.$$

And hence, in light of (20),

$$\lambda_1^* \geq 0.$$

Analogous arguments show that

$$\lambda_j^* \geq 0$$

for $j = 2, 3, \ldots, \bar{m}$ as well, completing the proof.

4.2 The Envelope Theorem

Proving a generalized version of the envelope theorem requires no new ideas, just repeated applications of the previous ones.

Consider, again, the constrained optimization problem with n choice variables and m constraints:

$$\max_{x_1,x_2,\ldots,x_n} F(x_1,x_2,\ldots,x_n) \text{ subject to } c_j \geq G_j(x_1,x_2,\ldots,x_n) \text{ for all } j=1,2,\ldots,m.$$

Now extend this problem by allowing the functions F and G_j, $j=1,2,\ldots,m$, to depend on a parameter $\theta \in \mathbf{R}$:

$$\begin{aligned} \max_{x_1,x_2,\ldots,x_n} \quad & F(x_1,x_2,\ldots,x_n,\theta) \text{ subject to} \\ & c_j \geq G_j(x_1,x_2,\ldots,x_n,\theta) \text{ for all } j=1,2,\ldots,m. \end{aligned}$$

Just as before, define the maximum value function $V:\mathbf{R}\to\mathbf{R}$ as

$$\begin{aligned} V(\theta) \quad = \quad & \max_{x_1,x_2,\ldots,x_n} F(x_1,x_2,\ldots,x_n,\theta) \\ & \text{subject to } c_j \geq G_j(x_1,x_2,\ldots,x_n,\theta) \text{ for all } j=1,2,\ldots,m. \end{aligned}$$

Note that V is still a function of the single parameter θ, since the n choice variables are "optimized out." Put another way, evaluating V requires the same two-step procedure as before:

First, given θ, find the values $x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta)$ that solve the constrained optimization problem.

Second, substitute these values $x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta)$, together with the given value of θ, into the objective function to obtain

$$V(\theta)=F(x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta),\theta).$$

And just as before, the envelope theorem tells us that we can calculate the derivative $V'(\theta)$ of the maximum value function while ignoring the dependence of $x_1^*,x_2^*,\ldots,x_n^*$ and $\lambda_1^*,\lambda_2^*,\ldots,\lambda_m^*$ on θ, provided we invoke the complementary slackness conditions (16) to add the sum of all of the multipliers times all of the constraints to the objective function before differentiating through by θ.

Theorem (Envelope) Let F and G_j, $j=1,2,\ldots,m$, be continuously differentiable functions of $x_1,x_2,\ldots,x_n$ and θ. For any value of θ, let $x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta)$ maximize $F(x_1,x_2,\ldots,x_n,\theta)$ subject to $c_j \geq G_j(x_1,x_2,\ldots,x_n,\theta)$ for all $j=1,2,\ldots,m$, and let $\lambda_1^*(\theta),\lambda_2^*(\theta),\ldots,\lambda_m^*(\theta)$ be the associated values of the Lagrange multipliers. Suppose, further, that $x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta)$ and $\lambda_1^*(\theta),\lambda_2^*(\theta),\ldots,\lambda_m^*(\theta)$ are all continuously differentiable functions, and that the $\bar{m}(\theta)\times m$ matrix of derivatives

$$\begin{bmatrix} G_{1,1}(x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta),\theta) & \ldots & G_{1,n}(x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta),\theta) \\ G_{2,1}(x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta),\theta) & \ldots & G_{2,n}(x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta),\theta) \\ \vdots & \ddots & \vdots \\ G_{\bar{m}(\theta),1}(x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta),\theta) & \ldots & G_{\bar{m}(\theta),n}(x_1^*(\theta),x_2^*(\theta),\ldots,x_n^*(\theta),\theta) \end{bmatrix}$$

associated with the $\bar{m}(\theta) \leq m$ binding constraints has rank $\bar{m}(\theta)$ for each value of θ. Then the maximum value function defined by

$$\begin{aligned} V(\theta) &= \max_{x_1, x_2, \ldots, x_n} F(x_1, x_2, \ldots, x_n, \theta) \\ &\qquad \text{subject to } c_j \geq G_j(x_1, x_2, \ldots, x_n, \theta) \text{ for all } j = 1, 2, \ldots, m \end{aligned}$$

satisfies

$$\begin{aligned} V'(\theta) = F_{n+1}(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta) \\ - \sum_{j=1}^{m} \lambda_j^*(\theta) G_{j,n+1}(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta). \end{aligned} \tag{23}$$

Proof The Kuhn-Tucker theorem implies that for any given value of θ,

$$F_i(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta) - \sum_{j=1}^{m} \lambda_j^*(\theta) G_{j,i}(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta) = 0 \tag{13}$$

for $i = 1, 2, \ldots, n$, and

$$\lambda_j^*(\theta)[c_j - G_j(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta)] = 0, \tag{16}$$

for $j = 1, 2, \ldots, m$ must hold.

In light of (16),

$$V(\theta) = F(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta) + \sum_{j=1}^{m} \lambda_j^*(\theta)[c_j - G_j(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta)].$$

Differentiating both sides of this expression by θ yields

$$\begin{aligned} V'(\theta) &= \sum_{i=1}^{n} F_i(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta) x^{*\prime}(\theta) \\ &\quad + F_{n+1}(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta) \\ &\quad + \sum_{j=1}^{m} \lambda_j^{*\prime}(\theta)[c_j - G_j(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta)] \\ &\quad - \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_j^*(\theta) G_{j,i}(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta) x^{*\prime}(\theta) \\ &\quad - \sum_{j=1}^{m} \lambda_j^*(\theta) G_{j,n+1}(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta). \end{aligned}$$

which shows that, in principle, we must take the dependence of $x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta)$ and $\lambda_1^*(\theta), \lambda_2^*(\theta), \ldots, \lambda_m^*(\theta)$ on θ into account when calculating $V'(\theta)$.

Note, however, that (13) implies that the sums in the first and fourth lines of this last expression together equal zero. Hence, to show that (23) holds, it only remains to show that

$$\sum_{j=1}^{m} \lambda_j^{*\prime}(\theta)[c_j - G_j(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta)] = 0$$

and this is true if

$$\lambda_j^{*\prime}(\theta)[c_j - G_j(x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta), \theta)] = 0 \tag{24}$$

for all $j = 1, 2, \ldots, m$.

Clearly, (24) holds for any θ such that constraint j is binding.

For θ such that constraint j is not binding, (16) implies that $\lambda_j^*(\theta) = 0$. Furthermore, by the continuity of G_j and $x_i(\theta)$, $i = 1, 2, \ldots, n$, if constraint j does not bind at θ, there exists an $\varepsilon^* > 0$ such that constraint j does not bind for all $\theta + \varepsilon$ with $\varepsilon^* > |\varepsilon|$. Hence,

$$\lambda_j^{*\prime}(\theta) = \lim_{\varepsilon \to 0} \frac{\lambda_j^*(\theta + \varepsilon) - \lambda_j^*(\theta)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{0}{\varepsilon} = 0,$$

and once again it becomes apparent that (24) must hold. Hence, (23) must hold as well.

The Maximum Principle

Peter Ireland*
ECON 772001 - Math for Economists
Boston College, Department of Economics

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Here, we will explore the connections between two ways of solving dynamic optimization problems, that is, problems that involve optimization over time. The first solution method is just a straightforward application of the Kuhn-Tucker theorem; the second solution method relies on the maximum principle. Although these two approaches might at first glance seem quite different, in fact and as we will see, they are closely related.

We'll begin by briefly noting the basic features that set dynamic optimization problems apart from purely static ones. Then we'll go on consider the connections between the Kuhn-Tucker theorem and the maximum principle in both discrete and continuous time.

References:

Dixit, Chapter 10.

Acemoglu, Chapter 7.

The maximum principle was developed in the 1950s and 1960s by Soviet mathematicians, one of the key original references being:

L.S. Pontryagin, with V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko, *The Mathematical Theory of Optimal Processes*, 1962.

1 Basic Elements of Dynamic Optimization Problems

Moving from the static optimization problems that we've considered so far to the dynamic optimization problems that are of primary interest here involves only a few minor changes.

a) We need to index the variables that enter into the problem by t, in order to keep track of changes in those variables that occur over time.

b) We need to distinguish between two types of variables:

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stock variables - e.g., stock of capital, assets, or wealth

flow variables - e.g., output, consumption, saving, or labor supply per unit of time

c) We need to introduce constraints that describe the evolution of stock variables over time: e.g., larger flows of savings or investment today will lead to larger stocks of wealth or capital tomorrow.

2 The Maximum Principle: Discrete Time

2.1 A Dynamic Optimization Problem in Discrete Time

Consider a dynamic optimization in discrete time, that is, in which time can be indexed by $t = 0, 1, \ldots, T$.

y_t = stock variable

z_t = flow variable

Objective function:

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

Following Dixit, we can allow for a wider range of possibilities by letting the functions as well as the variables depend on the time index t.

$1 \geq \beta > 0$ = discount factor

Constraint describing the evolution of the stock variable:

$$Q(y_t, z_t; t) \geq y_{t+1} - y_t$$

or

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

for all $t = 0, 1, \ldots, T$

Constraint applying to variables within each period:

$$c \geq G(y_t, z_t; t)$$

for all $t = 0, 1, \ldots, T$

Constraints on initial and terminal values of stock:

$$y_0 \text{ given}$$

$$y_{T+1} \geq y^*$$

The dynamic optimization problem can now be stated as: choose sequences $\{z_t\}_{t=0}^T$ and $\{y_t\}_{t=1}^{T+1}$ to maximize the objective function subject to all of the constraints.

Notes:

a) It is important for the application of the maximum principle that the problem be additively time separable: that is, the values of F, Q, and G at time t must depend on the values of y_t and z_t only at time t.

b) Although the constraints describing the evolution of the stock variable and applying to the variables within each period can each be written in the form of a single equation, it must be emphasized that these constraints must hold for all $t = 0, 1, \dots, T$. That is, each of these equations actually describes $T+1$ constraints.

2.2 The Kuhn-Tucker Formulation

Let's begin by applying the Kuhn-Tucker Theorem to solve this problem. That is, let's set up the Lagrangian and take first-order conditions.

Set up the Lagrangian, recognizing that the constraints must hold for all $t = 0, 1, \dots, T$:

$$\begin{aligned} L &= \sum_{t=0}^{T} \beta^t F(y_t, z_t; t) \\ &\quad + \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}] + \sum_{t=0}^{T} \lambda_t [c - G(y_t, z_t; t)] + \phi(y_{T+1} - y^*) \end{aligned}$$

The Kuhn-Tucker theorem tells us that the solution to this problem must satisfy the FOC for the choice variables z_t for $t = 0, 1, \dots, T$ and y_t for $t = 1, 2, \dots, T+1$.

FOC for z_t, $t = 0, 1, \dots, T$:

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \tag{1}$$

for all $t = 0, 1, \dots, T$.

FOC for y_t, $t = 1, 2, \dots, T$:

$$\beta^t F_y(y_t, z_t; t) + \pi_{t+1} + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t) - \pi_t = 0$$

or

$$\pi_{t+1} - \pi_t = -[\beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t)] \tag{2}$$

for all $t = 1, 2, \dots, T$.

FOC for y_{T+1}:

$$-\pi_{T+1} + \phi = 0$$

Let's assume that the problem is such that the constraint governing the evolution of the stock variable always holds with equality, as will typically be the case in economic applications. Then another condition describing the solution to the problem is

$$y_{t+1} - y_t = Q(y_t, z_t; t) \tag{3}$$

for all $t = 0, 1, \ldots, T$.

Finally, let's write down the initial condition for the stock variable and the complementary slackness condition for the constraint on the terminal value of the stock:

$$y_0 \text{ given} \tag{4}$$

$$\phi(y_{T+1} - y^*) = 0$$

or, using the FOC for y_{T+1}:

$$\pi_{T+1}(y_{T+1} - y^*) = 0 \tag{5}$$

Notes:

a) Together with the complementary slackness condition

$$\lambda_t[c - G(y_t, z_t; t)] = 0,$$

which implies either

$$\lambda_t = 0 \text{ or } c = G(y_t, z_t; t),$$

we can think of (1)-(3) as forming a system of four equations in four unknowns y_t, z_t, π_t, λ_t. This system of equations determines the problem's solution.

b) Equations (2) and (3), linking the values of y_t and π_t at adjacent points in time, are examples of difference equations. They must be solved subject to two boundary conditions:

The initial condition (4).

The terminal, or transversality, condition (5).

c) The analysis can also be applied to the case of an infinite time horizon, where $T = \infty$. In this case, (1) must hold for all $t = 0, 1, 2, \ldots$, (2) must hold for all $t = 1, 2, 3, \ldots$, (3) must hold for all $t = 0, 1, 2, \ldots$, and (5) becomes a condition on the limiting behavior of π_t and y_t:

$$\lim_{T\to\infty} \pi_{T+1}(y_{T+1} - y^*) = 0. \tag{6}$$

2.3 An Alternative Formulation

Now let's consider the problem in a slightly different way.

Begin by defining the Hamiltonian for time t:

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t) \tag{7}$$

Strictly speaking, (7) defines what Dixit and Acemoglu refer to more correctly as the "maximized Hamilton." In particular, the "Hamiltonian" should be defined as a function of the three variables y_t, π_{t+1}, and z_t as

$$\hat{H}(y_t, \pi_{t+1}, z_t; t) = \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t).$$

Then the maximized Hamiltonian coincides with the Hamiltonian after z_t is "maximized out," subject to the constraint:

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \hat{H}(y_t, \pi_{t+1}, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t).$$

Note that the maximized Hamiltonian, as defined in (7), is a maximum value function.

Note also that the maximization problem on the right-hand side of (7) is a static optimization problem, involving no dynamic elements.

By the Kuhn-Tucker theorem:

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) + \lambda_t [c - G(y_t, z_t; t)]$$

And by the envelope theorem:

$$H_y(y_t, \pi_{t+1}; t) = \beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t) \tag{8}$$

and

$$H_\pi(y_t, \pi_{t+1}; t) = Q(y_t, z_t; t) \tag{9}$$

where z_t solves the optimization problem on the right-hand side of (7) and must therefore satisfy the FOC:

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \tag{10}$$

Now notice the following:

a) Equation (10) coincides with (1).

b) In light of (8) and (9), (2) and (3) can be written more compactly as

$$\pi_{t+1} - \pi_t = -[\beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t)] \tag{2}$$

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) \tag{11}$$

and

$$y_{t+1} - y_t = Q(y_t, z_t; t) \tag{3}$$

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t). \tag{12}$$

This establishes the following result.

Theorem (Maximum Principle) Consider the discrete time dynamic optimization problem of choosing sequences $\{z_t\}_{t=0}^T$ and $\{y_t\}_{t=1}^{T+1}$ to maximize the objective function

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

subject to the constraints

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

for all $t = 0, 1, \dots, T$,

$$c \geq G(y_t, z_t; t)$$

for all $t = 0, 1, \dots, T$,

$$y_0 \text{ given}$$

and

$$y_{T+1} \geq y^*.$$

Associated with this problem, define the maximized Hamiltonian

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t). \quad (7)$$

Then the solution to the dynamic optimization problem must satisfy

a) The first-order condition for the static optimization problem on the right-hand side of (7):

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \quad (10)$$

for all $t = 0, 1, \dots, T$.

b) The pair of difference equations:

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) \quad (11)$$

for all $t = 1, 2, \dots, T$ and

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t) \quad (12)$$

for all $t = 0, 1, \dots, T$, where the derivatives of H can be calculated using the envelope theorem.

c) The initial condition

$$y_0 \text{ given} \quad (4)$$

d) The terminal, or transversality, condition

$$\pi_{T+1}(y_{T+1} - y^*) = 0 \quad (5)$$

in the case where $T < \infty$ or

$$\lim_{T \to \infty} \pi_{T+1}(y_{T+1} - y^*) = 0 \quad (6)$$

in the case where $T = \infty$.

Thus, according to the maximum principle, there are two ways of solving discrete time dynamic optimization problems, both of which lead to the same answer:

a) Set up the Lagrangian for the dynamic optimization problem and take first-order conditions for all $t = 0, 1, \ldots, T$.

b) Set up the maximized Hamiltonian for the problem and derive the first-order and envelope conditions (10)-(12) for the static optimization problem that appears in the definition of that maximized Hamiltonian.

3 The Maximum Principle: Continuous Time

3.1 A Dynamic Optimization Problem in Continuous Time

Like the extension from static to dynamic optimization, the extension from discrete to continuous time requires no new substantive ideas, but does require some changes in notation.

Accordingly, suppose now that the variable t, instead of taking on discrete values $t = 0, 1, \ldots, T$, takes on continuous values $t \in [0, T]$, where as before, T can be finite or infinite.

It is most convenient now to regard the variables as functions of time:

$y(t)$ = stock variable

$z(t)$ = flow variable

The obvious analog to the objective function from before is:

$$\int_0^T e^{-\rho t} F(y(t), z(t); t)\, \mathrm{d}t$$

$\rho \geq 0$ = discount rate

Example:

$\beta = 0.95$

$\rho = 0.05$

β^t for $t = 1$ is 0.95

$e^{-\rho t}$ for $t = 1$ is 0.951, or approximately 0.95

Consider next the constraint describing the evolution of the stock variable.

In the discrete time case, the interval between time periods is just $\Delta t = 1$.

Hence, the constraint might be written as

$$Q(y(t), z(t); t)\Delta t \geq y(t + \Delta t) - y(t)$$

or

$$Q(y(t), z(t); t) \geq \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

In the limit as the interval Δt goes to zero, this last expression simplifies to

$$Q(y(t), z(t); t) \geq \dot{y}(t)$$

for all $t \in [0, T]$, where $\dot{y}(t)$ denotes the derivative of $y(t)$ with respect to t.

The constraint applying to variables at a given point in time remains the same:

$$c \geq G(y(t), z(t); t)$$

for all $t \in [0, T]$.

Note once again that these constraints must hold for all $t \in [0, T]$. Thus, each of the two equations from above actually represents an entire continuum of constraints.

Finally, the initial and terminal constraints for the stock variable remain unchanged:

$$y(0) \text{ given}$$

$$y(T) \geq y^*$$

The dynamic optimization problem can now be stated as: choose functions $z(t)$ for $t \in [0, T]$ and $y(t)$ for $t \in (0, T]$ to maximize the objective function subject to all of the constraints.

3.2 The Kuhn-Tucker Formulation

Once again, let's begin by setting up the Lagrangian and taking first-order conditions:

$$\begin{aligned} L &= \int_0^T e^{-\rho t} F(y(t), z(t); t)\, \mathrm{d}t + \int_0^T \pi(t)[Q(y(t), z(t); t) - \dot{y}(t)]\, \mathrm{d}t \\ &\quad + \int_0^T \lambda(t)[c - G(y(t), z(t); t)]\, \mathrm{d}t + \phi[y(T) - y^*] \end{aligned}$$

Now we are faced with a problem: $y(t)$ is a choice variable for all $t \in [0, T]$, but $\dot{y}(t)$ appears in the Lagrangian.

To solve this problem, use integration by parts:

$$\int_0^T \left\{ \frac{d}{dt} [\pi(t)y(t)] \right\} \mathrm{d}t = \int_0^T \dot{\pi}(t)y(t)\, \mathrm{d}t + \int_0^T \pi(t)\dot{y}(t)\, \mathrm{d}t$$

$$\pi(T)y(T) - \pi(0)y(0) = \int_0^T \dot{\pi}(t)y(t)\,\mathrm{d}t + \int_0^T \pi(t)\dot{y}(t)\,\mathrm{d}t$$

$$-\int_0^T \pi(t)\dot{y}(t)\,\mathrm{d}t = \int_0^T \dot{\pi}(t)y(t)\,\mathrm{d}t + \pi(0)y(0) - \pi(T)y(T)$$

Use this result to rewrite the Lagrangian as

$$\begin{aligned} L \;=\; & \int_0^T e^{-\rho t}F(y(t), z(t); t)\,\mathrm{d}t + \int_0^T \pi(t)Q(y(t), z(t); t)\,\mathrm{d}t \\ & + \int_0^T \dot{\pi}(t)y(t)\,\mathrm{d}t + \pi(0)y(0) - \pi(T)y(T) \\ & + \int_0^T \lambda(t)[c - G(y(t), z(t); t)]\,\mathrm{d}t + \phi[y(T) - y^*] \end{aligned}$$

Before taking first-order conditions, note that the multipliers $\pi(t)$ and $\lambda(t)$ are functions of t and that the corresponding constraints appear in the form of integrals. These features of the Lagrangian reflect the fact that the constraints must hold for all $t \in [0, T]$.

Next, we have to ask: how do we differentiate the Lagrangian L with respect to the objects of choice: $z(t)$ and $y(t)$? Providing a full answer to this question is part of what makes the theory of continuous-time optimization difficult. We can get some intuition, however, from considering the following, very simple, "variational argument;" for more details, see Acemoglu's Section 7.1 and the references he cites on page 276.

Consider the problem of choosing a continuously differentiable function $y(t)$, defined for $t \in [0, T]$, to maximize the integral

$$F(y) = \int_0^T f(y(t))\mathrm{d}t,$$

where f is two times continuously differentiable. Let $y^*(t)$ be a solution to this problem, and consider an "admissible variation"

$$y(t) = y^*(t) + \varepsilon\eta(t),$$

where $\varepsilon \in \mathbf{R}$ and $\eta(t)$ is another continuously differentiable function, so that $y(t)$ is also continuously differentiable. Now we can evaluate

$$F(y^* + \varepsilon\eta) = \int_0^T f(y^*(t) + \varepsilon\eta(t))\mathrm{d}t.$$

If F is maximized by y^*, it should not be possible to increase the value of this objective function, starting from y^* and hence $\varepsilon = 0$, by choosing a value of ε that is small in absolute value but either strictly positive or strictly negative. Hence, the derivative

$$\frac{dF(y^* + \varepsilon\eta)}{d\varepsilon} = \int_0^T f'(y^*(t) + \varepsilon\eta(t))\eta(t)\mathrm{d}t$$

must equal zero when evaluated at $\varepsilon = 0$:

$$\left.\frac{dF(y^* + \varepsilon\eta)}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^T f'(y^*(t))\eta(t)\mathrm{d}t = 0.$$

Since this optimality condition must hold for any continuously differentiable function $\eta(t)$, consider in particular the choice $\eta(t) = f'(y^*(t))$ for all $t \in [0, T]$, so that the optimality condition becomes

$$\int_0^T [f'(y^*(t))]^2 \mathrm{d}t = 0,$$

which can only hold if

$$f'(y^*(t)) = 0 \text{ for all } t \in [0, T].$$

What this suggests is that, to characterize the solution to continuous-time problems such as this, we can pretend that the integral in the objective function

$$F(y) = \int_0^T f(y(t))\mathrm{d}t$$

is really a sum, fix an arbitrary value of $t \in [0, T]$, and differentiate the entire objective function with respect to $y(t)$ for that value of t to get the

$$f'(y^*(t)) = 0 \text{ for all } t \in [0, T].$$

This is how we will now proceed, in deriving the first-order conditions for our more general, constrained optimization in continuous time using the Lagrangian

$$\begin{aligned} L &= \int_0^T e^{-\rho t} F(y(t), z(t); t)\,\mathrm{d}t + \int_0^T \pi(t) Q(y(t), z(t); t)\,\mathrm{d}t \\ &\quad + \int_0^T \dot{\pi}(t) y(t)\,\mathrm{d}t + \pi(0)y(0) - \pi(T)y(T) \\ &\quad + \int_0^T \lambda(t)[c - G(y(t), z(t); t)]\,\mathrm{d}t + \phi[y(T) - y^*] \end{aligned}$$

FOC for $z(t)$, $t \in [0, T]$:

$$e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) - \lambda(t) G_z(y(t), z(t); t) = 0 \tag{13}$$

for all $t \in [0, T]$

FOC for $y(t)$, $t \in (0, T)$:

$$e^{-\rho t} F_y(y(t), z(t); t) + \pi(t) Q_y(y(t), z(t); t) + \dot{\pi}(t) - \lambda(t) G_y(y(t), z(t); t) = 0$$

or

$$\dot{\pi}(t) = -[e^{-\rho t} F_y(y(t), z(t); t) + \pi(t) Q_y(y(t), z(t); t) - \lambda(t) G_y(y(t), z(t); t)]$$

for all $t \in (0, T)$.

If we require all functions of t to be continuously differentiable, then this last equation will also hold for $t = 0$ and $t = T$, and we can write

$$\dot{\pi}(t) = -[e^{-\rho t}F_y(y(t), z(t); t) + \pi(t)Q_y(y(t), z(t); t) - \lambda(t)G_y(y(t), z(t); t)] \tag{14}$$

for all $t \in [0, T]$.

FOC for $y(T)$:

$$\begin{aligned} 0 \;=\; & e^{-\rho T}F_y(y(T), z(T); T) + \pi(T)Q_y(y(T), z(T); T) \\ & + \dot{\pi}(T) - \pi(T) - \lambda(T)G_y(y(T), z(T); T) + \phi \end{aligned}$$

or, using (14),

$$\pi(T) = \phi$$

Assume, as before, that the constraint governing $\dot{y}(t)$ holds with equality:

$$\dot{y}(t) = Q(y(t), z(t); t) \tag{15}$$

for all $t \in [0, T]$.

Finally, write down the initial condition

$$y(0) \text{ given} \tag{16}$$

and the complementary slackness, or transversality condition

$$\phi[y(T) - y^*] = 0$$

or

$$\pi(T)[y(T) - y^*] = 0 \tag{17}$$

or in the infinite-horizon case

$$\lim_{T \to \infty} \pi(T)[y(T) - y^*] = 0. \tag{18}$$

Notes:

a) Together with the complementary slackness condition

$$\lambda(t)[c - G(y(t), z(t); t)] = 0,$$

we can think of (13)-(15) as a system of four equations in four unknowns $y(t)$, $z(t)$, $\pi(t)$, and $\lambda(t)$. This system of equations determines the problem's solution.

b) Equations (14) and (15), describing the behavior of $\dot{y}(t)$ and $\dot{\pi}(t)$, are examples of differential equations. They must be solved subject to two boundary conditions: (16) and either (17) or (18).

3.3 An Alternative Formulation

As before, define the maximized Hamiltonian for this problem as

$$H(y(t),\pi(t);t) = \max_{z(t)} e^{-\rho t}F(y(t),z(t);t) + \pi(t)Q(y(t),z(t);t) \quad \text{subject to } c \geq G(y(t),z(t);t) \tag{19}$$

As before, the maximized Hamiltonian is a maximum value function. And as before, the maximization problem of the right-hand side is a static one.

By the Kuhn-Tucker theorem:

$$H(y(t),\pi(t);t) = \max_{z(t)} e^{-\rho t}F(y(t),z(t);t) + \pi(t)Q(y(t),z(t);t) + \lambda(t)[c - G(y(t),z(t);t)]$$

And by the envelope theorem:

$$H_y(y(t),\pi(t);t) = e^{-\rho t}F_y(y(t),z(t);t) + \pi(t)Q_y(y(t),z(t);t) - \lambda(t)G_y(y(t),z(t);t) \tag{20}$$

and

$$H_\pi(y(t),\pi(t);t) = Q(y(t),z(t);t) \tag{21}$$

where $z(t)$ solves the optimization problem on the right-hand side of (19) and must therefore satisfy the FOC:

$$e^{-\rho t}F_z(y(t),z(t);t) + \pi(t)Q_z(y(t),z(t);t) - \lambda(t)G_z(y(t),z(t);t) = 0. \tag{22}$$

Now notice the following:

a) Equation (22) coincides with (13).

b) In light of (20) and (21), (14) and (15) can be written more compactly as

$$\dot{\pi}(t) = -H_y(y(t),\pi(t);t) \tag{23}$$

and

$$\dot{y}(t) = H_\pi(y(t),\pi(t);t). \tag{24}$$

This establishes the following result.

Theorem (Maximum Principle) Consider the continuous time dynamic optimization problem of choosing continuously differentiable functions $z(t)$ and $y(t)$ for $t \in [0,T]$ to maximize the objective function

$$\int_0^T e^{-\rho t}F(y(t),z(t);t)\,\mathrm{d}t$$

subject to the constraints

$$Q(y(t),z(t);t) \geq \dot{y}(t)$$

for all $t \in [0, T]$,

$$c \geq G(y(t), z(t); t)$$

for all $t \in [0, T]$,

$$y(0) \text{ given,}$$

and

$$y(T) \geq y^*.$$

Associated with this problem, define the maximized Hamiltonian

$$\begin{aligned} H(y(t), \pi(t); t) = &\max_{z(t)} e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \\ &\text{subject to } c \geq G(y(t), z(t); t) \end{aligned} \tag{19}$$

Then the solution to the dynamic optimization problem must satisfy

a) The first-order condition for the static optimization problem on the right-hand side of (19):

$$e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) - \lambda(t) G_z(y(t), z(t); t) = 0 \tag{22}$$

for all $t \in [0, T]$.

b) The pair of differential equations

$$\dot{\pi}(t) = -H_y(y(t), \pi(t); t) \tag{23}$$

and

$$\dot{y}(t) = H_\pi(y(t), \pi(t); t) \tag{24}$$

for all $t \in [0, T]$, where the derivatives of H can be calculated using the envelope theorem.

c) The initial condition

$$y(0) \text{ given.} \tag{16}$$

d) The terminal, or transversality, condition

$$\pi(T)[y(T) - y^*] = 0 \tag{17}$$

in the case where $T < \infty$ or

$$\lim_{T \to \infty} \pi(T)[y(T) - y^*] = 0. \tag{18}$$

in the case where $T = \infty$.

Once again, according to the maximum principle, there are two ways of solving continuous time dynamic optimization problems, both of which lead to the same answer:

a) Set up the Lagrangian for the dynamic optimization problem and take first-order conditions for all $t \in [0, T]$.

b) Set up the maximized Hamiltonian for the problem and derive the first-order and envelope conditions (22)-(24) for the static optimization problem that appears in that definition of the maximized Hamiltonian.

4 Two Examples

4.1 Life-Cycle Saving

Consider a consumer who is employed for $T+1$ years: $t = 0, 1, \dots, T$.

w = constant annual labor income

k_t = stock of assets at the beginning of period $t = 0, 1, \dots, T+1$

$k_0 = 0$

k_t can be negative for $t = 1, 2, \dots, T$, so that the consumer is allowed borrow.

However, k_{T+1} must satisfy

$$k_{T+1} \geq k^* > 0$$

where k^* denotes saving required for retirement.

r = constant interest rate

total income during period $t = w + rk_t$

c_t = consumption

Hence,

$$k_{t+1} = k_t + w + rk_t - c_t$$

or equivalently,

$$k_t + Q(k_t, c_t; t) \geq k_{t+1}$$

where

$$Q(k_t, c_t; t) = Q(k_t, c_t) = w + rk_t - c_t$$

for all $t = 0, 1, \dots, T$

Utility function:

$$\sum_{t=0}^{T} \beta^t \ln(c_t)$$

The consumer's problem: choose sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ to maximize the utility function subject to all of the constraints.

For this problem:

k_t = stock variable
c_t = flow variable

To solve this problem, set up the maximized Hamiltonian:

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(w + rk_t - c_t)$$

FOC for c_t:

$$\frac{\beta^t}{c_t} = \pi_{t+1} \tag{25}$$

Difference equations for π_t and k_t:

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1} r \tag{26}$$

and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = w + rk_t - c_t \tag{27}$$

Equations (25)-(27) represent a system of three equations in the three unknowns c_t, π_t, and k_t. They must be solved subject to the boundary conditions

$$k_0 = 0 \text{ given} \tag{28}$$

and

$$\pi_{T+1}(k_{T+1} - k^*) = 0 \tag{29}$$

We can use (25)-(29) to deduce some key properties of the solution even without solving the system completely.

Note first that (25) implies that

$$\pi_{T+1} = \frac{\beta^T}{c_T} > 0.$$

Hence, it follows from (29) that

$$k_{T+1} = k^*.$$

Thus, the consumer saves just enough for retirement and no more.

Next, note that (26) implies

$$\pi_{t+1} - \pi_t = -\pi_{t+1} r$$
$$(1+r)\pi_{t+1} = \pi_t \tag{30}$$

Use (25) to obtain

$$\pi_{t+1} = \frac{\beta^t}{c_t} \text{ and } \pi_t = \frac{\beta^{t-1}}{c_{t-1}}$$

and substitute these expressions into (30) to obtain

$$(1+r)\frac{\beta^t}{c_t} = \frac{\beta^{t-1}}{c_{t-1}}$$

$$(1+r)\frac{\beta}{c_t} = \frac{1}{c_{t-1}}$$

$$\frac{c_t}{c_{t-1}} = \beta(1+r) \tag{31}$$

Equation (31) reveals that the optimal growth rate of consumption is constant, and is faster for a more patient consumer, with a higher value of β, and a consumer that faces a higher interest rate r.

But now let's go a step further, to characterize the solution more fully. Equation (27),

$$k_{t+1} - k_t = w + rk_t - c_t, \tag{27}$$

and (31)

$$\frac{c_t}{c_{t-1}} = \beta(1+r) \tag{31}$$

now form a system of 2 difference equations in two unknowns c_t and k_t. In effect, we've used the first order condition (25) to eliminate the unknown multiplier π_t from the system, in much the same way that in our earlier static constrained optimization problems, we worked to simplify the optimality conditions by solving out for the unknown Lagrange multiplier.

Let's look at the simpler of these two equations, (31), first. Note that (31) implies that once c_0 is chosen, the entire sequence $\{c_t\}_{t=0}^T$ is pinned down:

$$c_1 = \beta(1+r)c_0,$$

$$c_2 = \beta(1+r)c_1 = [\beta(1+r)]^2 c_0,$$
$$c_3 = \beta(1+r)c_2 = [\beta(1+r)]^3 c_0,$$

or

$$c_t = [\beta(1+r)]^t c_0$$

for all $t = 0, 1, \ldots, T$. Hence, once we determine the optimal value of c_0, we can iterate forward on (31) to construct recursively the entire sequence $\{c_t\}_{t=0}^T$.

Next, let's turn to (27), which can be rewritten as

$$k_{t+1} = (1+r)k_t + w - c_t$$

For $t = 0$,

$$k_1 = (1+r)k_0 + w - c_0.$$

And, for $t = 1$,

$$\begin{aligned} k_2 &= (1+r)k_1 + w - c_1 \\ &= (1+r)^2 k_0 + (1+r)(w - c_0) + w - c_1. \end{aligned}$$

Likewise, for $t = 2$,

$$\begin{aligned} k_3 &= (1+r)k_2 + w - c_2 \\ &= (1+r)^3 k_0 + (1+r)^2(w - c_0) + (1+r)(w - c_1) + w - c_2. \end{aligned}$$

Repeating this procedure eventually yields

$$k_{T+1} = (1+r)^{T+1}k_0 + \sum_{t=0}^{T}(1+r)^{T-t}(w - c_t)$$

or

$$k_{T+1} = (1+r)^{T+1}k_0 + \sum_{t=0}^{T}(1+r)^{T-t}w - \sum_{t=0}^{T}(1+r)^{T-t}[\beta(1+r)]^t c_0$$

or

$$k_{T+1} = (1+r)^{T+1}k_0 + (1+r)^T \sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t w - (1+r)^T \sum_{t=0}^{T}\beta^t c_0.$$

Now, use the initial condition k_0 and the terminal condition $k_{T+1} = k^*$ to find the optimal value of c_0:

$$\frac{k^*}{(1+r)^T} = \sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t w - \sum_{t=0}^{T}\beta^t c_0$$

or

$$c_0 = \left(\sum_{t=0}^{T}\beta^t\right)^{-1}\left[\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t w - \frac{k^*}{(1+r)^T}\right].$$

Note, finally, that all of the items on the right-hand-side of this last expression for c_0 are parameters: β, r, w, k^*, and T. So given numerical settings for these parameters, one could numerically compute c_0 and, from there, go back to (27), (31), and $k_0 = 0$ to recursively construct the optimal sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$.

4.2 Optimal Growth

Consider an economy in which output is produced with capital according to the production function

$$F(k_t) = k_t^\alpha,$$

where $0 < \alpha < 1$.

c_t = consumption

δ = depreciation rate for capital, $0 < \delta < 1$

Then the evolution of the capital stock is governed by

$$k_{t+1} = k_t^\alpha + (1-\delta)k_t - c_t$$

or

$$k_{t+1} - k_t = k_t^\alpha - \delta k_t - c_t$$

Our first example had a finite horizon and was cast in discrete time. So for the sake of variety, make this second example have an infinite horizon in continuous time.

The continuous time analog to the capital accumulation constraint shown above is just

$$k(t)^{\alpha} - \delta k(t) - c(t) \geq \dot{k}(t)$$

or

$$Q(k(t), c(t); t) \geq \dot{k}(t),$$

where

$$Q(k(t), c(t); t) = Q(k(t), c(t)) = k(t)^{\alpha} - \delta k(t) - c(t)$$

for all $t \in [0, \infty)$

Initial condition:

$$k(0) \text{ given}$$

Objective of a benevolent social planner or the utility of an infinitely-lived representative consumer:

$$\int_0^{\infty} e^{-\rho t} \ln(c(t)) \, \mathrm{d}t,$$

where $\rho > 0$ is the discount rate.

The problem: choose continuously differentiable functions $c(t)$ and $k(t)$ for $t \in [0, \infty)$ to maximize utility subject to all of the constraints.

For this problem:

$$k(t) = \text{stock variable}$$
$$c(t) = \text{flow variable}$$

To solve this problem, set up the maximized Hamiltonian:

$$H(k(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)[k(t)^{\alpha} - \delta k(t) - c(t)]$$

FOC for $c(t)$:

$$e^{-\rho t} = c(t)\pi(t) \tag{32}$$

Differential equations for $\pi(t)$ and $k(t)$:

$$\dot{\pi}(t) = -H_k(k(t), \pi(t); t) = -\pi(t)[\alpha k(t)^{\alpha-1} - \delta] \tag{33}$$

and

$$\dot{k}(t) = H_{\pi}(k(t), \pi(t); t) = k(t)^{\alpha} - \delta k(t) - c(t). \tag{34}$$

Equations (32)-(34) form a system of three equations in the three unknowns $c(t)$, $\pi(t)$, and $k(t)$. How can we solve them?

Start by differentiating both sides of (32) with respect to t:

$$e^{-\rho t} = c(t)\pi(t) \tag{32}$$

$$-\rho e^{-\rho t} = \dot{c}(t)\pi(t) + c(t)\dot{\pi}(t)$$
$$-\rho c(t)\pi(t) = \dot{c}(t)\pi(t) + c(t)\dot{\pi}(t)$$

Next, use (33)

$$\dot{\pi}(t) = -\pi(t)[\alpha k(t)^{\alpha-1} - \delta] \tag{33}$$

to rewrite this last equation as

$$-\rho c(t)\pi(t) = \dot{c}(t)\pi(t) - c(t)\pi(t)[\alpha k(t)^{\alpha-1} - \delta]$$

$$-\rho c(t) = \dot{c}(t) - c(t)[\alpha k(t)^{\alpha-1} - \delta]$$
$$\dot{c}(t) = c(t)[\alpha k(t)^{\alpha-1} - \delta - \rho] \tag{35}$$

Collect (34) and (35):

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t). \tag{34}$$
$$\dot{c}(t) = c(t)[\alpha k(t)^{\alpha-1} - \delta - \rho] \tag{35}$$

and notice that these two differential equations depend only on $k(t)$ and $c(t)$.

Equation (35) implies that $\dot{c}(t) = 0$ when

$$\alpha k(t)^{\alpha-1} - \delta - \rho = 0$$

or

$$k(t) = \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)} = k^*$$

And since $\alpha - 1 < 0$, (35) also implies that $\dot{c}(t) < 0$ when $k(t) > k^*$ and $\dot{c}(t) > 0$ when $k(t) < k^*$.

Equation (34) implies that $\dot{k}(t) = 0$ when

$$k(t)^{\alpha} - \delta k(t) - c(t) = 0$$

or

$$c(t) = k(t)^{\alpha} - \delta k(t).$$

Moreover, (34) implies that $\dot{k}(t) < 0$ when

$$c(t) > k(t)^{\alpha} - \delta k(t)$$

and $\dot{k}(t) > 0$ when

$$c(t) < k(t)^{\alpha} - \delta k(t)$$

We can illustrate these conditions graphically using the phase diagram below, which reveals that:

The economy has a steady state at (k^*, c^*).

For each possible value of k_0, there exists a unique value of c_0 such that, starting from (k_0, c_0), the economy converges to the steady state (k^*, c^*).

Starting from all other values of c_0, either k becomes negative or c approaches zero.

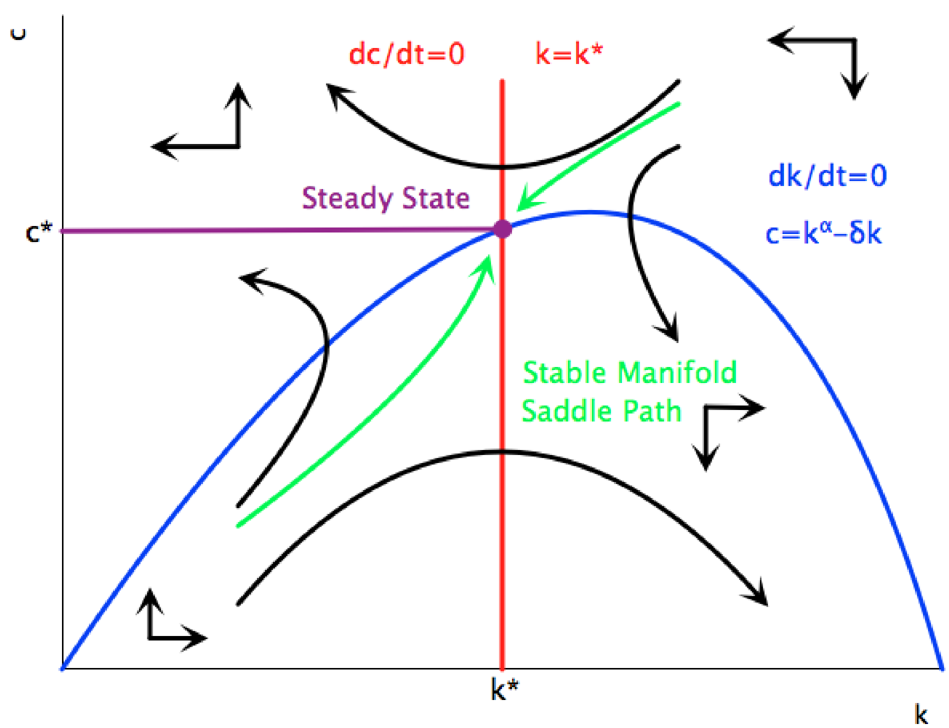
Trajectories that lead to negative values of k violate the nonnegativity condition for capital, and hence cannot represent a solution.

Trajectories that lead towards zero values of c violate the transversality condition

$$\lim_{T\to\infty} \pi(T)k(T) = \lim_{T\to\infty} \frac{e^{-\rho T}}{c(T)} k(T) = 0$$

and hence cannot represent a solution. (*Note:* for a proof of the necessity of this transversality condition for the optimal growth model, see Takashi Kamihigashi, "Necessity of Transversality Conditions for Infinite Horizon Problems," *Econometrica* vol. 69, July 2001, pp. 995-1012.)

Hence, the phase diagram allows us to identify the model's unique solution.



It is possible to see these same results numerically, using the discrete-time version of the model:

$$\max_{\{c_t\}_{t=0}^{\infty}, \{k_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to k_0 given and

$$k_t^\alpha - \delta k_t - c_t \geq k_{t+1} - k_t \text{ for all } t = 0, 1, 2, \ldots.$$

Set up the maximized Hamiltonian:

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(k_t^\alpha - \delta k_t - c_t)$$

FOC for c_t:

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

Difference equations for π_t and k_t:

$$\begin{aligned}\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) &= -\pi_{t+1}(\alpha k_t^{\alpha-1} - \delta) \\ k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) &= k_t^\alpha - \delta k_t - c_t \end{aligned} \tag{36}$$

Just as in the continuous-time case, use the first-order condition to solve for π_{t+1} and π_t, and substitute the results into the difference equation for π_t to obtain

$$\frac{\beta^t(\alpha k_t^{\alpha-1} + 1 - \delta)}{c_t} = \frac{\beta^{t-1}}{c_{t-1}}$$

or, simplified and rolled forward one period:

$$c_{t+1} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)c_t \tag{37}$$

Equations (36) and (37) are the discrete-time analogs to (34) and (35). Once numerical values are assigned to the parameters α, δ, and β, (36) and (37) can be used to construct the entire sequences $\{c_t\}_{t=0}^\infty$ and $\{k_t\}_{t=0}^\infty$ starting from the given value for k_0 and a conjectured value for c_0. Numerical analysis will reveal that for each value of k_0, there is a unique value of c_0 such that, starting from (k_0, c_0), the economy converges to the steady state (k^*, c^*). Starting from all other values of c_0, the sequences violate either the nonnegativity condition for capital or the transversality condition

$$\lim_{T\to\infty} \pi_{T+1} k_{T+1} = \lim_{T\to\infty} \frac{\beta^T}{c_T} k_{T+1} = 0$$

and hence cannot represent a solution.

5 One Final Note on the Maximum Principle

In applying the maximum principle in discrete time, we defined the maximized Hamiltonian as

$$\begin{aligned} H(y_t, \pi_{t+1}; t) &= \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t) \\ &= \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) + \lambda_t [c - G(y_t, z_t; t)] \end{aligned} \tag{7}$$

and used this definition to derive the optimality conditions (10)-(12) and either (5) or (6), depending on whether the horizon is finite or infinite.

The Hamiltonian

$$\hat{H}(y_t, \pi_{t+1}, z_t; t) = \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t).$$

that enters into the definition of the maximized Hamiltonian in (7) is often called the *present-value* Hamiltonian, because $\beta^t F(y_t, z_t; t)$ measures the present value at time $t = 0$ of the payoff $F(y_t, z_t; t)$ received at time $t > 0$.

The present-value Hamilton stands in contrast to the *current-value* Hamiltonian, defined by multiplying both sides of (7) by β^{-t}:

$$\begin{aligned} \beta^{-t} H(y_t, \pi_{t+1}; t) &= \max_{z_t} F(y_t, z_t; t) + \beta^{-t}\pi_{t+1} Q(y_t, z_t; t) + \beta^{-t}\lambda_t[c - G(y_t, z_t; t)] \\ &= \max_{z_t} F(y_t, z_t; t) + \theta_{t+1} Q(y_t, z_t; t) + \mu_t[c - G(y_t, z_t; t)] \\ &= \tilde{H}(y_t, \theta_{t+1}; t), \end{aligned}$$

where the last line states the definition of the maximized current-value Hamiltonian $\tilde{H}(y_t, \theta_{t+1}; t)$, where

$$\theta_{t+1} = \beta^{-t}\pi_{t+1} \Rightarrow \pi_{t+1} = \beta^t \theta_{t+1}$$

and

$$\mu_t = \beta^{-t}\lambda_t \Rightarrow \lambda_t = \beta^t \mu_t,$$

and where the current-value Hamiltonian itself is

$$\hat{\tilde{H}}(y_t, \theta_{t+1}, z_t) = F(y_t, z_t; t) + \theta_{t+1} Q(y_t, z_t; t),$$

and therefore depends on the current value at t of the payoff $F(y_t, z_t; t)$ received at time t.

Let's consider rewriting the optimality conditions (10)-(12) and (5) in terms of the maximized current value Hamiltonian $\tilde{H}(y_t, \theta_{t+1}; t)$.

To do this, note first that by definition

$$H(y_t, \pi_{t+1}; t) = \beta^t \tilde{H}(y_t, \theta_{t+1}; t) = \beta^t \tilde{H}(y_t, \beta^{-t}\pi_{t+1}; t)$$

Hence

$$H_y(y_t, \pi_{t+1}; t) = \beta^t \tilde{H}_y(y_t, \theta_{t+1}; t)$$

and

$$\begin{aligned} H_\pi(y_t, \pi_{t+1}; t) &= \frac{\partial}{\partial \pi_{t+1}}[\beta^t \tilde{H}(y_t, \beta^{-t}\pi_{t+1}; t)] \\ &= \beta^t \beta^{-t} \tilde{H}_\theta(y_t, \beta^{-t}\pi_{t+1}; t) \\ &= \tilde{H}_\theta(y_t, \theta_{t+1}; t) \end{aligned}$$

In light of these results, (10) can be rewritten

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \tag{10}$$

$$F_z(y_t, z_t; t) + \beta^{-t}\pi_{t+1} Q_z(y_t, z_t; t) - \beta^{-t}\lambda_t G_z(y_t, z_t; t) = 0$$

$$F_z(y_t, z_t; t) + \theta_{t+1} Q_z(y_t, z_t; t) - \mu_t G_z(y_t, z_t; t) = 0 \tag{10$'$}$$

(11) can be rewritten

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) \tag{11}$$

$$\beta^t \theta_{t+1} - \beta^{t-1}\theta_t = -\beta^t \tilde{H}_y(y_t, \theta_{t+1}; t)$$

$$\theta_{t+1} - \beta^{-1}\theta_t = -\tilde{H}_y(y_t, \theta_{t+1}; t) \tag{11$'$}$$

(12) can be rewritten

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t) \tag{12}$$

$$y_{t+1} - y_t = \tilde{H}_\theta(y_t, \theta_{t+1}; t) \tag{12$'$}$$

(5) can be rewritten

$$\pi_{T+1}(y_{T+1} - y^*) = 0 \tag{5}$$

$$\beta^T \theta_{T+1}(y_{T+1} - y^*) = 0 \tag{5$'$}$$

(6) can be rewritten

$$\lim_{T\to\infty} \pi_{T+1}(y_{T+1} - y^*) = 0 \tag{6}$$

$$\lim_{T\to\infty} \beta^T \theta_{T+1}(y_{T+1} - y^*) = 0 \tag{6$'$}$$

Thus, when the maximum principle in discrete time is stated in terms of the current-value Hamiltonian instead of the present-value Hamiltonian, (10)-(12) and (5) or (6) are replaced by (10′)-(12′) and (5′) or (6′).

We can use the same types of transformations in the case of continuous time, where the maximized present-value Hamiltonian is defined by

$$\begin{aligned} H(y(t), \pi(t); t) = &\max_{z(t)} e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \\ &\text{subject to } c \geq G(y(t), z(t); t) \\ = &\max_{z(t)} e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \\ &+ \lambda(t)[c - G(y(t), z(t); t)] \end{aligned} \tag{19}$$

Define the current-value Hamiltonian by multiplying both sides of (19) by $e^{\rho t}$:

$$\begin{aligned} e^{\rho t} H(y(t), \pi(t); t) &= \max_{z(t)} F(y(t), z(t); t) + e^{\rho t}\pi(t) Q(y(t), z(t); t) \\ &\quad + e^{\rho t}\lambda(t)[c - G(y(t), z(t); t)] \\ &= \max_{z(t)} F(y(t), z(t); t) + \theta(t) Q(y(t), z(t); t) \\ &\quad + \mu(t)[c - G(y(t), z(t); t)] \\ &= \tilde{H}(y(t), \theta(t); t) \end{aligned}$$

where the last line defines the maximized current-value Hamiltonian $\tilde{H}(y(t), \theta(t); t)$, where

$$\theta(t) = e^{\rho t}\pi(t) \Rightarrow \pi(t) = e^{-\rho t}\theta(t)$$

and

$$\mu(t) = e^{\rho t}\lambda(t) \Rightarrow \lambda(t) = e^{-\rho t}\mu(t),$$

and where the current-value Hamiltonian itself is

$$\hat{\tilde{H}}(y(t), \theta(t), z(t)) = F(y(t), z(t); t) + \theta(t)Q(y(t), z(t); t).$$

In the case of continuous time, the optimality conditions derived from (19) are (22)-(24) and either (17) or (18). Let's rewrite these conditions in terms of the maximized current-value Hamiltonian $\tilde{H}(y(t), \theta(t); t)$.

To begin, note that

$$H(y(t), \pi(t); t) = e^{-\rho t}\tilde{H}(y(t), \theta(t); t) = e^{-\rho t}\tilde{H}(y(t), e^{\rho t}\pi(t); t)$$

Hence

$$H_y(y(t), \pi(t); t) = e^{-\rho t}\tilde{H}_y(y(t), \theta(t); t)$$

and

$$\begin{aligned} H_\pi(y(t), \pi(t); t) &= \frac{\partial}{\partial \pi(t)}[e^{-\rho t}\tilde{H}(y(t), e^{\rho t}\pi(t); t)] \\ &= e^{-\rho t}e^{\rho t}\tilde{H}_\theta(y(t), e^{\rho t}\pi(t); t) \\ &= \tilde{H}_\theta(y(t), \theta(t); t) \end{aligned}$$

and, finally,

$$\dot{\pi}(t) = \frac{\partial}{\partial t}[e^{-\rho t}\theta(t)] = -\rho e^{-\rho t}\theta(t) + e^{-\rho t}\dot{\theta}(t)$$

In light of these results, (22) can be rewritten

$$e^{-\rho t}F_z(y(t), z(t); t) + \pi(t)Q_z(y(t), z(t); t) - \lambda(t)G_z(y(t), z(t); t) = 0 \tag{22}$$

$$F_z(y(t), z(t); t) + e^{\rho t}\pi(t)Q_z(y(t), z(t); t) - e^{\rho t}\lambda(t)G_z(y(t), z(t); t) = 0$$

$$F_z(y(t), z(t); t) + \theta(t)Q_z(y(t), z(t); t) - \mu(t)G_z(y(t), z(t); t) = 0 \tag{22'}$$

(23) can be rewritten

$$\dot{\pi}(t) = -H_y(y(t), \pi(t); t) \tag{23}$$

$$-\rho e^{-\rho t}\theta(t) + e^{-\rho t}\dot{\theta}(t) = -e^{-\rho t}\tilde{H}_y(y(t), \theta(t); t)$$

$$\dot{\theta}(t) = \rho\theta(t) - \tilde{H}_y(y(t), \theta(t); t) \tag{23'}$$

(24) can be rewritten

$$\dot{y}(t) = H_\pi(y(t), \pi(t); t) \tag{24}$$

$$\dot{y}(t) = \tilde{H}_\theta(y(t), \theta(t); t) \tag{24'}$$

(17) can be rewritten

$$\pi(T)[y(T) - y^*] = 0 \tag{17}$$

$$e^{-\rho T}\theta(T)[y(T) - y^*] = 0 \tag{17'}$$

(18) can be rewritten

$$\lim_{T\to\infty} \pi(T)[y(T) - y^*] = 0 \tag{18}$$

$$\lim_{T\to\infty} e^{-\rho T}\theta(T)[y(T) - y^*] = 0 \tag{18'}$$

Thus, when the maximum principle in continuous time is stated in terms of the current-value Hamiltonian instead of the present-value Hamiltonian, (22)-(24) and (17) or (18) are replaced by (22′)-(24′) and (17′) or (18′).

Dynamic Programming

Peter Ireland*
ECON 772001 - Math for Economists
Boston College, Department of Economics

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We have now studied two ways of solving dynamic optimization problems, one based on the Kuhn-Tucker theorem and the other based on the maximum principle. These two methods both lead us to the same sets of optimality conditions; they differ only in terms of how those optimality conditions are derived.

Here, we will consider a third way of solving dynamic optimization problems: the method of dynamic programming. We will see, once again, that dynamic programming leads us to the same set of optimality conditions that the Kuhn-Tucker theorem does; once again, this new method differs from the others only in terms of how the optimality conditions are derived.

While the maximum principle lends itself equally well to dynamic optimization problems set in both discrete time and continuous time, dynamic programming is easiest to apply in discrete time settings. On the other hand, dynamic programing, unlike the Kuhn-Tucker theorem and the maximum principle, can be used quite easily to solve problems in which optimal decisions must be made under conditions of uncertainty.

Thus, in our discussion of dynamic programming, we will begin by considering dynamic programming under certainty; later, we will move on to consider stochastic dynamic programming.

References:

Dixit, Chapter 11.

Acemoglu, Chapters 6 and 16.

Dynamic programming was invented by Richard Bellman in the late 1950s, around the same time that Pontryagin and his colleagues were working out the details of the maximum principle. A famous early reference is:

Richard Bellman. *Dynamic Programming*, 1957.

A very comprehensive reference with many economic examples is

Nancy L. Stokey and Robert E. Lucas, Jr. with Edward C. Prescott. *Recursive Methods in Economic Dynamics*, 1989.

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1 Dynamic Programming Under Certainty

1.1 A Perfect Foresight Dynamic Optimization Problem in Discrete Time

No uncertainty

Discrete time, infinite horizon: $t = 0, 1, 2, \ldots$

y_t = stock, or state, variable

z_t = flow, or control, variable

Objective function:

$$\sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

$1 > \beta > 0$ discount factor

Constraint describing the evolution of the state variable

$$Q(y_t, z_t; t) \geq y_{t+1} - y_t$$

or

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$

Constraint applying to variables within each period:

$$c \geq G(y_t, z_t; t)$$

for all $t = 0, 1, 2, \ldots$

Constraint on initial value of the state variable:

$$y_0 \text{ given}$$

The problem: choose sequences $\{z_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ to maximize the objective function subject to all of the constraints.

Notes:

a) It is important for the application of dynamic programming that the problem is additively time separable: that is, the values of F, Q, and G at time t must depend only on the values of y_t and z_t at time t.

b) Once again, it must be emphasized that although the constraints describing the evolution of the state variable and that apply to the variables within each period can each be written in the form of a single equation, these constraints must hold for all $t = 0, 1, 2, \ldots$. Thus, each equation actually represents an infinite number of constraints.

1.2 The Kuhn-Tucker Formulation

Let's being our analysis of this problem by applying the Kuhn-Tucker theorem. That is, let's begin by setting up the Lagrangian and taking first order conditions.

Set up the Lagrangian, recognizing that the constraints must hold for all $t = 0, 1, 2, \ldots$:

$$L = \sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t) + \sum_{t=0}^{\infty} \tilde{\mu}_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}] + \sum_{t=0}^{\infty} \tilde{\lambda}_t [c - G(y_t, z_t; t)]$$

It will be convenient to define

$$\mu_{t+1} = \beta^{-(t+1)}\tilde{\mu}_{t+1} \Rightarrow \tilde{\mu}_{t+1} = \beta^{t+1}\mu_{t+1}$$
$$\lambda_t = \beta^{-t}\tilde{\lambda}_t \Rightarrow \tilde{\lambda}_t = \beta^t \lambda_t$$

and to rewrite the Lagrangian in terms of μ_{t+1} and λ_t instead of $\tilde{\mu}_{t+1}$ and $\tilde{\lambda}_t$:

$$L = \sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t) + \sum_{t=0}^{\infty} \beta^{t+1}\mu_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}] + \sum_{t=0}^{\infty} \beta^t \lambda_t [c - G(y_t, z_t; t)]$$

FOC for z_t, $t = 0, 1, 2, \ldots$:

$$\beta^t F_2(y_t, z_t; t) + \beta^{t+1}\mu_{t+1} Q_2(y_t, z_t; t) - \beta^t \lambda_t G_2(y_t, z_t; t) = 0$$

FOC for y_t, $t = 1, 2, 3, \ldots$:

$$\beta^t F_1(y_t, z_t; t) + \beta^{t+1}\mu_{t+1}[1 + Q_1(y_t, z_t; t)] - \beta^t \lambda_t G_1(y_t, z_t; t) - \beta^t \mu_t = 0$$

Now, let's suppose that somehow we could solve for μ_t as a function of the state variable y_t:

$$\mu_t = W(y_t; t)$$
$$\mu_{t+1} = W(y_{t+1}; t+1) = W[y_t + Q(y_t, z_t; t); t+1]$$

Then we could rewrite the FOC as:

$$F_2(y_t, z_t; t) + \beta W[y_t + Q(y_t, z_t; t); t+1] Q_2(y_t, z_t; t) - \lambda_t G_2(y_t, z_t; t) = 0 \quad (1)$$

$$W(y_t; t) = F_1(y_t, z_t; t) + \beta W[y_t + Q(y_t, z_t; t); t+1][1 + Q_1(y_t, z_t; t)] - \lambda_t G_1(y_t, z_t; t) \quad (2)$$

And together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t; t) \quad (3)$$

and the complementary slackness condition

$$\lambda_t [c - G(y_t, z_t; t)] = 0 \quad (4)$$

we can think of (1) and (2) as forming a system of four equations in three unknown variables y_t, z_t, and λ_t and one unknown function $W(\cdot; t)$. This system of equations determines the problem's solution.

Note that since (3) is in the form of a difference equation, finding the problem's solution involves solving a difference equation.

1.3 An Alternative Formulation

Now let's consider the same problem in a slightly different way.

For any given value of the initial state variable y_0, define the value function

$$v(y_0;0) = \max_{\{z_t\}_{t=0}^{\infty},\{y_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

subject to

$$y_0 \text{ given}$$
$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, 2, \ldots$$
$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, 2, \ldots$$

More generally, for any period t and any value of y_t, define

$$v(y_t;t) = \max_{\{z_{t+j}\}_{j=0}^{\infty},\{y_{t+j}\}_{j=1}^{\infty}} \sum_{j=0}^{\infty} \beta^j F(y_{t+j}, z_{t+j}; t+j)$$

subject to

$$y_t \text{ given}$$
$$y_{t+j} + Q(y_{t+j}, z_{t+j}; t+j) \geq y_{t+j+1} \text{ for all } j = 0, 1, 2, \ldots$$
$$c \geq G(y_{t+j}, z_{t+j}; t+j) \text{ for all } j = 0, 1, 2, \ldots$$

Note that the value function is a maximum value function.

Now consider expanding the definition of the value function by separating out the time t components:

$$v(y_t;t) = \max_{z_t, y_{t+1}} [F(y_t, z_t; t) + \max_{\{z_{t+j}\}_{j=1}^{\infty},\{y_{t+j}\}_{j=2}^{\infty}} \sum_{j=1}^{\infty} \beta^j F(y_{t+j}, z_{t+j}; t+j)]$$

subject to

$$y_t \text{ given}$$
$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$
$$y_{t+j} + Q(y_{t+j}, z_{t+j}; t+j) \geq y_{t+j+1} \text{ for all } j = 1, 2, 3, \ldots$$
$$c \geq G(y_t, z_t; t)$$
$$c \geq G(y_{t+j}, z_{t+j}; t+j) \text{ for all } j = 1, 2, 3, \ldots$$

Next, relabel the time indices:

$$v(y_t;t) = \max_{z_t, y_{t+1}} [F(y_t, z_t; t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}; t+1+j)]$$

subject to

$$y_t \text{ given}$$

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

$$y_{t+j+1} + Q(y_{t+1+j}, z_{t+1+j}; t+1+j) \geq y_{t+1+j+1} \text{ for all} j = 0, 1, 2, \ldots$$

$$c \geq G(y_t, z_t; t)$$

$$c \geq G(y_{t+1+j}, z_{t+1+j}; t+1+j) \text{ for all } j = 0, 1, 2, \ldots$$

Now notice that together, the components for $t+1+j$, $j = 0, 1, 2, \ldots$ define $v(y_{t+1}; t+1)$, enabling us to simplify the statement considerably:

$$v(y_t;t) = \max_{z_t, y_{t+1}} F(y_t, z_t; t) + \beta v(y_{t+1}; t+1)$$

subject to

$$y_t \text{ given}$$

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

$$c \geq G(y_t, z_t; t)$$

Or, even more simply:

$$v(y_t;t) = \max_{z_t} F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] \tag{5}$$

subject to

$$y_t \text{ given}$$

$$c \geq G(y_t, z_t; t)$$

Equation (5) is called the Bellman equation for this problem, and lies at the heart of the dynamic programming approach.

Note that the maximization on the right-hand side of (5) is a static optimization problem, involving no dynamic elements.

By the Kuhn-Tucker theorem:

$$v(y_t;t) = \max_{z_t} F(y_t, z_t; t) + \beta v[y_t + Q(y_t, z_t; t); t+1] + \lambda_t[c - G(y_t, z_t; t)]$$

The FOC for z_t is

$$F_2(y_t, z_t; t) + \beta v'[y_t + Q(y_t, z_t; t); t+1] Q_2(y_t, z_t; t) - \lambda_t G_2(y_t, z_t; t) = 0 \tag{6}$$

And by the envelope theorem:

$$v'(y_t;t) = F_1(y_t,z_t;t) + \beta v'[y_t + Q(y_t,z_t;t);t+1][1 + Q_1(y_t,z_t;t)] - \lambda_t G_1(y_t,z_t;t) \quad (7)$$

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t,z_t;t) \quad (3)$$

and complementary slackness condition

$$\lambda_t[c - G(y_t,z_t;t)] = 0, \quad (4)$$

we can think of (6) and (7) as forming a system of four equations in three unknown variables y_t, z_t, and λ_t and one unknown function $v(:,t)$. This system of equations determines the problem's solution.

Note once again that since (3) is in the form of a difference equation, finding the problem's solution involves solving a difference equation.

But more important, notice that (6) and (7) are equivalent to (1) and (2) with

$$v'(y_t;t) = W(y_t;t).$$

Thus, we have two ways of solving this discrete time dynamic optimization problem, both of which lead us to the same set of optimality conditions:

a) Set up the Lagrangian for the dynamic optimization problem and take first order conditions for z_t, $t = 0,1,2,\ldots$ and y_t, $t = 1,2,3,\ldots$.

b) Set up the Bellman equation and take the first order condition for z_t and then derive the envelope condition for y_t.

One question remains: How, in practice, can we solve for the unknown value functions $v(:,t)$?

To see how to answer this question, consider two examples:

Example 1: Optimal Growth - Here, it will be possible to solve for v explicitly.

Example 2: Saving Under Certainty - Here, it will not be possible to solve for v explicitly, yet we can learn enough about the properties of v to obtain some useful economic insights.

2 Example 1: Optimal Growth

Here, we will modify the optimal growth example that we solved earlier using the maximum principle in two ways:

a) We will switch to discrete time in order to facilitate the use of dynamic programming.

b) Set the depreciation rate for capital equal to $\delta = 1$ in order to obtain a very special case in which an explicit solution for the value function can be found.

Production function:

$$F(k_t) = k_t^\alpha$$

where $0 < \alpha < 1$

k_t = capital (state variable)

c_t = consumption (control variable)

Evolution of the capital stock:

$$k_{t+1} = k_t^\alpha - c_t$$

for all $t = 0, 1, 2, \ldots$

Initial condition:

$$k_0 \text{ given}$$

Utility or social welfare:

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

The social planner's problem: choose sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function subject to all of the constraints.

To solve this problem via dynamic programming, use

k_t = state variable

c_t = control variable

Set up the Bellman equation:

$$v(k_t; t) = \max_{c_t} \ln(c_t) + \beta v(k_t^\alpha - c_t; t+1)$$

Now guess that the value function takes the time-invariant form

$$v(k_t; t) = v(k_t) = E + F \ln(k_t),$$

where E and F are constants to be determined.

Using the guess for v, the Bellman equation becomes

$$E + F \ln(k_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(k_t^\alpha - c_t) \tag{8}$$

FOC for c_t:

$$\frac{1}{c_t} - \frac{\beta F}{k_t^\alpha - c_t} = 0 \tag{9}$$

Envelope condition for k_t:

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1}}{k_t^\alpha - c_t} \tag{10}$$

Together with the binding constraint

$$k_{t+1} = k_t^\alpha - c_t,$$

(8)-(10) form a system of four equations in 4 unknowns: c_t, k_t, E, and F.

Equation (9) implies

$$k_t^\alpha - c_t = \beta F c_t$$

or

$$c_t = \left(\frac{1}{1+\beta F}\right) k_t^\alpha \tag{11}$$

Substitute (11) into the envelope condition (10):

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1}}{k_t^\alpha - c_t} \tag{10}$$

$$F k_t^\alpha - F\left(\frac{1}{1+\beta F}\right) k_t^\alpha = \alpha\beta F k_t^\alpha$$

$$1 - \left(\frac{1}{1+\beta F}\right) = \alpha\beta$$

Hence

$$\frac{1}{1+\beta F} = 1 - \alpha\beta \tag{12}$$

Or, equivalently,

$$1 + \beta F = \frac{1}{1-\alpha\beta}$$

$$\beta F = \frac{1}{1-\alpha\beta} - 1 = \frac{\alpha\beta}{1-\alpha\beta}$$

$$F = \frac{\alpha}{1-\alpha\beta} \tag{13}$$

Substitute (12) into (11) to obtain

$$c_t = (1-\alpha\beta) k_t^\alpha \tag{14}$$

which shows that it is optimal to consume the fixed fraction $1 - \alpha\beta$ of output.

Evolution of capital:

$$k_{t+1} = k_t^\alpha - c_t = k_t^\alpha - (1-\alpha\beta)k_t^\alpha = \alpha\beta k_t^\alpha \tag{15}$$

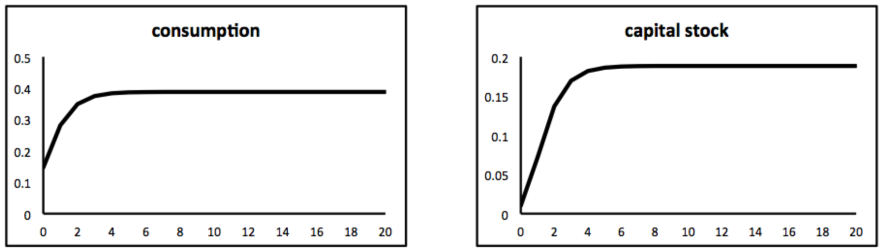
which is in the form of a difference equation for k_t.

Equations (14) and (15) show how the optimal values of c_t and k_{t+1} depend on the state variable k_t and the parameters α and β. Given a value for k_0, these two equations can be used to construct the optimal sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$.

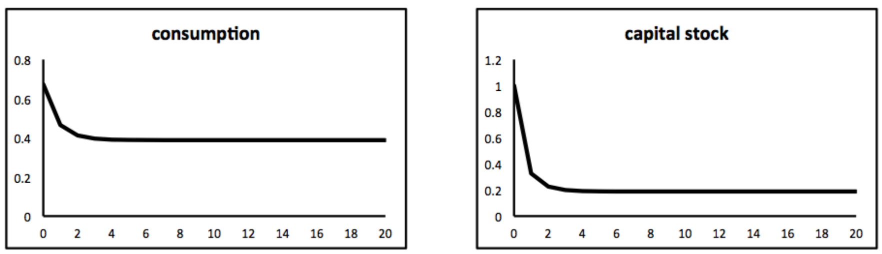
Numerical Solutions to the Optimal Growth Model with Complete Depreciation

Generated using equations (14) and (15). Each example sets α = 0.33 and β = 0.99.

Example 1: k(0) = 0.01



Example 2: k(0) = 1



In both examples, c(t) converges to its steady state value of 0.388 and k(t) converges to its steady-state value of 0.188.

For the sake of completeness, substitute (14) and (15) back into (8) to solve for E:

$$E + F\ln(k_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(k_t^{\alpha} - c_t) \tag{8}$$

$$E + F\ln(k_t) = \ln(1-\alpha\beta) + \alpha\ln(k_t) + \beta E + \beta F\ln(\alpha\beta) + \alpha\beta F\ln(k_t)$$

Since (13) implies that

$$F = \alpha + \alpha\beta F,$$

this last equality reduces to

$$E = \ln(1-\alpha\beta) + \beta E + \beta F\ln(\alpha\beta)$$

which leads directly to the solution

$$E = \frac{\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)}{1-\beta}$$

3 Example 2: Saving Under Certainty

Here, a consumer maximizes utility over an infinite horizon, $t = 0, 1, 2, \ldots$, earning income from labor and from investments.

A_t = beginning-of-period assets

A_t can be negative, that is, the consumer is allowed to borrow

y_t = labor income (exogenous)

c_t = consumption

saving $= s_t = A_t + y_t - c_t$

r = constant interest rate

Evolution of assets:

$$A_{t+1} = (1+r)s_t = (1+r)(A_t + y_t - c_t)$$

Note:

$$A_t + y_t - c_t = \left(\frac{1}{1+r}\right) A_{t+1}$$

$$A_t = \left(\frac{1}{1+r}\right) A_{t+1} + c_t - y_t$$

Similarly,

$$A_{t+1} = \left(\frac{1}{1+r}\right) A_{t+2} + c_{t+1} - y_{t+1}$$

Combining these last two equalities yields

$$A_t = \left(\frac{1}{1+r}\right)^2 A_{t+2} + \left(\frac{1}{1+r}\right)(c_{t+1} - y_{t+1}) + (c_t - y_t)$$

Continuing in this manner yields

$$A_t = \left(\frac{1}{1+r}\right)^T A_{t+T} + \sum_{j=0}^{T-1} \left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j}).$$

Now assume that the sequence $\{A_t\}_{t=0}^{\infty}$ must remain bounded (while borrowing is allowed, unlimited borrowing is ruled out), and take the limit as $T \to \infty$ to obtain

$$A_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j})$$

or

$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j c_{t+j}. \tag{16}$$

Equation (16) takes the form of an infinite horizon budget constraint, indicating that over the infinite horizon beginning at any period t, the consumer's sources of funds include assets A_t and the present value of current and future labor income, while the consumer's use of funds is summarized by the present value of current and future consumption.

The consumer's problem: choose the sequences $\{s_t\}_{t=0}^{\infty}$ and $\{A_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

subject to the constraints

$$A_0 \text{ given}$$

and

$$(1+r)s_t \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$

To solve the problem via dynamic programming, note first that

A_t = state variable

s_t = control variable

Set up the Bellman equation

$$v(A_t; t) = \max_{s_t} u(A_t + y_t - s_t) + \beta v(A_{t+1}; t+1) \text{ subject to } (1+r)s_t \geq A_{t+1}$$

$$v(A_t; t) = \max_{s_t} u(A_t + y_t - s_t) + \beta v[(1+r)s_t; t+1]$$

FOC for s_t:

$$-u'(A_t + y_t - s_t) + \beta(1+r)v'[(1+r)s_t; t+1] = 0$$

Envelope condition for A_t:

$$v'(A_t; t) = u'(A_t + y_t - s_t)$$

Use the constraints to rewrite these optimality conditions as

$$u'(c_t) = \beta(1+r)v'(A_{t+1}; t+1) \tag{17}$$

and

$$v'(A_t; t) = u'(c_t) \tag{18}$$

Since (18) must hold for all $t = 0, 1, 2, \ldots$, it implies

$$v'(A_{t+1}; t+1) = u'(c_{t+1})$$

Substitute this result into (17) to obtain:

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{19}$$

Now make 2 extra assumptions:

a) $\beta(1+r) = 1$ or $1 + r = 1/\beta$, the interest rate equals the discount rate

b) u is strictly concave

Under these 2 additional assumptions, (19) implies

$$u'(c_t) = u'(c_{t+1})$$

or

$$c_t = c_{t+1}$$

And since this last equation must hold for all $t = 0, 1, 2, \ldots$, it implies

$$c_t = c_{t+j} \text{ for all } j = 0, 1, 2, \ldots$$

Now, return to (16):

$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j c_{t+j}. \tag{16}$$

$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} = c_t \sum_{j=0}^{\infty} \beta^j \tag{20}$$

FACT: Since $|\beta| < 1$,

$$\sum_{j=0}^{\infty} \beta^j = \frac{1}{1-\beta}$$

To see why this is true, multiply both sides by $1 - \beta$:

$$\begin{aligned} 1 &= \frac{1-\beta}{1-\beta} \\ &= (1-\beta) \sum_{j=0}^{\infty} \beta^j \\ &= (1 + \beta + \beta^2 + \ldots) - \beta(1 + \beta + \beta^2 + \ldots) \\ &= (1 + \beta + \beta^2 + \ldots) - (\beta + \beta^2 + \beta^3 + \ldots) \\ &= 1 \end{aligned}$$

Use this fact to rewrite (20):

$$A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} = \left(\frac{1}{1-\beta}\right) c_t$$

or

$$c_t = (1-\beta) \left[A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} \right] \tag{21}$$

Equation (21) indicates that it is optimal to consume a fixed fraction $1-\beta$ of wealth at each date t, where wealth consists of value of current asset holdings and the present discounted value of future labor income. Thus, (21) describes a version of the permanent income hypothesis.

4 Stochastic Dynamic Programming

4.1 A Dynamic Stochastic Optimization Problem

Discrete time, infinite horizon: $t = 0, 1, 2, \ldots$

y_t = state variable

z_t = control variable

ε_{t+1} = random shock, which is observed at the beginning of $t+1$

Thus, when z_t is chosen:

ε_t is known ...

... but ε_{t+1} is still viewed as random.

The shock ε_{t+1} may be serially correlated, but will be assumed to have the Markov property (i.e., to be generated by a Markov process): the distribution of ε_{t+1} depends on ε_t, but not on $\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \ldots$.

For example, ε_{t+1} may follow a first-order autoregressive process:

$$\varepsilon_{t+1} = \rho\varepsilon_t + \eta_{t+1}.$$

Now, the full state of the economy at the beginning of each period is described jointly by the pair of values for y_t and ε_t, since the value for ε_t is relevant for forecasting, that is, forming expectations of, future values of ε_{t+j}, $j = 1, 2, 3, \ldots$.

Objective function:

$$E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

$1 > \beta > 0$ discount factor

E_0 = expected value as of $t = 0$

Constraint describing the evolution of the state variable

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and for all possible realizations of ε_{t+1}

Note that the constraint implies that the randomness in ε_{t+1} induces randomness into y_{t+1} as well: in particular, the value of y_{t+1} does not become known until ε_{t+1} is observed at the beginning of $t+1$ for all $t = 0, 1, 2, \ldots$

Note, too, that the sequential, period-by-period, revelation of values for ε_t, $t = 0, 1, 2, \ldots$, also generates sequential growth in the information set available to the agent solving the problem:

At the beginning of period $t = 0$, the agent knows

$$I_0 = \{y_0, \varepsilon_0\}$$

At the beginning of period $t = 1$, the agent's information set expands to

$$I_1 = \{y_1, \varepsilon_1, y_0, \varepsilon_0\}$$

And, more generally, at the beginning of period $t = 0, 1, 2, \ldots$, the agent's information set is given by

$$I_t = \{y_t, \varepsilon_t, y_{t-1}, \varepsilon_{t-1}, \ldots, y_0, \varepsilon_0\}$$

so that conditional expectations of future variables are defined implicitly with respect to this growing information set: for any variable X_{t+j} whose value becomes known at time $t+j$, $j = 0, 1, 2, \ldots$:

$$E_t X_{t+j} = E(X_{t+j}|I_t) = E(X_{t+j}|y_t, \varepsilon_t, y_{t-1}, \varepsilon_{t-1}, \ldots, y_0, \varepsilon_0)$$

The role of the additive time separability of the objective function, the similar "additive time separability" that is built into the constraints, and the Markov property of the shocks is to make the most recent values of y_t and ε_t sufficient statistics for I_t, so that within the confines of this problem,

$$E_t(X_{t+j}|y_t, \varepsilon_t, y_{t-1}, \varepsilon_{t-1}, \ldots, y_0, \varepsilon_0) = E_t(X_{t+j}|y_t, \varepsilon_t).$$

Note, finally, that the randomness in y_{t+1} induced by the randomness in ε_{t+1} also introduces randomness into the choice of z_{t+1} from the perspective of time t:

Given (y_t, ε_t), choose z_t

$\Rightarrow$ Given (y_t, z_t) the realization of ε_{t+1} determines $(y_{t+1}, \varepsilon_{t+1})$

$\Rightarrow$ Given $(y_{t+1}, \varepsilon_{t+1})$, choose z_{t+1}

This makes the number of choice variables, as well as the number of constraints, quite large.

The problem: choose contingency plans for z_t, $t = 0, 1, 2, \ldots$, and y_t, $t = 1, 2, 3, \ldots$, to maximize the objective function subject to all of the constraints.

Notes:

a) In order to incorporate uncertainty, we have really only made two adjustments to the problem:

First, we have added the shock ε_t to the objective function for period t and the shock ε_{t+1} to the constraint linking periods t and $t+1$.

And second, we have assumed that the planner cares about the expected value of the objective function.

b) For simplicity, the functions F and Q are now assumed to be time-invariant, although now they depend on the shock as well as on the state and control variable.

c) For simplicity, we have also dropped the second set of constraints, $c \geq G(y_t, z_t)$. Adding them back is straightforward, but complicates the algebra.

d) In the presence of uncertainty, the constraint

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$$

must hold, not only for all $t = 0, 1, 2, \ldots$, but for all possible realizations of ε_{t+1} as well. Thus, this single equation can actually represent a very large number of constraints.

e) The Kuhn-Tucker theorem can still be used to solve problems that feature uncertainty. But because problems with uncertainty can have a very large number of choice variables and constraints, the Kuhn-Tucker theorem can become very cumbersome to apply in practice, since one may have to introduce a very large number of Lagrange multipliers. We will return to the Kuhn-Tucker theorem to see how it works under uncertainty in section 6 of the notes, below. But for now, let's see how dynamic programming can be an easier and more convenient way to solve dynamic stochastic optimization problems.

4.2 The Dynamic Programming Formulation

Once again, for any values of y_0 and ε_0, define

$$v(y_0, \varepsilon_0) = \max_{\{z_t\}_{t=0}^{\infty}, \{y_t\}_{t=1}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

subject to

$$y_0 \text{ and } \varepsilon_0 \text{ given}$$

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all} t = 0, 1, 2, \ldots \text{ and all } \varepsilon_{t+1}$$

More generally, for any period t and any values of y_t and ε_t, define

$$v(y_t, \varepsilon_t) = \max_{\{z_{t+j}\}_{j=0}^{\infty}, \{y_{t+j}\}_{j=1}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j F(y_{t+j}, z_{t+j}, \varepsilon_{t+j})$$

subject to

$$y_t \text{ and } \varepsilon_t \text{ given}$$

$$y_{t+j} + Q(y_{t+j}, z_{t+j}, \varepsilon_{t+j+1}) \geq y_{t+j+1} \text{ for all} j = 0, 1, 2, \ldots \text{ and all } \varepsilon_{t+j+1}$$

Note once again that the value function is a maximum value function.

Now separate out the time t components:

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} [F(y_t, z_t, \varepsilon_t) + \max_{\{z_{t+j}\}_{j=1}^{\infty}, \{y_{t+j}\}_{j=2}^{\infty}} E_t \sum_{j=1}^{\infty} \beta^j F(y_{t+j}, z_{t+j}, \varepsilon_{t+j})]$$

subject to

$$y_t \text{ and } \varepsilon_t \text{ given}$$

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all } \varepsilon_{t+1}$$

$$y_{t+j} + Q(y_{t+j}, z_{t+j}, \varepsilon_{t+j+1}) \geq y_{t+j+1} \text{ for all} j = 1, 2, 3, \ldots \text{ and all } \varepsilon_{t+j+1}$$

Relabel the time indices:

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} [F(y_t, z_t, \varepsilon_t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+1+j})]$$

subject to

$$y_t \text{ and } \varepsilon_t \text{ given}$$

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all } \varepsilon_{t+1}$$

$$y_{t+j+1} + Q(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+1+j+1}) \geq y_{t+1+j+1} \text{ for all } j = 0, 1, 2, \ldots \text{ and all } \varepsilon_{t+1+j+1}$$

FACT (Law of Iterated Expectations): For any random variable X_{t+j}, realized at time $t+j$, $j = 0, 1, 2, \ldots$:

$$E_t E_{t+1} X_{t+j} = E_t X_{t+j}.$$

To see why this fact holds true, consider the following example:

Suppose ε_{t+1} follows the first-order autoregression:

$$\varepsilon_{t+1} = \rho \varepsilon_t + \eta_{t+1}, \text{ with } E_t \eta_{t+1} = 0$$

Hence

$$\varepsilon_{t+2} = \rho \varepsilon_{t+1} + \eta_{t+2}, \text{ with } E_{t+1} \eta_{t+2} = 0$$

or

$$\varepsilon_{t+2} = \rho^2 \varepsilon_t + \rho \eta_{t+1} + \eta_{t+2}.$$

It follows that

$$E_{t+1} \varepsilon_{t+2} = E_{t+1}(\rho^2 \varepsilon_t + \rho \eta_{t+1} + \eta_{t+2}) = \rho^2 \varepsilon_t + \rho \eta_{t+1}$$

and therefore

$$E_t E_{t+1} \varepsilon_{t+2} = E_t(\rho^2 \varepsilon_t + \rho \eta_{t+1}) = \rho^2 \varepsilon_t.$$

It also follows that

$$E_t \varepsilon_{t+2} = E_t(\rho^2 \varepsilon_t + \rho \eta_{t+1} + \eta_{t+2}) = \rho^2 \varepsilon_t.$$

So that in this case as in general

$$E_t E_{t+1} \varepsilon_{t+2} = E_t \varepsilon_{t+2}$$

Using this fact:

$$\begin{aligned} & v(y_t, \varepsilon_t) \\ = \; & \max_{z_t, y_{t+1}} [F(y_t, z_t, \varepsilon_t) + \beta \max_{\{z_{t+1+j}\}_{j=0}^{\infty}, \{y_{t+1+j}\}_{j=1}^{\infty}} E_t E_{t+1} \sum_{j=0}^{\infty} \beta^j F(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+1+j})] \end{aligned}$$

subject to

$$y_t \text{ and } \varepsilon_t \text{ given}$$

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all } \varepsilon_{t+1}$$

$$y_{t+j+1} + Q(y_{t+1+j}, z_{t+1+j}, \varepsilon_{t+1+j+1}) \geq y_{t+1+j+1} \text{ for all } j = 0, 1, 2, \ldots \text{ and all } \varepsilon_{t+1+j+1}$$

Now use the definition of $v(y_{t+1}, \varepsilon_{t+1})$ to simplify:

$$v(y_t, \varepsilon_t) = \max_{z_t, y_{t+1}} F(y_t, z_t, \varepsilon_t) + \beta E_t v(y_{t+1}, \varepsilon_{t+1})$$

subject to

$$y_t \text{ and } \varepsilon_t \text{ given}$$

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1} \text{ for all } \varepsilon_{t+1}$$

Or, even more simply:

$$v(y_t, \varepsilon_t) = \max_{z_t} F(y_t, z_t, \varepsilon_t) + \beta E_t v[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}] \tag{22}$$

Equation (22) is the Bellman equation for this stochastic problem.

Thus, in order to incorporate uncertainty into the dynamic programming framework, we only need to make two modifications to the Bellman equation:

a) Include the shock ε_t as an additional argument of the value function.

b) Add the expectation term E_t in front of the value function for $t+1$ on the right-hand side.

Note that the maximization on the right-hand side of (22) is a static optimization problem, involving no dynamic elements.

Note also that by substituting the constraints into the value function, we are left with an unconstrained problem. Unlike the Kuhn-Tucker approach, which requires many constraints and many multipliers, dynamic programming in this case has no constraints and no multipliers.

The FOC for z_t is

$$F_2(y_t, z_t, \varepsilon_t) + \beta E_t\{v_1[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}]Q_2(y_t, z_t, \varepsilon_{t+1})\} = 0 \quad (23)$$

The envelope condition for y_t is:

$$v_1(y_t, \varepsilon_t) = F_1(y_t, z_t, \varepsilon_t) + \beta E_t\{v_1[y_t + Q(y_t, z_t, \varepsilon_{t+1}), \varepsilon_{t+1}][1 + Q_1(y_t, z_t, \varepsilon_{t+1})]\} \quad (24)$$

Equations (23)-(24) coincide exactly with the first-order conditions for z_t and y_t that we would have derived through a direct application of the Kuhn-Tucker theorem to the original, dynamic stochastic optimization problem.

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t, \varepsilon_{t+1}) \quad (25)$$

we can think of (23) and (24) as forming a system of three equations in two unknown variables y_t and z_t and one unknown function v. This system of equations determines the problem's solution, given the behavior of the exogenous shocks ε_t.

Note that (25) is in the form of a difference equation; once again, solving a dynamic optimization problem involves solving a difference equation.

5 Example 3: Saving with Multiple Random Returns

This example extends example 2 by:

a) Introducing $n \geq 1$ assets

b) Allowing returns on each asset to be random

As in example 2, we will not be able to solve explicitly for the value function, but we will be able to learn enough about its properties to derive some useful economic results.

Since we are extending the example in two ways, assume for simplicity that the consumer receives no labor income, and therefore must finance all of his or her consumption by investing.

A_t = beginning-of-period financial wealth

c_t = consumption

s_{it} = savings allocated to asset $i = 1, 2, \ldots, n$

Hence,

$$A_t = c_t + \sum_{i=1}^{n} s_{it}$$

R_{it+1} = random gross return on asset i, not known until $t+1$

Hence, when s_{it} is chosen:

R_{it} is known . . .

. . . but R_{it+1} is still viewed as random.

Hence

$$A_{t+1} = \sum_{i=1}^{n} R_{it+1} s_{it}$$

does not become known until the beginning of $t+1$, even though the s_{it} must be chosen during t.

Utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t u(A_t - \sum_{i=1}^{n} s_{it})$$

The problem can now be stated as: choose contingency plans for s_{it} for all $i = 1, 2, \ldots, n$ and $t = 0, 1, 2, \ldots$ and A_t for all $t = 1, 2, 3, \ldots$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(A_t - \sum_{i=1}^{n} s_{it})$$

subject to

$$A_0 \text{ given}$$

and

$$\sum_{i=1}^{n} R_{it+1} s_{it} \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of R_{it+1} for each $i = 1, 2, \ldots, n$.

As in the general case, the returns can be serially correlated, but must have the Markov property.

To solve this problem via dynamic programming, let

A_t = state variable

s_{it}, $i = 1, 2, \ldots, n$ = control variables

$R_t = [R_{1t}, R_{2t}, \ldots, R_{nt}]$ = vector of random returns

The Bellman equation is

$$v(A_t, R_t) = \max_{\{s_{it}\}_{i=1}^{n}} u(A_t - \sum_{i=1}^{n} s_{it}) + \beta E_t v(\sum_{i=1}^{n} R_{it+1} s_{it}, R_{t+1})$$

FOC:

$$-u'(A_t - \sum_{i=1}^{n} s_{it}) + \beta E_t R_{it+1} v_1(\sum_{i=1}^{n} R_{it+1} s_{it}, R_{t+1}) = 0$$

for all $i = 1, 2, \ldots, n$

Envelope condition:

$$v_1(A_t, R_t) = u'(A_t - \sum_{i=1}^{n} s_{it})$$

Use the constraints to rewrite the FOC and envelope conditions more simply as

$$u'(c_t) = \beta E_t R_{it+1} v_1(A_{t+1}, R_{t+1})$$

for all $i = 1, 2, \ldots, n$ and

$$v_1(A_t, R_t) = u'(c_t)$$

Since the envelope condition must hold for all $t = 0, 1, 2, \ldots$, it implies

$$v_1(A_{t+1}, R_{t+1}) = u'(c_{t+1})$$

Hence, the FOC imply that

$$u'(c_t) = \beta E_t R_{it+1} u'(c_{t+1}) \tag{26}$$

must hold for all $i = 1, 2, \ldots, n$

Equation (26) generalizes (19) to the case where there is more than one asset and where the asset returns are random. It must hold for all assets $i = 1, 2, \ldots, n$, even though each asset may pay a different return ex-post.

In example 2, we combined (19) with some additional assumptions to derive a version of the permanent income hypothesis. Similarly, we can use (26) to derive a version of the famous capital asset pricing model.

For simplicity, let

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

denote the consumer's intertemporal marginal rate of substitution.

Then (26) can be written more simply as

$$1 = E_t R_{it+1} m_{t+1} \tag{27}$$

Keeping in mind that (27) must hold for all assets, suppose that there is a risk-free asset, with return R^f_{t+1} that is known during period t. Then R^f_{t+1} must satisfy

$$1 = R^f_{t+1} E_t m_{t+1}$$

or

$$E_t m_{t+1} = \frac{1}{R^f_{t+1}} \tag{28}$$

FACT: For any two random variables x and y,

$$cov(x,y) = E[(x-\mu_x)(y-\mu_y)], \text{ where } \mu_x = E(x) \text{ and } \mu_y = E(y).$$

Hence,

$$\begin{aligned} cov(x,y) &= E[xy - \mu_x y - x\mu_y + \mu_x\mu_y] \\ &= E(xy) - \mu_x\mu_y - \mu_x\mu_y + \mu_x\mu_y \\ &= E(xy) - \mu_x\mu_y \\ &= E(xy) - E(x)E(y) \end{aligned}$$

Or, by rearranging,

$$E(xy) = E(x)E(y) + cov(x,y)$$

Using this fact, (27) can be rewritten as

$$1 = E_t R_{it+1} m_{t+1} = E_t R_{it+1} E_t m_{t+1} + cov_t(R_{it+1}, m_{t+1})$$

or, using (28),

$$R^f_{t+1} = E_t R_{it+1} + R^f_{t+1} cov_t(R_{it+1}, m_{t+1})$$

$$E_t R_{it+1} - R^f_{t+1} = -R^f_{t+1} cov_t(R_{it+1}, m_{t+1}) \tag{29}$$

Equation (29) indicates that the expected return on asset i exceeds the risk-free rate only if R_{it+1} is negatively correlated with m_{t+1}.

Does this make sense?

Consider that an asset that acts like insurance pays a high return R_{it+1} during bad economic times, when consumption c_{t+1} is low. Therefore, for this asset:

$$\begin{aligned} cov_t(R_{it+1}, c_{t+1}) < 0 &\Rightarrow cov_t[R_{it+1}, u'(c_{t+1})] > 0 \\ &\Rightarrow cov_t(R_{it+1}, m_{t+1}) > 0 \\ &\Rightarrow E_t R_{it+1} < R^f_{t+1}. \end{aligned}$$

This implication seems reasonable: assets that work like insurance often have expected returns below the risk-free return.

Consider that common stocks tend to pay a high return R_{it+1} during good economic times, when consumption c_{t+1} is high. Therefore, for stocks:

$$\begin{aligned} cov_t(R_{it+1}, c_{t+1}) > 0 &\Rightarrow cov_t[R_{it+1}, u'(c_{t+1})] < 0 \\ &\Rightarrow cov_t(R_{it+1}, m_{t+1}) < 0 \\ &\Rightarrow E_t R_{it+1} > R^f_{t+1}. \end{aligned}$$

This implication also seems to hold true: historically, stocks have had expected returns above the risk-free return.

Recalling once more that (29) must hold for all assets, consider in particular the asset whose return happens to coincide exactly with the representative consumer's intertemporal marginal rate of substitution:

$$R^m_{t+1} = m_{t+1}.$$

For this asset, equation (29) implies

$$E_t R^m_{t+1} - R^f_{t+1} = -R^f_{t+1} cov_t(R^m_{t+1}, m_{t+1})$$

$$E_t m_{t+1} - R^f_{t+1} = -R^f_{t+1} cov_t(m_{t+1}, m_{t+1}) = -R^f_{t+1} var_t(m_{t+1})$$

or

$$-R^f_{t+1} = \frac{E_t m_{t+1} - R^f_{t+1}}{var_t(m_{t+1})} \tag{30}$$

Substitute (30) into the right-hand side of (29) to obtain

$$E_t R_{it+1} - R^f_{t+1} = \frac{cov_t(R_{it+1}, m_{t+1})}{var_t(m_{t+1})}(E_t m_{t+1} - R^f_{t+1})$$

or

$$E_t R_{it+1} - R^f_{t+1} = b_{it}(E_t m_{t+1} - R^f_{t+1}), \tag{31}$$

where

$$b_{it} = \frac{cov_t(R_{it+1}, m_{t+1})}{var_t(m_{t+1})}$$

is like the slope coefficient from a regression of R_{it+1} on m_{t+1}.

Equation (31) is a statement of the consumption-based capital asset pricing model, or consumption CAPM. This model links the expected return on each asset to the risk-free rate and the representative consumer's intertemporal marginal rate of substitution.

6 Dynamic Programming In Continuous Time

Although dynamic programming is most useful in the discrete-time case, particularly as a way of handling stochastic problems, it can also be applied in continuous time. Working through the derivations in continuous time shows why the maximum principle provides what is typically the more convenient approach. But doing so is still useful, partly because it yields some additional insights into the maximized Hamiltonian and its derivatives and what they measure.

6.1 A Perfect Foresight Dynamic Optimization Problem in Continuous Time

Let's return to the case of perfect foresight, and start with a discrete-time formulation.

Infinite horizon $t = 0, 1, 2, \ldots$

y_t = stock, or state, variable

z_t = flow, or control, variable

Objective function:

$$\sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

$1 > \beta > 0$ discount factor

Constraint describing the evolution of the state variable:

$$Q(y_t, z_t; t) \geq y_{t+1} - y_t$$

for all $t = 0, 1, 2, \ldots$

For simplicity, drop the constraint $c \geq G(y_t, z_t; t)$ imposed on the choice of z_t given y_t for each $t = 0, 1, 2, \ldots$.

Constraint on the initial value of the state variable:

$$y_0 \text{ given}$$

The discrete-time problem is then to choose sequences $\{z_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ to maximize the objective function

$$\sum_{t=0}^{\infty} \beta^t F(y_t, z_t; t)$$

subject to the constraints y_0 given and

$$Q(y_t, z_t; t) \geq y_{t+1} - y_t$$

for all $t = 0, 1, 2, \ldots$

The continuous-time analog to this problem has

$y(t)$ = stock, or state, variable

$z(t)$ = flow, or control, variable

Objective function:

$$\int_0^{\infty} e^{-\rho t} F(y(t), z(t); t)\, dt$$

$\rho > 0$ discount rate

Constraint describing the evolution of the state variable:

$$Q(y(t), z(t); t) \geq \dot{y}(t)$$

for all $t \in [0, \infty)$

Initial condition

$$y(0) \text{ given}$$

The continuous-time problem is to choose continuously differentiable functions $z(t)$ and $y(t)$ for $t \in [0, \infty)$ to maximize the objective function

$$\int_0^\infty e^{-\rho t} F(y(t), z(t); t)\, \mathrm{d}t$$

subject to the constraints $y(0)$ given and

$$Q(y(t), z(t); t) \geq \dot{y}(t)$$

for all $t \in [0, \infty)$

6.2 The Dynamic Programming Formulation

For the discrete-time problem, the Bellman equation is

$$v(y_t; t) = \max_{z_t} F(y_t, z_t; t) + \beta v(y_{t+1}; t+1),$$

where

$$y_{t+1} = y_t + Q(y_t, z_t; t)$$

To translate the Bellman equation into continuous time, consider that in discrete-time, the interval between time periods is $\Delta t = 1$. Hence

$$v(y_t; t) = \max_{z_t} F(y_t, z_t; t)\Delta t + e^{-\rho \Delta t} v(y_{t+\Delta t}; t + \Delta t),$$

where

$$y_{t+\Delta t} = y_t + Q(y_t, z_t; t)\Delta t$$

Consider a first-order Taylor approximation of the second term on the right-hand side of the discrete-time Bellman equation, viewed as a function of $y_{t+\Delta t}$ and Δt, around y_t and $\Delta t = 0$:

$$\begin{aligned} e^{-\rho \Delta t} v(y_{t+\Delta t}; t + \Delta t) &\approx v(y_t; t) + v_y(y_t; t)(y_{t+\Delta t} - y_t) + v_t(y_t; t)\Delta t - \rho v(y_t; t)\Delta t \\ &= v(y_t; t) + v_y(y_t; t) Q(y_t, z_t; t)\Delta t + v_t(y_t; t)\Delta t - \rho v(y_t; t)\Delta t \end{aligned}$$

where v_y and v_t denote the partial derivatives of the value function v with respect to its first and second arguments.

Substitute this expression into the Bellman equation

$$v(y_t; t) \approx \max_{z_t} F(y_t, z_t; t)\Delta t + v(y_t; t) + v_y(y_t; t) Q(y_t, z_t; t)\Delta t + v_t(y_t; t)\Delta t - \rho v(y_t; t)\Delta t$$

Subtract $v(y_t, t)$ from both sides

$$0 \approx \max_{z_t} F(y_t, z_t; t)\Delta t + v_y(y_t; t)Q(y_t, z_t; t)\Delta t + v_t(y_t; t)\Delta t - \rho v(y_t; t)\Delta t$$

Divide through by Δt:

$$0 \approx \max_{z_t} F(y_t, z_t; t) + v_y(y_t; t)Q(y_t, z_t; t) + v_t(y_t; t) - \rho v(y_t; t)$$

And move the terms that don't depend on the control variable z_t to the left-hand side:

$$\rho v(y_t; t) - v_t(y_t; t) \approx \max_{z_t} F(y_t, z_t; t) + v_y(y_t; t)Q(y_t, z_t; t)$$

Finally, note that as $\Delta t \to 0$, the Taylor approximation for $y_{t+\Delta t}$ becomes exact. Use this fact to replace the $\approx$ with an equal sign, and convert the rest of the notation to continuous time to obtain

$$\rho v(y(t); t) - v_t(y(t); t) = \max_{z(t)} F(y(t), z(t); t) + v_y(y(t); t)Q(y(t), z(t); t) \tag{32}$$

Equation (32) is called the Hamilton-Jacobi-Bellman (HJB) equation for the continuous-time problem. But as we have just seen, it is really just the continuous-time analog to the discrete-time Bellman equation.

The HJB equation takes the form of a partial differential equation, linking the unknown value function $v(y(t), t)$ to its partial derivatives $v_y(y(t), t)$ and $v_t(y(t), t)$. Since partial differential equations are often quite difficult to work with, the dynamic programming approach is not used as frequently as the maximum principle to solve continuous-time dynamic optimization in most areas of economics. In financial economics, however, the stochastic version of (32) has served quite usefully as a starting point for numerous analyses that draw on analytic and computational methods for characterizing the solutions to stochastic differential equations in which randomness is asset prices is driven by continuous-time random walks, or Brownian motions. The Nobel prize-winning economist Robert Merton was a pioneer in initiating this line of research.

Recall that the dynamic programming approach takes a "current value" view of the dynamic optimization problem. For this same problem, the current-value formulation of the maximized Hamiltonian is

$$\tilde{H}(y(t), \theta(t); t) = \max_{z(t)} F(y(t), z(t); t) + \theta(t)Q(y(t), z(t); t) \tag{33}$$

Comparing (32) and (33) suggests that across the two formulations

$$\theta(t) = v_y(y(t); t), \tag{34}$$

consistent with our earlier interpretation of $\theta(t)$ as measuring the current value, at time t, if having an additional unit of the stock variable $y(t)$.

Comparing (32) and (33) also suggests that

$$\tilde{H}(y(t), \theta(t); t) = \rho v(y(t); t) - v_t(y(t); t). \tag{35}$$

To see that this equality also holds, consider the definition of the value function $v(y(t), t)$ as the maximized value of the objective function from time t forward, given the predetermined value $y(t)$ of the stock or state variable:

$$v(y(t), t) = \max_{z(s), s \in [t, \infty)} \int_t^\infty e^{-\rho(s-t)} F(y(s), z(s); s) \, \mathrm{d}s$$

subject to $y(t)$ given and

$$Q(y(s), z(s), s) \geq \dot{y}(s)$$

for all $s \in [t, \infty)$.

Differentiate $v(y(t), t)$ by t to obtain

$$\begin{aligned} & v_y(y(t), t)\dot{y}(t) + v_t(y(t); t) \\ = & \max_{z(s), s \in [t, \infty)} \left[-F(y(t), z(t); t) + \rho \int_t^\infty e^{-\rho(s-t)} F(y(s), z(s); s) \, \mathrm{d}s \right] \end{aligned}$$

or in light of (34) and the definition of $v(y(t), t)$,

$$\theta(t)Q(y(t), z(t); t) + v_t(y(t); t) = \max_{z(s), s \in [t, \infty)} -F(y(t), z(t); t) + \rho v(y(t); t).$$

Note: in these expressions, "$\max_{z(s), s \in [t, \infty)}$" simply means that the functions that follow are being evaluated at the values of $z(s), s \in [t, \infty)$ that solve the problem, not that these functions are themselves being maximized by choice of $z(s)$.

Rearranging terms and simplifying yields

$$\rho v(y(t); t) - v_t(y(t); t) = \max_{z(t)} F(y(t), z(t); t) + \theta(t)Q(y(t), z(t); t)$$

which, in light of (33), coincides with (35).

Returning to the Hamilton-Jacobi-Bellman equation

$$\rho v(y(t); t) - v_t(y(t); t) = \max_{z(t)} F(y(t), z(t); t) + v_y(y(t); t)Q(y(t), z(t); t), \tag{32}$$

the first-order condition for the optimal choice of $z(t)$ on the right-hand side is

$$F_z(y(t), z(t); t) + v_y(y(t); t)Q_z(y(t), z(t); t) = 0, \tag{36}$$

which in light of (34) coincides with the first-order condition that one could derive, instead, with the help of the maximum principle.

Now differentiate both sides of (32) with respect to $y(t)$, invoking the envelope theorem to ignore the dependence of the optimal $z(t)$ on $y(t)$:

$$\begin{aligned}&\rho v_y(y(t);t) - v_{ty}(y(t);t)\\ &= F_y(y(t),z(t);t) + v_{yy}(y(t);t)Q(y(t),z(t);t) + v_y(y(t);t)Q_y(y(t),z(t);t).\end{aligned} \tag{37}$$

Using (34) once again,

$$\theta(t) = v_y(y(t);t), \tag{34}$$

implies

$$\dot{\theta}(t) = v_{yy}(y(t);t)\dot{y}(t) + v_{yt}(y(t);t) = v_{yy}(y(t);t)Q(y(t),z(t);t) + v_{yt}(y(t);t) \tag{38}$$

Equation (34) and (38) reveal that (37) is equivalent to

$$\rho\theta(t) = F_y(y(t),z(t);t) + \dot{\theta}(t) + \theta(t)Q_y(y(t),z(t);t)$$

or

$$\dot{\theta}(t) = \rho\theta(t) - \tilde{H}_y(y(t),\theta(t);t),$$

which again coincides with the optimality conditions that one could derive with the help of the maximum principle.

Yet again we see that the Kuhn-Tucker theorem, the maximum principle, and dynamic programming are just different approaches to deriving the same optimality conditions that characterize the solution to any given constrained optimization problem.

7 Example 4: Optimal Growth

Although the maximum principle is easier to apply in this case, let's consider solving the optimal growth model in continuous time using dynamic programming instead.

In this model, a benevolent social planner or a representative consumer chooses continuously differentiable functions $c(t)$ and $k(t)$ for $t \in [0,\infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t}\ln(c(t))\mathrm{d}t$$

subject to the constraints $k(0)$ given and

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0,\infty)$.

For this problem, the value function depends on $k(t)$ and not separately on t. Hence, the Hamilton-Jacobi-Bellman equation specializes from

$$\rho v(y(t);t) - v_t(y(t);t) = \max_{z(t)} F(y(t),z(t);t) + v_y(y(t);t)Q(y(t),z(t);t) \tag{32}$$

to

$$\rho v(k(t)) = \max_{c(t)} \ln(c(t)) + v'(k(t))[k(t)^\alpha - \delta k(t) - c(t)]$$

The first-order condition for $c(t)$ is

$$\frac{1}{c(t)} = v'(k(t))$$

and the envelope condition for $k(t)$ is

$$\rho v'(k(t)) = v''(k(t))[k(t)^{\alpha} - \delta k(t) - c(t)] + v'(k(t))[\alpha k(t)^{\alpha-1} - \delta]$$

As always, the problem with these optimality conditions is that they make reference to the unknown value function $v(k(t))$ – in this case, the first and second derivatives of $v(k(t))$. But the first-order condition gives us a "solution" for $v'(k(t))$. And if we differentiate the first-order condition with respect to t to obtain

$$0 = v''(k(t))c(t)\dot{k}(t) + v'(k(t))\dot{c}(t)$$

or, using the binding constraint to eliminate $\dot{k}(t)$,

$$0 = v''(k(t))[k(t)^{\alpha} - \delta k(t) - c(t)] + v'(k(t))\left[\frac{\dot{c}(t)}{c(t)}\right].$$

This last equation can be substituted into the envelope condition to yield

$$\rho v'(k(t)) = -v'(k(t))\left[\frac{\dot{c}(t)}{c(t)}\right] + v'(k(t))[\alpha k(t)^{\alpha-1} - \delta]$$

Divide through by $v'(k(t))$ and rearrange to obtain

$$\dot{c}(t) = c(t)[\alpha k(t)^{\alpha-1} - \delta - \rho]$$

which together with the constraint

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t)$$

yields the same system of two differential equations in $c(t)$ and $k(t)$ that we analyzed previously with the help of the phase diagram.

8 Applying the Kuhn-Tucker Theorem in the Case of Uncertainty

Although dynamic programming is usually the most convenient approach to solving dynamic, stochastic optimization problems, the Kuhn-Tucker is applicable, too. For the sake of completeness, let's see how this can be done with an investment in some additional notation.

Consider again the discrete-time, stochastic problem: choose contingency plans for z_t, $t = 0, 1, 2, \ldots$, and y_t, $t = 1, 2, 3, \ldots$, to maximize the objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t F(y_t, z_t, \varepsilon_t)$$

subject to the constraints that y_0 and ε_0 are given and

$$y_t + Q(y_t, z_t, \varepsilon_{t+1}) \geq y_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of ε_{t+1}.

To describe the "contingency plans" in more detail, without getting too deeply into the measure-theoretic foundations, let

$$H_t = \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_t\}$$

denote a realized history of shocks through time t.

Then let

$p(H_t)$ = the probability of history H_t

$y_t(H_t)$ = the value of the stock variable at time t following history H_t

$z_t(H_t)$ = the value of the flow variable at time t following the history H_t

$\varepsilon_t(H_t)$ = the value of the shock ε_t realized at the end of history H_t

Ω_t = the set of all possible histories H_t through time t

$H_{t+1} = (H_t, \varepsilon_{t+1})$ = a history through time $t+1$ that follows the history H_t through time t

$\Omega_{t+1}|H_t$ = the set of all possible histories $H_{t+1} = (H_t, \varepsilon_{t+1})$ at time $t+1$ that follow the history H_t through t

Note that the period t variables $y_t(H_t)$ and $z_t(H_t)$ depend on H_t, implying that they must be chosen before the realization of ε_{t+1} is observed.

Now restate the problem with the help of this new notation: choose values for $z_t(H_t)$ for all $t = 0, 1, 2, \ldots$ and all $H_t \in \Omega_t$ and $y_t(H_t)$ for all $t = 1, 2, 3, \ldots$ and all $H_t \in \Omega_t$ to maximize

$$\sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)]$$

subject to the constraints that $H_0 = \varepsilon_0$ and $y_t(H_0)$ are given and

$$y_t(H_t) + Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] \geq y_{t+1}(H_t, \varepsilon_{t+1})$$

for all $t = 0, 1, 2, \ldots$, all $H_t \in \Omega_t$, and all $H_{t+1} = (H_t, \varepsilon_{t+1}) \in \Omega_{t+1}|H_t$.

Although the new notation hints at the technical details involved in formalizing the choices of $z_t(H_t)$ and $y_t(H_t)$ as random variables "adapted" to the expanding information set that accumulates as time passes and shocks are realized, it also makes clear that the dynamic, stochastic problem is complicated mainly by the fact that there are many choice variables and many constraints.

Still, one can form the Lagrangian in the usual way, taking care to introduce a separate multiplier for each of the many constraints:

$$\begin{aligned} L \;=\; & \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\ & + \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1}|H_t} \tilde{\mu}_{t+1}(H_t, \varepsilon_{t+1}) y_t(H_t) \\ & + \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1}|H_t} \tilde{\mu}_{t+1}(H_t, \varepsilon_{t+1}) Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] \\ & - \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1}|H_t} \tilde{\mu}_{t+1}(H_t, \varepsilon_{t+1}) y_{t+1}(H_t, \varepsilon_{t+1}) \end{aligned}$$

Next, let

$$\mu_{t+1}(H_t, \varepsilon_{t+1}) = \frac{\tilde{\mu}_{t+1}(H_t, \varepsilon_{t+1})}{\beta^{t+1} p(H_t, \varepsilon_{t+1})}$$

This transformation extends the similar change in variables that we used in section 1.2 in the perfect foresight case. It puts the Lagrange multipliers in "current value" form so as to make the links to objects from the dynamic programming approach, which also takes a current-value view of the problem, more apparent. The notation, $\mu_{t+1}(H_t, \varepsilon_{t+1})$ also makes clear that the multipliers are also random variables in the stochastic case.

Rewrite the Langrangian as

$$\begin{aligned} & L = \\ & \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \beta^t p(H_t) F[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\ & + \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1}|H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) y_t(H_t) \\ & + \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1}|H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) Q[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] \\ & - \sum_{t=0}^{\infty} \sum_{H_t \in \Omega_t} \sum_{(H_t, \varepsilon_{t+1}) \in \Omega_{t+1}|H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) y_{t+1}(H_t, \varepsilon_{t+1}) \end{aligned}$$

For a given value of $t = 0, 1, 2, \ldots$ and a given history $H_t \in \Omega_t$, the first-order condition for $z_t(H_t)$ is

$$\begin{aligned}
&\beta^t p(H_t) F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
&+ \sum_{(H_t,\varepsilon_{t+1})\in\Omega_{t+1}|H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0
\end{aligned}$$

For a given value of $t = 1, 2, 3, \ldots$ and a given history $H_t \in \Omega_t$, the first-order condition for $y_t(H_t)$ is

$$\begin{aligned}
&\beta^t p(H_t) F_1[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
&+ \sum_{(H_t,\varepsilon_{t+1})\in\Omega_{t+1}|H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\
&+ \sum_{(H_t,\varepsilon_{t+1})\in\Omega_{t+1}|H_t} \beta^{t+1} p(H_t, \varepsilon_{t+1}) \mu_{t+1}(H_t, \varepsilon_{t+1}) Q_1[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] \\
&- \beta^t p(H_t) \mu_t(H_t) = 0
\end{aligned}$$

Divide the first-order condition z_t through by $\beta^t p(H_t)$, and recall that

$$p(\varepsilon_{t+1}|H_t) = \frac{p(H_t, \varepsilon_{t+1})}{p(H_t)}$$

defines the conditional probability that the period $t+1$ shock ε_{t+1} will follow the history H_t:

$$\begin{aligned}
&F_2[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
&+\beta \sum_{(H_t,\varepsilon_{t+1})\in\Omega_{t+1}|H_t} p(\varepsilon_{t+1}|H_t) \mu_{t+1}(H_t, \varepsilon_{t+1}) Q_2[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})] = 0
\end{aligned}$$

for all $t = 0, 1, 2, \ldots$ and all $H_t \in \Omega_t$.

Or, more compactly,

$$F_2(y_t, z_t, \varepsilon_t) + \beta E_t[\mu_{t+1} Q_2(y_t, z_t, \varepsilon_{t+1})] = 0 \tag{39}$$

Divide the first-order condition for y_t through by $\beta^t p(H_t)$:

$$\begin{aligned}
&\mu_t(H_t) \\
= \; &F_1[y_t(H_t), z_t(H_t), \varepsilon_t(H_t)] \\
&+\beta \sum_{(H_t,\varepsilon_{t+1})\in\Omega_{t+1}|H_t} p(\varepsilon_{t+1}|H_t) \mu_{t+1}(H_t, \varepsilon_{t+1}) \\
&+\beta \sum_{(H_t,\varepsilon_{t+1})\in\Omega_{t+1}|H_t} p(\varepsilon_{t+1}|H_t) \mu_{t+1}(H_t, \varepsilon_{t+1}) Q_1[y_t(H_t), z_t(H_t), \varepsilon_{t+1}(H_t, \varepsilon_{t+1})]
\end{aligned}$$

for all $t = 1, 2, 3, \ldots$ and all $H_t \in \Omega_t$.

Or, again more compactly,

$$\mu_t = F_1(y_t, z_t, \varepsilon_t) + \beta E_t\{\mu_{t+1}[1 + Q_1(y_t, z_t, \varepsilon_{t+1})]\} \tag{40}$$

Together with the binding constraint

$$y_{t+1} = y_t + Q(y_t, z_t, \varepsilon_{t+1}), \tag{41}$$

equations (39) and (40) form a system of three equations in the three unknown variables, z_t, y_t, and μ_t. This system of equations determines the problem's solution, given the behavior of the exogenous shocks ε_t.

Moreover, the three-equation system (39)-(41) that we just derived from the Lagrangian coincides with the three-equation system (23)-(25) that we derived previously using the Bellman equation, but with

$$\mu_t = v_1(y_t, \varepsilon_t)$$

for all $t = 0, 1, 2, \ldots$, and all possible values of ε_t.

Once again, the Kuhn-Tucker theorem and the dynamic programming approach lead to exactly the same results. One can choose whichever approach one finds most convenient to characterize the solution to the dynamic, stochastic optimization problem

9 Example 5: Saving Under Uncertainty

This last example presents a dynamic, stochastic optimization problem that is simple enough to allow a relatively straightforward application of the Kuhn-Tucker theorem. The optimality conditions derived with the help of the Lagrangian and the Kuhn-Tucker theorem can then be compared with those that can be derived with the help of the Bellman equation and dynamic programming.

9.1 The Problem

Consider the simplest possible dynamic, stochastic optimization problem with:

Two periods, $t = 0$ and $t = 1$

No uncertainty at $t = 0$

Two possible states at $t = 1$:

Good, or high, state H occurs with probability p

Bad, or low, state L occurs with probability $1 - p$

Notation for a consumer's problem:

y_0 = income at $t = 0$

c_0 = consumption at $t = 0$

s = savings at $t = 0$, carried into $t = 1$ (s can be negative, that is, the consumer is allowed to borrow)

r = interest rate on savings

y_1^H = income at $t = 1$ in the high state

y_1^L = income at $t = 1$ in the low state

$y_1^H > y_1^L$ makes H the good state and L the bad state

c_1^H = consumption at $t = 1$ in the high state

c_1^L = consumption at $t = 1$ in the low state

Expected utility:

$$u(c_0) + \beta E[u(c_1)] = u(c_0) + \beta p u(c_1^H) + \beta(1-p)u(c_1^L)$$

Constraints:

$$y_0 \geq c_0 + s$$

$$(1+r)s + y_1^H \geq c_1^H$$

$$(1+r)s + y_1^L \geq c_1^L$$

The problem:

$$\max_{c_0, s, c_1^H, c_1^L} u(c_0) + \beta p u(c_1^H) + \beta(1-p)u(c_1^L)$$

subject to

$$y_0 \geq c_0 + s$$

$$(1+r)s + y_1^H \geq c_1^H$$

and

$$(1+r)s + y_1^L \geq c_1^L$$

Notes:

There are two constraints for period $t = 1$: one for each possible realization of y_1.

What makes the problem interesting is that savings s at $t = 0$ must be chosen before income y_1 at $t = 1$ is known.

From the viewpoint of $t = 0$, uncertainty about y_1 induces uncertainty about c_1: the consumer must choose a "contingency plan" for c_1.

In this simple case, it's not really that much of a problem to deal with all of the constraints in forming a Lagrangian.

In this simple case, it's relatively easy to describe the contingency plan using the notation c_1^H and c_1^L to distinguish between consumption at $t = 1$ in each of the two states.

But, as we've already seen, when the number of periods and/or the number of possible states grow, these notational burdens become increasing tedious, which is what motivates our interest in dynamic programming as a way of dealing with stochastic problems.

9.2 The Kuhn-Tucker Formulation

Set up the Lagrangian, using separate multipliers μ_1^H and μ_1^L for each constraint at $t = 1$:

$$\begin{aligned} L(c_0, s, c_1^H, c_1^L, \mu_0, \mu_1^H, \mu_1^L) &= u(c_0) + \beta p u(c_1^H) + \beta(1-p)u(c_1^L) + \mu_0(y_0 - c_0 - s) \\ &\quad + \mu_1^H[(1+r)s + y_1^H - c_1^H] + \mu_1^L[(1+r)s + y_1^L - c_1^L] \end{aligned}$$

FOC for c_0:

$$u'(c_0) - \mu_0 = 0$$

FOC for s:

$$-\mu_0 + \mu_1^H(1+r) + \mu_1^L(1+r) = 0$$

FOC for c_1^H:

$$\beta p u'(c_1^H) - \mu_1^H = 0$$

FOC for c_1^L:

$$\beta(1-p)u'(c_1^L) - \mu_1^L = 0$$

Use the FOC's for c_0, c_1^H, and c_1^L to eliminate reference to the multipliers μ_0, μ_1^H, and μ_1^L in the FOC for s:

$$u'(c_0) = \beta p u'(c_1^H)(1+r) + \beta(1-p)u'(c_1^L)(1+r) \tag{42}$$

Together with the binding constraints

$$y_0 = c_0 + s$$

$$(1+r)s + y_1^H = c_1^H$$

and

$$(1+r)s + y_1^L = c_1^L$$

(42) gives us a system of 4 equations in the 4 unknowns: c_0, s, c_1^H, and c_1^L.

Note also that (42) can be written more compactly as

$$u'(c_0) = \beta(1+r)E[u'(c_1)],$$

which is a special case of the more general optimality condition that we derived previously in the "saving with multiple random returns" example, reflecting that in this simple example:

There are only two periods.

The return on the single asset is known.

9.3 The Dynamic Programming Formulation

Consider "restarting" the problem at $t = 1$, in state H, given that s has already been chosen and y_1^H already determined.

The consumer solves the static problem:

$$\max_{c_1^H} u(c_1^H)$$

subject to

$$(1+r)s + y_1^H \geq c_1^H.$$

The solution is trivial: set

$$c_1^H = (1+r)s + y_1^H$$

Hence, if we define the maximum value function

$$v(s, y_1^H) = \max_{c_1^H} u(c_1^H) \text{ subject to } (1+r)s + y_1^H \geq c_1^H$$

then we know right away that

$$v(s, y_1^H) = u[(1+r)s + y_1^H]$$

and hence

$$v_1(s, y_1^H) = (1+r)u'[(1+r)s + y_1^H] = (1+r)u'(c_1^H) \tag{43}$$

Likewise, if we restart the problem at $t = 1$ in state L, given that s has already been chosen and y_1^L already determined, then

$$v(s, y_1^L) = \max_{c_1^L} u(c_1^L) \text{ subject to } (1+r)s + y_1^L \geq c_1^L$$

and we know right away that

$$v(s, y_1^L) = u[(1+r)s + y_1^L]$$

and

$$v_1(s, y_1^L) = (1+r)u'[(1+r)s + y_1^L] = (1+r)u'(c_1^L) \tag{44}$$

Now back up to $t = 0$, and consider the problem

$$\max_{c_0, s} u(c_0) + \beta E v(s, y_1) \text{ subject to } y_0 \geq c_0 + s$$

or, even more simply

$$\max_{s} u(y_0 - s) + \beta E v(s, y_1) \tag{45}$$

Equation (45) is like the Bellman equation for the consumer's problem:

The problem described on the right-hand-side is a static problem: the dynamic programming approach breaks the dynamic program down into a sequence of static problems.

Note, too, that the problem is an unconstrained optimization problem.

And note that in (45), the "maximize with respect to c_1^H and c_1^L" part of the original dynamic problem has been moved inside the expectation term, sidestepping the need to talk explicitly about "contingency plans" for the future.

Take the FOC for the value of s that solves the problem in (45):

$$-u'(y_0 - s) + \beta E v_1(s, y_1) = 0$$

and rewrite it using (43) and (44) as

$$u'(c_0) = \beta E[(1+r)u'(c_1)] = \beta(1+r)pu'(c_1^H) + \beta(1+r)(1-p)u'(c_1^L) \tag{46}$$

Notes:

Together with the binding constraints

$$y_0 = c_0 + s$$

$$(1+r)s + y_1^H = c_1^H$$

and

$$(1+r)s + y_1^L = c_1^L$$

(46) gives us a system of 4 equations in the 4 unknowns: c_0, s, c_1^H, and c_1^L.

This system of equations is exactly the same one that we derived earlier with the help of the Lagrangian and the Kuhn-Tucker theorem.

The Necessity of the Transversality Condition at Infinity: A (Very) Special Case

Peter Ireland*
ECON 772001 - Math for Economists
Boston College, Department of Economics

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Consider a discrete-time, infinite horizon model that characterizes the optimal consumption of an exhaustible resource (like oil or coal). Let time periods be indexed by $t = 0, 1, 2, \ldots$, and let s_t, $t = 0, 1, 2, \ldots$, denote the stock of the exhaustible resource that remains at the beginning of period t. Let c_t, $t = 0, 1, 2, \ldots$, denote the amount of this resource that is consumed during period t. Since no new units of the resource are ever created, the amount consumed simply subtracts from the available stock according to

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, \ldots$, where the inequality constraint (which will always bind at the optimum) simply recognizes that the resource can be freely disposed of. The optimization problem then involves choosing sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize utility from consuming the resource over the infinite horizon, given by

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where the discount factor lies between zero and one, $0 < \beta < 1$, subject to the constraints $s_t - c_t \geq s_{t+1}$ for all $t = 0, 1, 2, \ldots$, taking as given the level of the initial resource stock $s_0 > 0$.

Strictly speaking, we could also add nonnegativity constraints $c_t \geq 0$ for all $t = 0, 1, 2, \ldots$ and $s_t \geq 0$ for all $t = 1, 2, 3, \ldots$ to the statement of the problem, but the assumption of log utility, which implies that the marginal utility of consumption becomes infinite as the level of consumption approaches zero, also implies that these constraints will never bind at the optimum.

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We now know that there are at least three ways of deriving the necessary conditions describing a solution to this problem: using the Kuhn-Tucker theorem and the Lagrangian, using the maximum principle and the Hamiltonian, or using dynamic programming and the Bellman equation.

If we choose to use the Kuhn-Tucker theorem, then we would start by defining the Lagrangian for the problem as

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \tilde{\lambda}_{t+1}(s_t - c_t - s_{t+1}).$$

This definition of the Lagrangian casts the problem in "present value" form, in the sense that $\tilde{\lambda}_t$ measures the present value at $t = 0$ of having an additional unit of the resource available at the end of period t or the beginning of period $t + 1$. Alternatively, we can use the new variable

$$\lambda_{t+1} = \beta^{-t}\tilde{\lambda}_{t+1},$$

to replace $\tilde{\lambda}_t$ with $\beta^t \lambda_t$ and write the Lagrangian in "current value" form as

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \beta^t \lambda_{t+1}(s_t - c_t - s_{t+1}).$$

According to the Kuhn-Tucker theorem, if the sequences $\{c_t^*\}_{t=0}^{\infty}$ and $\{s_t^*\}_{t=1}^{\infty}$ solve the dynamic optimization problem, then there exists a sequence $\{\lambda_t^*\}_{t=1}^{\infty}$ of value for the Lagrange multipliers such that together, these three sequences satisfy:

a) The first-order condition for c_t^*,

$$\frac{\beta^t}{c_t^*} - \beta^t \lambda_{t+1}^* = 0$$

or more simply

$$\frac{1}{c_t^*} = \lambda_{t+1}^* \tag{1}$$

for all $t = 0, 1, 2, \ldots$.

b) The first-order condition for s_t^*,

$$\beta^t \lambda_{t+1}^* - \beta^{t-1}\lambda_t^* = 0$$

or more simply

$$\beta\lambda_{t+1}^* = \lambda_t^* \tag{2}$$

for all $t = 1, 2, 3, \ldots$.

c) The constraint

$$s_t^* - c_t^* \geq s_{t+1}^* \tag{3}$$

d) The nonnegativity condition

$$\lambda_{t+1}^* \geq 0 \tag{4}$$

for all $t = 0, 1, 2, \ldots$.

e) The complementary slackness condition

$$\lambda_{t+1}^*(s_t^* - c_t^* - s_{t+1}^*) = 0 \tag{5}$$

for all $t = 0, 1, 2, \ldots$.

Note that the first-order condition (1) for consumption implies that

$$\lambda_{t+1}^* = \frac{1}{c_t^*} > 0$$

and so, by extension, the complementary slackness condition (5) implies that the constraint (3) will always bind.

Note also that we can use (1) to "solve out" for the Lagrange multipliers in (2), using

$$\lambda_{t+1}^* = \frac{1}{c_t^*}$$

and

$$\lambda_t^* = \frac{1}{c_{t-1}^*}.$$

Hence, the solution to the original dynamic optimization problem can be characterized by finding solutions to a system of two difference equations in the two unknown sequences $\{c_t^*\}_{t=0}^\infty$ and $\{s_t^*\}_{t=1}^\infty$. The first difference equation comes from the first-order conditions and can be written as

$$\beta\lambda_{t+1}^* = \lambda_t^*$$

or

$$\frac{\beta}{c_t^*} = \frac{1}{c_{t-1}^*}$$

or

$$c_t^* = \beta c_{t-1}^*$$

or

$$c_{t+1}^* = \beta c_t^* \tag{6}$$

This optimality condition can be interpreted as one that indicates that it is optimal to equate the marginal rate of intertemporal substitution

$$\frac{\beta/c_t^*}{1/c_{t-1}^*}$$

to the intertemporal price, which is fixed at unity by the technological assumption that the exhaustible resource can be stored across periods without depreciation.

The second difference equation comes from the binding constraint and can be written as

$$s^*_{t+1} = s^*_t - c^*_t. \tag{7}$$

We know that in general, two boundary conditions are needed to pin down a unique solution to this system of two difference equations.

One boundary condition is the initial condition

$$s_0 \text{ given.} \tag{8}$$

In a finite-horizon version of the problem, second boundary condition would be given by the complementary slackness condition on the nonnegativity constraint $s^*_{T+1} \geq 0$ for the terminal value of the stock, which we know from our more general analysis can be written as

$$\beta^T \lambda^*_{T+1} s^*_{T+1} = 0. \tag{9}$$

Moreover, in the finite-horizon version of the problem, we could show that this transversality condition will hold because the nonnegativity constraint on the terminal value of the stock binds at the optimum:

$$s^*_{T+1} = 0.$$

Intuitively, with a finite horizon, if a strictly positive amount of the exhaustible resource remains at the end of period T, then a higher level of utility could be achieved by consuming that positive amount of the resource at one or more periods $t = 0, 1, \ldots, T$.

For the infinite-horizon version of the problem, our more general analysis suggests that the relevant terminal, or transversality, condition is given by

$$\lim_{T\to\infty} \beta^T \lambda^*_{T+1} s^*_{T+1} = 0. \tag{10}$$

Notice that the first-order condition (2) implies that $\beta^T \lambda^*_{T+1}$ is going to be constant at the optimum. So in this special case, (10) will hold because

$$\lim_{T\to\infty} s^*_{T+1} = 0.$$

Intuitively, with an infinite horizon, it will never be optimal for the stock of the exhaustible resource to be run all the way down to zero over any finite period of time, since that would entail zero consumption from that point onward. On the other hand, if the stock is not exhausted in the limit, then there is a sense in which the resource is not being consumed "fast enough," in a way that parallels our argument for the finite-horizon case when some amount of the resource remains at the end of the horizon.

This model is simple enough, in fact, that we can prove formally that (10) is a necessary condition for the infinite-horizon case. The proof involves two steps.

Step one is to argue that $\{\beta^t\lambda^*_{t+1}s^*_{t+1}\}^\infty_{t=0}$ is a nonincreasing sequence. To show this, note that for all $t = 1, 2, 3, \ldots$, (2), (3), (4), and the nonnegativity of c^*_t imply that

$$\beta^t\lambda^*_{t+1}s^*_{t+1} - \beta^{t-1}\lambda^*_t s^*_t = \beta^t\lambda^*_{t+1}(s^*_{t+1} - s^*_t) \leq \beta^t\lambda^*_{t+1}(s^*_t - c^*_t - s^*_t) = -\beta^t\lambda^*_{t+1}c^*_t \leq 0.$$

Step two is to argue that

$$\inf_t \beta^t\lambda^*_{t+1}s^*_{t+1} = 0.$$

To show this, suppose to the contrary that there exists an $\varepsilon > 0$ such that

$$\beta^t\lambda^*_{t+1}s^*_{t+1} \geq \varepsilon$$

for all $t = 0, 1, 2, \ldots$. Since (1) implies that $\lambda^*_{t+1} > 0$ for all $t = 0, 1, 2, \ldots$, (2) implies that this last condition requires that

$$s^*_{t+1} \geq \gamma$$

for all $t = 0, 1, 2, \ldots$, where γ equals ε divided by the constant, positive value of $\beta^t\lambda^*_{t+1}$ along the optimal path. But, in this case, we can define new sequences $\{c^*_t\}^\infty_{t=0}$ and $\{s^{**}_t\}^\infty_{t=1}$ with

$$c^{**}_0 = c^*_0 + \gamma,$$

$$c^{**}_t = c^*_t \text{ for all } t = 1, 2, 3, \ldots,$$

and

$$s^{**}_t = s^*_t - \gamma \text{ for all } t = 1, 2, 3, \ldots$$

that satisfy all of the constraints from the original problem, but yield a higher level of utility, contradicting the assumption that $\{c^*_t\}^\infty_{t=0}$ and $\{s^*_t\}^\infty_{t=1}$ solve the problem.

Taken together, $\{\beta^t\lambda^*_{t+1}s^*_{t+1}\}^\infty_{t=0}$ and $\inf_t \beta^t\lambda^*_{t+1}s^*_{t+1} = 0$ require that (10) hold at the optimum, completing the proof.

The proof turns out to be relative straightforward for this simple problem, but becomes much more difficult to apply in other cases that are only slightly more complicated. Even for the Ramsey model with log utility and Cobb-Douglas production, for instance, this proof does not generalize.

For a much more elaborate proof that does apply to that version of the Ramsey model, see Ivar Ekeland and Jose Alexandre Scheinkman, "Transversality Conditions for Some Infinite Horizon Discrete Time Optimization Problems," *Mathematics of Operations Research*, Vol. 11 (May 1986), pp.216-229.

For the sake of completeness, let's wrap up by completely characterizing the solution to the original dynamic optimization problem.

Once again, that solution must satisfy the difference equations

$$c^*_{t+1} = \beta c^*_t \tag{6}$$

and

$$s^*_{t+1} = s^*_t - c^*_t \tag{7}$$

together with the initial condition

$$s_0 \text{ given} \tag{8}$$

and the transversality condition

$$\lim_{T\to\infty} \beta^T \lambda^*_{T+1} s^*_{T+1} = 0 \tag{10}$$

where the latter requires that

$$\lim_{T\to\infty} s^*_{T+1} = 0.$$

To start, consider (7), which implies

$$s^*_1 = s_0 - c^*_0,$$
$$s^*_2 = s^*_1 - c^*_1 = s_0 - c^*_0 - c^*_1,$$
$$s^*_3 = s^*_2 - c^*_2 = s_0 - c^*_0 - c^*_1 - c^*_2$$

or, after repeating T times,

$$s^*_T = s_0 - \sum_{t=0}^{T-1} c^*_t$$

or, after repeating infinitely many times and using the transversality condition

$$0 = s_0 - \sum_{t=0}^{\infty} c^*_t.$$

Rewritten as

$$\sum_{t=0}^{\infty} c^*_t = s_0,$$

this result confirms our intuition about the implications of the transversality condition: it shows that over the infinite horizon, it is optimal to consume the entire resource stock, otherwise, a higher level of utility could be achieved while still satisfying all of the constraints.

Now use (6), which implies that

$$c^*_t = \beta^t c^*_0$$

to pin down the level of the consumption path from

$$s_0 = \sum_{t=0}^{\infty} c^*_t = c^*_0 \sum_{t=0}^{\infty} \beta^t = \frac{c^*_0}{1-\beta}.$$

Evidently, it is optimal to start by consuming

$$c^*_0 = (1-\beta)s_0$$

at $t = 0$ and then to allow consumption to decrease proportionally according to (6) for all $t = 1, 2, 3, \ldots$.