The Maximum Principle

Peter Ireland*
ECON 772001 - Math for Economists
Boston College, Department of Economics

Fall 2020

Here, we will explore the connections between two ways of solving dynamic optimization problems, that is, problems that involve optimization over time. The first solution method is just a straightforward application of the Kuhn-Tucker theorem; the second solution method relies on the maximum principle. Although these two approaches might at first glance seem quite different, in fact and as we will see, they are closely related.

We'll begin by briefly noting the basic features that set dynamic optimization problems apart from purely static ones. Then we'll go on consider the connections between the Kuhn-Tucker theorem and the maximum principle in both discrete and continuous time.

References:

Dixit, Chapter 10.

Acemoglu, Chapter 7.

The maximum principle was developed in the 1950s and 1960s by Soviet mathematicians, one of the key original references being:

L.S. Pontryagin, with V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko, *The Mathematical Theory of Optimal Processes*, 1962.

1 Basic Elements of Dynamic Optimization Problems

Moving from the static optimization problems that we've considered so far to the dynamic optimization problems that are of primary interest here involves only a few minor changes.

a) We need to index the variables that enter into the problem by t, in order to keep track of changes in those variables that occur over time.

b) We need to distinguish between two types of variables:

stock variables - e.g., stock of capital, assets, or wealth

flow variables - e.g., output, consumption, saving, or labor supply per unit of time

c) We need to introduce constraints that describe the evolution of stock variables over time: e.g., larger flows of savings or investment today will lead to larger stocks of wealth or capital tomorrow.

2 The Maximum Principle: Discrete Time

2.1 A Dynamic Optimization Problem in Discrete Time

Consider a dynamic optimization in discrete time, that is, in which time can be indexed by $t = 0, 1, \dots, T$.

y_t = stock variable

z_t = flow variable

Objective function:

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

Following Dixit, we can allow for a wider range of possibilities by letting the functions as well as the variables depend on the time index t.

$1 \geq \beta > 0$ = discount factor

Constraint describing the evolution of the stock variable:

$$Q(y_t, z_t; t) \geq y_{t+1} - y_t$$

or

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

for all $t = 0, 1, \dots, T$

Constraint applying to variables within each period:

$$c \geq G(y_t, z_t; t)$$

for all $t = 0, 1, \dots, T$

Constraints on initial and terminal values of stock:

$$y_0 \text{ given}$$

$$y_{T+1} \geq y^*$$

The dynamic optimization problem can now be stated as: choose sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ to maximize the objective function subject to all of the constraints.

Notes:

a) It is important for the application of the maximum principle that the problem be additively time separable: that is, the values of F, Q, and G at time t must depend on the values of y_t and z_t only at time t.

b) Although the constraints describing the evolution of the stock variable and applying to the variables within each period can each be written in the form of a single equation, it must be emphasized that these constraints must hold for all $t = 0, 1, \ldots, T$. That is, each of these equations actually describes $T+1$ constraints.

2.2 The Kuhn-Tucker Formulation

Let's begin by applying the Kuhn-Tucker Theorem to solve this problem. That is, let's set up the Lagrangian and take first-order conditions.

Set up the Lagrangian, recognizing that the constraints must hold for all $t = 0, 1, \ldots, T$:

$$\begin{aligned} L &= \sum_{t=0}^{T} \beta^t F(y_t, z_t; t) \\ &\quad + \sum_{t=0}^{T} \pi_{t+1}[y_t + Q(y_t, z_t; t) - y_{t+1}] + \sum_{t=0}^{T} \lambda_t [c - G(y_t, z_t; t)] + \phi(y_{T+1} - y^*) \end{aligned}$$

The Kuhn-Tucker theorem tells us that the solution to this problem must satisfy the FOC for the choice variables z_t for $t = 0, 1, \ldots, T$ and y_t for $t = 1, 2, \ldots, T+1$.

FOC for z_t, $t = 0, 1, \ldots, T$:

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \tag{1}$$

for all $t = 0, 1, \ldots, T$.

FOC for y_t, $t = 1, 2, \ldots, T$:

$$\beta^t F_y(y_t, z_t; t) + \pi_{t+1} + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t) - \pi_t = 0$$

or

$$\pi_{t+1} - \pi_t = -[\beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t)] \tag{2}$$

for all $t = 1, 2, \ldots, T$.

FOC for y_{T+1}:

$$-\pi_{T+1} + \phi = 0$$

Let's assume that the problem is such that the constraint governing the evolution of the stock variable always holds with equality, as will typically be the case in economic applications. Then another condition describing the solution to the problem is

$$y_{t+1} - y_t = Q(y_t, z_t; t) \tag{3}$$

for all $t = 0, 1, \ldots, T$.

Finally, let's write down the initial condition for the stock variable and the complementary slackness condition for the constraint on the terminal value of the stock:

$$y_0 \text{ given} \tag{4}$$

$$\phi(y_{T+1} - y^*) = 0$$

or, using the FOC for y_{T+1}:

$$\pi_{T+1}(y_{T+1} - y^*) = 0 \tag{5}$$

Notes:

a) Together with the complementary slackness condition

$$\lambda_t[c - G(y_t, z_t; t)] = 0,$$

which implies either

$$\lambda_t = 0 \text{ or } c = G(y_t, z_t; t),$$

we can think of (1)-(3) as forming a system of four equations in four unknowns y_t, z_t, π_t, λ_t. This system of equations determines the problem's solution.

b) Equations (2) and (3), linking the values of y_t and π_t at adjacent points in time, are examples of difference equations. They must be solved subject to two boundary conditions:

The initial condition (4).

The terminal, or transversality, condition (5).

c) The analysis can also be applied to the case of an infinite time horizon, where $T = \infty$. In this case, (1) must hold for all $t = 0, 1, 2, \ldots$, (2) must hold for all $t = 1, 2, 3, \ldots$, (3) must hold for all $t = 0, 1, 2, \ldots$, and (5) becomes a condition on the limiting behavior of π_t and y_t:

$$\lim_{T\to\infty} \pi_{T+1}(y_{T+1} - y^*) = 0. \tag{6}$$

2.3 An Alternative Formulation

Now let's consider the problem in a slightly different way.

Begin by defining the Hamiltonian for time t as

$$\hat{H}(y_t, \pi_{t+1}, z_t; t) = \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t).$$

Next, define the "maximized Hamiltonian" as

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \hat{H}(y_t, \pi_{t+1}, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t).$$

Putting these two definitions together, the maximized Hamiltonian can also be written as

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t) \tag{7}$$

Note that the maximized Hamiltonian is a maximum value function.

Note also that the maximization problem on the right-hand side of (7) is a static optimization problem, involving no dynamic elements.

By the Kuhn-Tucker theorem:

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) + \lambda_t [c - G(y_t, z_t; t)]$$

And by the envelope theorem:

$$H_y(y_t, \pi_{t+1}; t) = \beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t) \tag{8}$$

and

$$H_\pi(y_t, \pi_{t+1}; t) = Q(y_t, z_t; t) \tag{9}$$

where z_t solves the optimization problem on the right-hand side of (7) and must therefore satisfy the FOC:

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \tag{10}$$

Now notice the following:

a) Equation (10) coincides with (1).

b) In light of (8) and (9), (2) and (3) can be written more compactly as

$$\pi_{t+1} - \pi_t = -[\beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t)] \tag{2}$$

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) \tag{11}$$

and

$$y_{t+1} - y_t = Q(y_t, z_t; t) \tag{3}$$

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t). \tag{12}$$

This establishes the following result.

Theorem (Maximum Principle) Consider the discrete time dynamic optimization problem of choosing sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ to maximize the objective function

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

subject to the constraints

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

for all $t = 0, 1, \ldots, T$,

$$c \geq G(y_t, z_t; t)$$

for all $t = 0, 1, \ldots, T$,

$$y_0 \text{ given}$$

and

$$y_{T+1} \geq y^*.$$

Associated with this problem, define the maximized Hamiltonian

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t). \tag{7}$$

Then the solution to the dynamic optimization problem must satisfy

a) The first-order and complementary slackness conditions for the static optimization problem on the right-hand side of (7):

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \tag{10}$$

and

$$\lambda_t [c - G(y_t, z_t; t)] = 0$$

for all $t = 0, 1, \ldots, T$.

b) The pair of difference equations:

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) \tag{11}$$

for all $t = 1, 2, \ldots, T$ and

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t) \tag{12}$$

for all $t = 0, 1, \ldots, T$, where the derivatives of H can be calculated using the envelope theorem.

c) The initial condition

$$y_0 \text{ given} \tag{4}$$

d) The terminal, or transversality, condition

$$\pi_{T+1}(y_{T+1} - y^*) = 0 \tag{5}$$

in the case where $T < \infty$ or

$$\lim_{T \to \infty} \pi_{T+1}(y_{T+1} - y^*) = 0 \tag{6}$$

in the case where $T = \infty$.

Thus, according to the maximum principle, there are two ways of solving discrete time dynamic optimization problems, both of which lead to the same answer:

a) Set up the Lagrangian for the dynamic optimization problem and take first-order conditions for all $t = 0, 1, \ldots, T$.

b) Set up the maximized Hamiltonian for the problem and derive the first-order and envelope conditions (10)-(12) for the static optimization problem that appears in the definition of that maximized Hamiltonian.

3 The Maximum Principle: Continuous Time

3.1 A Dynamic Optimization Problem in Continuous Time

Like the extension from static to dynamic optimization, the extension from discrete to continuous time requires no new substantive ideas, but does require some changes in notation.

Accordingly, suppose now that the variable t, instead of taking on discrete values $t = 0, 1, \ldots, T$, takes on continuous values $t \in [0, T]$, where as before, T can be finite or infinite.

It is most convenient now to regard the variables as functions of time:

$y(t)$ = stock variable

$z(t)$ = flow variable

The obvious analog to the objective function from before is:

$$\int_0^T e^{-\rho t} F(y(t), z(t); t)\, \mathrm{d}t$$

$\rho \geq 0$ = discount rate

Example:

$\beta = 0.95$

$\rho = 0.05$

β^t for $t = 1$ is 0.95

$e^{-\rho t}$ for $t = 1$ is 0.951, or approximately 0.95

Consider next the constraint describing the evolution of the stock variable.

In the discrete time case, the interval between time periods is just $\Delta t = 1$.

Hence, the constraint might be written as

$$Q(y(t), z(t); t)\Delta t \geq y(t + \Delta t) - y(t)$$

or

$$Q(y(t), z(t); t) \geq \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

In the limit as the interval Δt goes to zero, this last expression simplifies to

$$Q(y(t), z(t); t) \geq \dot{y}(t)$$

for all $t \in [0, T]$, where $\dot{y}(t)$ denotes the derivative of $y(t)$ with respect to t.

The constraint applying to variables at a given point in time remains the same:

$$c \geq G(y(t), z(t); t)$$

for all $t \in [0, T]$.

Note once again that these constraints must hold for all $t \in [0, T]$. Thus, each of the two equations from above actually represents an entire continuum of constraints.

Finally, the initial and terminal constraints for the stock variable remain unchanged:

$$y(0) \text{ given}$$

$$y(T) \geq y^*$$

The dynamic optimization problem can now be stated as: choose functions $z(t)$ for $t \in [0, T]$ and $y(t)$ for $t \in (0, T]$ to maximize the objective function subject to all of the constraints.

3.2 The Kuhn-Tucker Formulation

Once again, let's begin by setting up the Lagrangian and taking first-order conditions:

$$\begin{aligned} L &= \int_0^T e^{-\rho t} F(y(t), z(t); t)\, \mathrm{d}t + \int_0^T \pi(t)[Q(y(t), z(t); t) - \dot{y}(t)]\, \mathrm{d}t \\ &\quad + \int_0^T \lambda(t)[c - G(y(t), z(t); t)]\, \mathrm{d}t + \phi[y(T) - y^*] \end{aligned}$$

Now we are faced with a problem: $y(t)$ is a choice variable for all $t \in [0, T]$, but $\dot{y}(t)$ appears in the Lagrangian.

To solve this problem, use integration by parts:

$$\int_0^T \left\{ \frac{d}{dt} [\pi(t) y(t)] \right\} \mathrm{d}t = \int_0^T \dot{\pi}(t) y(t) \, \mathrm{d}t + \int_0^T \pi(t) \dot{y}(t) \, \mathrm{d}t$$

$$\pi(T) y(T) - \pi(0) y(0) = \int_0^T \dot{\pi}(t) y(t) \, \mathrm{d}t + \int_0^T \pi(t) \dot{y}(t) \, \mathrm{d}t$$

$$-\int_0^T \pi(t) \dot{y}(t) \, \mathrm{d}t = \int_0^T \dot{\pi}(t) y(t) \, \mathrm{d}t + \pi(0) y(0) - \pi(T) y(T)$$

Use this result to rewrite the Lagrangian as

$$\begin{aligned} L \quad = \quad & \int_0^T e^{-\rho t} F(y(t), z(t); t) \, \mathrm{d}t + \int_0^T \pi(t) Q(y(t), z(t); t) \, \mathrm{d}t \\ & + \int_0^T \dot{\pi}(t) y(t) \, \mathrm{d}t + \pi(0) y(0) - \pi(T) y(T) \\ & + \int_0^T \lambda(t) [c - G(y(t), z(t); t)] \, \mathrm{d}t + \phi [y(T) - y^*] \end{aligned}$$

Before taking first-order conditions, note that the multipliers $\pi(t)$ and $\lambda(t)$ are functions of t and that the corresponding constraints appear in the form of integrals. These features of the Lagrangian reflect the fact that the constraints must hold for all $t \in [0, T]$.

Next, we have to ask: how do we differentiate the Lagrangian L with respect to the objects of choice: $z(t)$ and $y(t)$? Providing a full answer to this question is part of what makes the theory of continuous-time optimization, or "optimal control," difficult. We can get some intuition, however, from considering the following, very simple, "variational argument;" for more details, see Acemoglu's Section 7.1 and the references he cites on page 276.

Consider the problem of choosing a continuously differentiable function $y(t)$, defined for $t \in [0, T]$, to maximize the integral

$$F(y) = \int_0^T f(y(t), t) \mathrm{d}t,$$

where f is two times continuously differentiable in both arguments, y and t. Let $y^*(t)$ be a solution to this problem, and consider an "admissible variation"

$$y(t) = y^*(t) + \varepsilon \eta(t),$$

where $\varepsilon \in \mathbf{R}$ and $\eta(t)$ is another continuously differentiable function, so that $y(t)$ is also continuously differentiable. Now we can evaluate

$$F(y^* + \varepsilon \eta) = \int_0^T f(y^*(t) + \varepsilon \eta(t), t) \mathrm{d}t.$$

If F is maximized by y^*, it should not be possible to increase the value of this objective function, starting from y^* and hence $\varepsilon = 0$, by choosing a value of ε that is small in absolute value but either strictly positive or strictly negative. Hence, the derivative

$$\frac{dF(y^* + \varepsilon\eta)}{d\varepsilon} = \int_0^T f_y(y^*(t) + \varepsilon\eta(t), t)\eta(t)\mathrm{d}t$$

must equal zero when evaluated at $\varepsilon = 0$:

$$\left.\frac{dF(y^* + \varepsilon\eta)}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^T f_y(y^*(t), t)\eta(t)\mathrm{d}t = 0.$$

Since this optimality condition must hold for any continuously differentiable function $\eta(t)$, consider in particular the choice $\eta(t) = f'(y^*(t), t)$ for all $t \in [0, T]$, so that the optimality condition becomes

$$\int_0^T [f_y(y^*(t), t)]^2 \mathrm{d}t = 0,$$

which can only hold if

$$f_y(y^*(t), t) = 0 \text{ for all } t \in [0, T].$$

What this suggests is that, to characterize the solution to continuous-time problems such as this, we can pretend that the integral in the objective function

$$F(y) = \int_0^T f(y(t), t)\mathrm{d}t$$

is really a sum, fix an arbitrary value of $t \in [0, T]$, and differentiate the entire objective function with respect to $y(t)$ for that value of t to get the

$$f_y(y^*(t), t) = 0 \text{ for all } t \in [0, T].$$

This is how we will now proceed, in deriving the first-order conditions for our more general, constrained optimization in continuous time using the Lagrangian

$$\begin{aligned} L \quad = \quad & \int_0^T e^{-\rho t} F(y(t), z(t); t)\,\mathrm{d}t + \int_0^T \pi(t) Q(y(t), z(t); t)\,\mathrm{d}t \\ & + \int_0^T \dot{\pi}(t) y(t)\,\mathrm{d}t + \pi(0)y(0) - \pi(T)y(T) \\ & + \int_0^T \lambda(t)[c - G(y(t), z(t); t)]\,\mathrm{d}t + \phi[y(T) - y^*] \end{aligned}$$

FOC for $z(t)$, $t \in [0, T]$:

$$e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) - \lambda(t) G_z(y(t), z(t); t) = 0 \tag{13}$$

for all $t \in [0, T]$

FOC for $y(t)$, $t \in (0, T)$:

$$e^{-\rho t}F_y(y(t), z(t); t) + \pi(t)Q_y(y(t), z(t); t) + \dot{\pi}(t) - \lambda(t)G_y(y(t), z(t); t) = 0$$

or

$$\dot{\pi}(t) = -[e^{-\rho t}F_y(y(t), z(t); t) + \pi(t)Q_y(y(t), z(t); t) - \lambda(t)G_y(y(t), z(t); t)]$$

for all $t \in (0, T)$.

If we require all functions of t to be continuously differentiable, then this last equation will also hold for $t = 0$ and $t = T$, and we can write

$$\dot{\pi}(t) = -[e^{-\rho t}F_y(y(t), z(t); t) + \pi(t)Q_y(y(t), z(t); t) - \lambda(t)G_y(y(t), z(t); t)] \tag{14}$$

for all $t \in [0, T]$.

FOC for $y(T)$:

$$\begin{aligned} 0 \;=\;& e^{-\rho T}F_y(y(T), z(T); T) + \pi(T)Q_y(y(T), z(T); T) \\ &+\dot{\pi}(T) - \pi(T) - \lambda(T)G_y(y(T), z(T); T) + \phi \end{aligned}$$

or, using (14),

$$\pi(T) = \phi$$

Assume, as before, that the constraint governing $\dot{y}(t)$ holds with equality:

$$\dot{y}(t) = Q(y(t), z(t); t) \tag{15}$$

for all $t \in [0, T]$.

Finally, write down the initial condition

$$y(0) \text{ given} \tag{16}$$

and the complementary slackness, or transversality condition

$$\phi[y(T) - y^*] = 0$$

or

$$\pi(T)[y(T) - y^*] = 0 \tag{17}$$

or in the infinite-horizon case

$$\lim_{T\to\infty} \pi(T)[y(T) - y^*] = 0. \tag{18}$$

Notes:

a) Together with the complementary slackness condition

$$\lambda(t)[c - G(y(t), z(t); t)] = 0,$$

we can think of (13)-(15) as a system of four equations in four unknowns $y(t)$, $z(t)$, $\pi(t)$, and $\lambda(t)$. This system of equations determines the problem's solution.

b) Equations (14) and (15), describing the behavior of $\dot{y}(t)$ and $\dot{\pi}(t)$, are examples of differential equations. They must be solved subject to two boundary conditions: (16) and either (17) or (18).

3.3 An Alternative Formulation

As before, define the Hamiltonian for this problem as

$$\hat{H}(y(t), \pi(t), z(t); t) = e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t)$$

and the maximized Hamiltonian as

$$H(y(t), \pi(t); t) = \max_{z(t)} \hat{H}(y(t), \pi(t), z(t); t) \text{ subject to } c \geq G(y(t), z(t); t).$$

And again as before, combine these definitions to obtain

$$\begin{aligned} H(y(t), \pi(t); t) = &\max_{z(t)} e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \\ &\text{subject to } c \geq G(y(t), z(t); t) \end{aligned} \tag{19}$$

As before, the maximized Hamiltonian is a maximum value function. And as before, the maximization problem of the right-hand side is a static one.

By the Kuhn-Tucker theorem:

$$H(y(t), \pi(t); t) = \max_{z(t)} e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) + \lambda(t)[c - G(y(t), z(t); t)]$$

And by the envelope theorem:

$$H_y(y(t), \pi(t); t) = e^{-\rho t} F_y(y(t), z(t); t) + \pi(t) Q_y(y(t), z(t); t) - \lambda(t) G_y(y(t), z(t); t) \tag{20}$$

and

$$H_\pi(y(t), \pi(t); t) = Q(y(t), z(t); t) \tag{21}$$

where $z(t)$ solves the optimization problem on the right-hand side of (19) and must therefore satisfy the FOC:

$$e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) - \lambda(t) G_z(y(t), z(t); t) = 0. \tag{22}$$

Now notice the following:

a) Equation (22) coincides with (13).

b) In light of (20) and (21), (14) and (15) can be written more compactly as

$$\dot{\pi}(t) = -H_y(y(t), \pi(t); t) \tag{23}$$

and

$$\dot{y}(t) = H_\pi(y(t), \pi(t); t). \tag{24}$$

This establishes the following result.

Theorem (Maximum Principle) Consider the continuous time dynamic optimization problem of choosing continuously differentiable functions $z(t)$ and $y(t)$ for $t \in [0, T]$ to maximize the objective function

$$\int_0^T e^{-\rho t} F(y(t), z(t); t) \, \mathrm{d}t$$

subject to the constraints

$$Q(y(t), z(t); t) \geq \dot{y}(t)$$

for all $t \in [0, T]$,

$$c \geq G(y(t), z(t); t)$$

for all $t \in [0, T]$,

$$y(0) \text{ given,}$$

and

$$y(T) \geq y^*.$$

Associated with this problem, define the maximized Hamiltonian

$$\begin{aligned} H(y(t), \pi(t); t) = \max_{z(t)} & \, e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \\ & \text{subject to } c \geq G(y(t), z(t); t) \end{aligned} \tag{19}$$

Then the solution to the dynamic optimization problem must satisfy

a) The first-order and complementary slackness conditions for the static optimization problem on the right-hand side of (19):

$$e^{-\rho t} F_z(y(t), z(t); t) + \pi(t) Q_z(y(t), z(t); t) - \lambda(t) G_z(y(t), z(t); t) = 0 \tag{22}$$

and

$$\lambda(t)[c - G(y(t), z(t); t)] = 0$$

for all $t \in [0, T]$.

b) The pair of differential equations

$$\dot{\pi}(t) = -H_y(y(t), \pi(t); t) \tag{23}$$

and

$$\dot{y}(t) = H_\pi(y(t), \pi(t); t) \tag{24}$$

for all $t \in [0, T]$, where the derivatives of H can be calculated using the envelope theorem.

c) The initial condition

$$y(0) \text{ given.} \tag{16}$$

d) The terminal, or transversality, condition

$$\pi(T)[y(T) - y^*] = 0 \tag{17}$$

in the case where $T < \infty$ or

$$\lim_{T\to\infty} \pi(T)[y(T) - y^*] = 0. \tag{18}$$

in the case where $T = \infty$.

Once again, according to the maximum principle, there are two ways of solving continuous time dynamic optimization problems, both of which lead to the same answer:

a) Set up the Lagrangian for the dynamic optimization problem and take first-order conditions for all $t \in [0, T]$.

b) Set up the maximized Hamiltonian for the problem and derive the first-order and envelope conditions (22)-(24) for the static optimization problem that appears in that definition of the maximized Hamiltonian.

4 Two Examples

4.1 Life-Cycle Saving

Consider a consumer who is employed for $T + 1$ years: $t = 0, 1, \dots, T$.

w = constant annual labor income

k_t = stock of assets at the beginning of period $t = 0, 1, \dots, T + 1$

$k_0 = 0$

k_t can be negative for $t = 1, 2, \dots, T$, so that the consumer is allowed borrow.

However, k_{T+1} must satisfy

$$k_{T+1} \geq k^* > 0$$

where k^* denotes saving required for retirement.

r = constant interest rate

total income during period $t = w + rk_t$

c_t = consumption

Hence,

$$k_{t+1} = k_t + w + rk_t - c_t$$

or equivalently,

$$k_t + Q(k_t, c_t; t) \geq k_{t+1}$$

where

$$Q(k_t, c_t; t) = Q(k_t, c_t) = w + rk_t - c_t$$

for all $t = 0, 1, \dots, T$

Utility function:

$$\sum_{t=0}^{T} \beta^t \ln(c_t)$$

The consumer's problem: choose sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ to maximize the utility function subject to all of the constraints.

For this problem:

$$k_t = \text{stock variable}$$
$$c_t = \text{flow variable}$$

To solve this problem, set up the maximized Hamiltonian:

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(w + rk_t - c_t)$$

FOC for c_t:

$$\frac{\beta^t}{c_t} = \pi_{t+1} \tag{25}$$

Difference equations for π_t and k_t:

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1} r \tag{26}$$

and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = w + rk_t - c_t \tag{27}$$

Equations (25)-(27) represent a system of three equations in the three unknowns c_t, π_t, and k_t. They must be solved subject to the boundary conditions

$$k_0 = 0 \text{ given} \tag{28}$$

and

$$\pi_{T+1}(k_{T+1} - k^*) = 0 \tag{29}$$

We can use (25)-(29) to deduce some key properties of the solution even without solving the system completely.

Note first that (25) implies that

$$\pi_{T+1} = \frac{\beta^T}{c_T} > 0.$$

Hence, it follows from (29) that

$$k_{T+1} = k^*.$$

Thus, the consumer saves just enough for retirement and no more.

Next, note that (26) implies

$$\pi_{t+1} - \pi_t = -\pi_{t+1} r$$
$$(1+r)\pi_{t+1} = \pi_t \tag{30}$$

Use (25) to obtain

$$\pi_{t+1} = \frac{\beta^t}{c_t} \text{ and } \pi_t = \frac{\beta^{t-1}}{c_{t-1}}$$

and substitute these expressions into (30) to obtain

$$(1+r)\frac{\beta^t}{c_t} = \frac{\beta^{t-1}}{c_{t-1}}$$

$$(1+r)\frac{\beta}{c_t} = \frac{1}{c_{t-1}}$$

$$\frac{c_t}{c_{t-1}} = \beta(1+r) \tag{31}$$

Equation (31) reveals that the optimal growth rate of consumption is constant, and is faster for a more patient consumer, with a higher value of β, and a consumer that faces a higher interest rate r.

But now let's go a step further, to characterize the solution more fully. Equation (27),

$$k_{t+1} - k_t = w + rk_t - c_t, \tag{27}$$

and (31)

$$\frac{c_t}{c_{t-1}} = \beta(1+r) \tag{31}$$

now form a system of 2 difference equations in two unknowns c_t and k_t. In effect, we've used the first order condition (25) to eliminate the unknown multiplier π_t from the system, in much the same way that in our earlier static constrained optimization problems, we worked to simplify the optimality conditions by solving out for the unknown Lagrange multiplier.

Let's look at the simpler of these two equations, (31), first. Note that (31) implies that once c_0 is chosen, the entire sequence $\{c_t\}_{t=0}^T$ is pinned down:

$$c_1 = \beta(1+r)c_0,$$

$$c_2 = \beta(1+r)c_1 = [\beta(1+r)]^2 c_0,$$
$$c_3 = \beta(1+r)c_2 = [\beta(1+r)]^3 c_0,$$

or

$$c_t = [\beta(1+r)]^t c_0$$

for all $t = 0, 1, \ldots, T$. Hence, once we determine the optimal value of c_0, we can iterate forward on (31) to construct recursively the entire sequence $\{c_t\}_{t=0}^T$.

Next, let's turn to (27), which can be rewritten as

$$k_{t+1} = (1+r)k_t + w - c_t$$

For $t = 0$,

$$k_1 = (1+r)k_0 + w - c_0.$$

And, for $t = 1$,

$$\begin{aligned} k_2 &= (1+r)k_1 + w - c_1 \\ &= (1+r)^2 k_0 + (1+r)(w - c_0) + w - c_1. \end{aligned}$$

Likewise, for $t = 2$,

$$\begin{aligned} k_3 &= (1+r)k_2 + w - c_2 \\ &= (1+r)^3 k_0 + (1+r)^2(w - c_0) + (1+r)(w - c_1) + w - c_2. \end{aligned}$$

Repeating this procedure eventually yields

$$k_{T+1} = (1+r)^{T+1} k_0 + \sum_{t=0}^{T} (1+r)^{T-t}(w - c_t)$$

or

$$k_{T+1} = (1+r)^{T+1} k_0 + \sum_{t=0}^{T} (1+r)^{T-t} w - \sum_{t=0}^{T} (1+r)^{T-t} [\beta(1+r)]^t c_0$$

or

$$k_{T+1} = (1+r)^{T+1} k_0 + (1+r)^T \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t w - (1+r)^T \sum_{t=0}^{T} \beta^t c_0.$$

Now, use the initial condition k_0 and the terminal condition $k_{T+1} = k^*$ to find the optimal value of c_0:

$$\frac{k^*}{(1+r)^T} = \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t w - \sum_{t=0}^{T} \beta^t c_0$$

or

$$c_0 = \left(\sum_{t=0}^{T} \beta^t\right)^{-1} \left[\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t w - \frac{k^*}{(1+r)^T}\right].$$

Note, finally, that all of the items on the right-hand-side of this last expression for c_0 are parameters: β, r, w, k^*, and T. So given numerical settings for these parameters, one could numerically compute c_0 and, from there, go back to (27), (31), and $k_0 = 0$ to recursively construct the optimal sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$.

4.2 Optimal Growth

Consider an economy in which output is produced with capital according to the production function

$$F(k_t) = k_t^\alpha,$$

where $0 < \alpha < 1$.

c_t = consumption

δ = depreciation rate for capital, $0 < \delta < 1$

Then the evolution of the capital stock is governed by

$$k_{t+1} = k_t^\alpha + (1-\delta)k_t - c_t$$

or

$$k_{t+1} - k_t = k_t^\alpha - \delta k_t - c_t$$

Our first example had a finite horizon and was cast in discrete time. So for the sake of variety, make this second example have an infinite horizon in continuous time.

The continuous time analog to the capital accumulation constraint shown above is just

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

or

$$Q(k(t), c(t); t) \geq \dot{k}(t),$$

where

$$Q(k(t), c(t); t) = Q(k(t), c(t)) = k(t)^\alpha - \delta k(t) - c(t)$$

for all $t \in [0, \infty)$

Initial condition:

$$k(0) \text{ given}$$

Objective of a benevolent social planner or the utility of an infinitely-lived representative consumer:

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t,$$

where $\rho > 0$ is the discount rate.

The problem: choose continuously differentiable functions $c(t)$ and $k(t)$ for $t \in [0, \infty)$ to maximize utility subject to all of the constraints.

For this problem:

$k(t)$ = stock variable

$c(t)$ = flow variable

To solve this problem, set up the maximized Hamiltonian:

$$H(k(t),\pi(t);t) = \max_{c(t)} e^{-\rho t}\ln(c(t)) + \pi(t)[k(t)^\alpha - \delta k(t) - c(t)]$$

FOC for $c(t)$:

$$e^{-\rho t} = c(t)\pi(t) \tag{32}$$

Differential equations for $\pi(t)$ and $k(t)$:

$$\dot{\pi}(t) = -H_k(k(t),\pi(t);t) = -\pi(t)[\alpha k(t)^{\alpha-1} - \delta] \tag{33}$$

and

$$\dot{k}(t) = H_\pi(k(t),\pi(t);t) = k(t)^\alpha - \delta k(t) - c(t). \tag{34}$$

Equations (32)-(34) form a system of three equations in the three unknowns $c(t)$, $\pi(t)$, and $k(t)$. How can we solve them?

Start by differentiating both sides of (32) with respect to t:

$$e^{-\rho t} = c(t)\pi(t) \tag{32}$$

$$-\rho e^{-\rho t} = \dot{c}(t)\pi(t) + c(t)\dot{\pi}(t)$$

$$-\rho c(t)\pi(t) = \dot{c}(t)\pi(t) + c(t)\dot{\pi}(t)$$

Next, use (33)

$$\dot{\pi}(t) = -\pi(t)[\alpha k(t)^{\alpha-1} - \delta] \tag{33}$$

to rewrite this last equation as

$$-\rho c(t)\pi(t) = \dot{c}(t)\pi(t) - c(t)\pi(t)[\alpha k(t)^{\alpha-1} - \delta]$$

$$-\rho c(t) = \dot{c}(t) - c(t)[\alpha k(t)^{\alpha-1} - \delta]$$

$$\dot{c}(t) = c(t)[\alpha k(t)^{\alpha-1} - \delta - \rho] \tag{35}$$

Collect (34) and (35):

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t). \tag{34}$$

$$\dot{c}(t) = c(t)[\alpha k(t)^{\alpha-1} - \delta - \rho] \tag{35}$$

and notice that these two differential equations depend only on $k(t)$ and $c(t)$.

Equation (35) implies that $\dot{c}(t) = 0$ when

$$\alpha k(t)^{\alpha-1} - \delta - \rho = 0$$

or

$$k(t) = \left(\frac{\delta+\rho}{\alpha}\right)^{1/(\alpha-1)} = k^*$$

And since $\alpha - 1 < 0$, (35) also implies that $\dot{c}(t) < 0$ when $k(t) > k^*$ and $\dot{c}(t) > 0$ when $k(t) < k^*$.

Equation (34) implies that $\dot{k}(t) = 0$ when

$$k(t)^\alpha - \delta k(t) - c(t) = 0$$

or

$$c(t) = k(t)^\alpha - \delta k(t).$$

Moreover, (34) implies that $\dot{k}(t) < 0$ when

$$c(t) > k(t)^\alpha - \delta k(t)$$

and $\dot{k}(t) > 0$ when

$$c(t) < k(t)^\alpha - \delta k(t)$$

We can illustrate these conditions graphically using the phase diagram below, which reveals that:

The economy has a steady state at (k^*, c^*).

For each possible value of k_0, there exists a unique value of c_0 such that, starting from (k_0, c_0), the economy converges to the steady state (k^*, c^*).

Starting from all other values of c_0, either k becomes negative or c approaches zero.

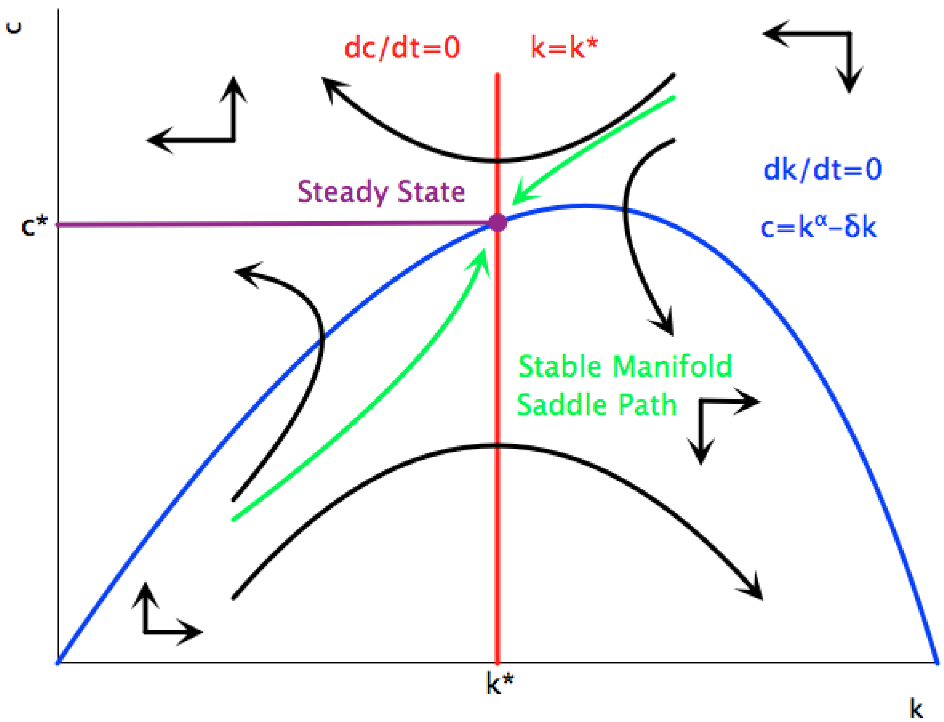
Trajectories that lead to negative values of k violate the nonnegativity condition for capital, and hence cannot represent a solution.

Trajectories that lead towards zero values of c violate the transversality condition

$$\lim_{T\to\infty} \pi(T)k(T) = \lim_{T\to\infty} \frac{e^{-\rho T}}{c(T)} k(T) = 0$$

and hence cannot represent a solution. (*Note:* for a proof of the necessity of this transversality condition for the optimal growth model, see Takashi Kamihigashi, "Necessity of Transversality Conditions for Infinite Horizon Problems," *Econometrica* vol. 69, July 2001, pp. 995-1012.)

Hence, the phase diagram allows us to identify the model's unique solution.



It is possible to see these same results numerically, using the discrete-time version of the model:

$$\max_{\{c_t\}_{t=0}^{\infty},\{k_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to k_0 given and

$$k_t^{\alpha} - \delta k_t - c_t \geq k_{t+1} - k_t \text{ for all } t = 0, 1, 2, \ldots.$$

Set up the maximized Hamiltonian:

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(k_t^{\alpha} - \delta k_t - c_t)$$

FOC for c_t:

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

Difference equations for π_t and k_t:

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}(\alpha k_t^{\alpha-1} - \delta)$$

$$k_{t+1} - k_t = H_{\pi}(k_t, \pi_{t+1}; t) = k_t^{\alpha} - \delta k_t - c_t \tag{36}$$

Just as in the continuous-time case, use the first-order condition to solve for π_{t+1} and π_t, and substitute the results into the difference equation for π_t to obtain

$$\frac{\beta^t(\alpha k_t^{\alpha-1} + 1 - \delta)}{c_t} = \frac{\beta^{t-1}}{c_{t-1}}$$

or, simplified and rolled forward one period:

$$c_{t+1} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)c_t \tag{37}$$

Equations (36) and (37) are the discrete-time analogs to (34) and (35). Once numerical values are assigned to the parameters α, δ, and β, (36) and (37) can be used to construct the entire sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$ starting from the given value for k_0 and a conjectured value for c_0. Numerical analysis will reveal that for each value of k_0, there is a unique value of c_0 such that, starting from (k_0, c_0), the economy converges to the steady state (k^*, c^*). Starting from all other values of c_0, the sequences violate either the nonnegativity condition for capital or the transversality condition

$$\lim_{T\to\infty} \pi_{T+1}k_{T+1} = \lim_{T\to\infty} \frac{\beta^T}{c_T}k_{T+1} = 0$$

and hence cannot represent a solution.

5 One Final Note on the Maximum Principle

In applying the maximum principle in discrete time, we defined the maximized Hamiltonian as

$$\begin{aligned} H(y_t, \pi_{t+1}; t) &= \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1}Q(y_t, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t) \\ &= \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1}Q(y_t, z_t; t) + \lambda_t[c - G(y_t, z_t; t)] \end{aligned} \tag{7}$$

and used this definition to derive the optimality conditions (10)-(12) and either (5) or (6), depending on whether the horizon is finite or infinite.

The Hamiltonian

$$\hat{H}(y_t, \pi_{t+1}, z_t; t) = \beta^t F(y_t, z_t; t) + \pi_{t+1}Q(y_t, z_t; t).$$

that enters into the definition of the maximized Hamiltonian in (7) is often called the *present-value* Hamiltonian, because $\beta^t F(y_t, z_t; t)$ measures the present value at time $t = 0$ of the payoff $F(y_t, z_t; t)$ received at time $t > 0$.

The present-value Hamilton stands in contrast to the *current-value* Hamiltonian, defined by multiplying both sides of (7) by β^{-t}:

$$\begin{aligned} \beta^{-t}H(y_t, \pi_{t+1}; t) &= \max_{z_t} F(y_t, z_t; t) + \beta^{-t}\pi_{t+1}Q(y_t, z_t; t) + \beta^{-t}\lambda_t[c - G(y_t, z_t; t)] \\ &= \max_{z_t} F(y_t, z_t; t) + \theta_{t+1}Q(y_t, z_t; t) + \mu_t[c - G(y_t, z_t; t)] \\ &= \tilde{H}(y_t, \theta_{t+1}; t), \end{aligned}$$

where the last line states the definition of the maximized current-value Hamiltonian $\tilde{H}(y_t, \theta_{t+1}; t)$, where

$$\theta_{t+1} = \beta^{-t}\pi_{t+1} \Rightarrow \pi_{t+1} = \beta^t\theta_{t+1}$$

and

$$\mu_t = \beta^{-t}\lambda_t \Rightarrow \lambda_t = \beta^t \mu_t,$$

and where the current-value Hamiltonian itself is

$$\hat{\tilde{H}}(y_t, \theta_{t+1}, z_t) = F(y_t, z_t; t) + \theta_{t+1} Q(y_t, z_t; t),$$

and therefore depends on the current value at t of the payoff $F(y_t, z_t; t)$ received at time t.

Let's consider rewriting the optimality conditions (10)-(12) and (5) in terms of the maximized current value Hamiltonian $\tilde{H}(y_t, \theta_{t+1}; t)$.

To do this, note first that by definition

$$H(y_t, \pi_{t+1}; t) = \beta^t \tilde{H}(y_t, \theta_{t+1}; t) = \beta^t \tilde{H}(y_t, \beta^{-t}\pi_{t+1}; t)$$

Hence

$$H_y(y_t, \pi_{t+1}; t) = \beta^t \tilde{H}_y(y_t, \theta_{t+1}; t)$$

and

$$\begin{aligned} H_\pi(y_t, \pi_{t+1}; t) &= \frac{\partial}{\partial \pi_{t+1}}[\beta^t \tilde{H}(y_t, \beta^{-t}\pi_{t+1}; t)] \\ &= \beta^t \beta^{-t} \tilde{H}_\theta(y_t, \beta^{-t}\pi_{t+1}; t) \\ &= \tilde{H}_\theta(y_t, \theta_{t+1}; t) \end{aligned}$$

In light of these results, (10) can be rewritten

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0 \tag{10}$$

$$F_z(y_t, z_t; t) + \beta^{-t}\pi_{t+1} Q_z(y_t, z_t; t) - \beta^{-t}\lambda_t G_z(y_t, z_t; t) = 0$$

$$F_z(y_t, z_t; t) + \theta_{t+1} Q_z(y_t, z_t; t) - \mu_t G_z(y_t, z_t; t) = 0 \tag{10$'$}$$

(11) can be rewritten

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) \tag{11}$$

$$\beta^t \theta_{t+1} - \beta^{t-1}\theta_t = -\beta^t \tilde{H}_y(y_t, \theta_{t+1}; t)$$

$$\theta_{t+1} - \beta^{-1}\theta_t = -\tilde{H}_y(y_t, \theta_{t+1}; t) \tag{11$'$}$$

(12) can be rewritten

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t) \tag{12}$$

$$y_{t+1} - y_t = \tilde{H}_\theta(y_t, \theta_{t+1}; t) \tag{12$'$}$$

(5) can be rewritten

$$\pi_{T+1}(y_{T+1} - y^*) = 0 \tag{5}$$

$$\beta^T \theta_{T+1}(y_{T+1} - y^*) = 0 \tag{5$'$}$$

(6) can be rewritten

$$\lim_{T\to\infty} \pi_{T+1}(y_{T+1} - y^*) = 0 \tag{6}$$

$$\lim_{T\to\infty} \beta^T \theta_{T+1}(y_{T+1} - y^*) = 0 \tag{6$'$}$$

Thus, when the maximum principle in discrete time is stated in terms of the current-value Hamiltonian instead of the present-value Hamiltonian, (10)-(12) and (5) or (6) are replaced by (10′)-(12′) and (5′) or (6′).

We can use the same types of transformations in the case of continuous time, where the maximized present-value Hamiltonian is defined by

$$\begin{aligned} H(y(t), \pi(t); t) &= \max_{z(t)} e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \\ &\quad \text{subject to } c \geq G(y(t), z(t); t) \\ &= \max_{z(t)} e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \\ &\quad + \lambda(t)[c - G(y(t), z(t); t)] \end{aligned} \tag{19}$$

Define the current-value Hamiltonian by multiplying both sides of (19) by $e^{\rho t}$:

$$\begin{aligned} e^{\rho t} H(y(t), \pi(t); t) &= \max_{z(t)} F(y(t), z(t); t) + e^{\rho t} \pi(t) Q(y(t), z(t); t) \\ &\quad + e^{\rho t} \lambda(t)[c - G(y(t), z(t); t)] \\ &= \max_{z(t)} F(y(t), z(t); t) + \theta(t) Q(y(t), z(t); t) \\ &\quad + \mu(t)[c - G(y(t), z(t); t)] \\ &= \tilde{H}(y(t), \theta(t); t) \end{aligned}$$

where the last line defines the maximized current-value Hamiltonian $\tilde{H}(y(t), \theta(t); t)$, where

$$\theta(t) = e^{\rho t} \pi(t) \Rightarrow \pi(t) = e^{-\rho t} \theta(t)$$

and

$$\mu(t) = e^{\rho t} \lambda(t) \Rightarrow \lambda(t) = e^{-\rho t} \mu(t),$$

and where the current-value Hamiltonian itself is

$$\hat{\tilde{H}}(y(t), \theta(t), z(t)) = F(y(t), z(t); t) + \theta(t) Q(y(t), z(t); t).$$

In the case of continuous time, the optimality conditions derived from (19) are (22)-(24) and either (17) or (18). Let's rewrite these conditions in terms of the maximized current-value Hamiltonian $\tilde{H}(y(t), \theta(t); t)$.

To begin, note that

$$H(y(t), \pi(t); t) = e^{-\rho t} \tilde{H}(y(t), \theta(t); t) = e^{-\rho t} \tilde{H}(y(t), e^{\rho t} \pi(t); t)$$

Hence

$$H_y(y(t),\pi(t);t) = e^{-\rho t}\tilde{H}_y(y(t),\theta(t);t)$$

and

$$\begin{aligned}
H_\pi(y(t),\pi(t);t) &= \frac{\partial}{\partial \pi(t)}[e^{-\rho t}\tilde{H}(y(t),e^{\rho t}\pi(t);t)] \\
&= e^{-\rho t}e^{\rho t}\tilde{H}_\theta(y(t),e^{\rho t}\pi(t);t) \\
&= \tilde{H}_\theta(y(t),\theta(t);t)
\end{aligned}$$

and, finally,

$$\dot{\pi}(t) = \frac{\partial}{\partial t}[e^{-\rho t}\theta(t)] = -\rho e^{-\rho t}\theta(t) + e^{-\rho t}\dot{\theta}(t)$$

In light of these results, (22) can be rewritten

$$e^{-\rho t}F_z(y(t),z(t);t) + \pi(t)Q_z(y(t),z(t);t) - \lambda(t)G_z(y(t),z(t);t) = 0 \tag{22}$$

$$F_z(y(t),z(t);t) + e^{\rho t}\pi(t)Q_z(y(t),z(t);t) - e^{\rho t}\lambda(t)G_z(y(t),z(t);t) = 0$$

$$F_z(y(t),z(t);t) + \theta(t)Q_z(y(t),z(t);t) - \mu(t)G_z(y(t),z(t);t) = 0 \tag{22$'$}$$

(23) can be rewritten

$$\dot{\pi}(t) = -H_y(y(t),\pi(t);t) \tag{23}$$

$$-\rho e^{-\rho t}\theta(t) + e^{-\rho t}\dot{\theta}(t) = -e^{-\rho t}\tilde{H}_y(y(t),\theta(t);t)$$

$$\dot{\theta}(t) = \rho\theta(t) - \tilde{H}_y(y(t),\theta(t);t) \tag{23$'$}$$

(24) can be rewritten

$$\dot{y}(t) = H_\pi(y(t),\pi(t);t) \tag{24}$$

$$\dot{y}(t) = \tilde{H}_\theta(y(t),\theta(t);t) \tag{24$'$}$$

(17) can be rewritten

$$\pi(T)[y(T) - y^*] = 0 \tag{17}$$

$$e^{-\rho T}\theta(T)[y(T) - y^*] = 0 \tag{17$'$}$$

(18) can be rewritten

$$\lim_{T\to\infty} \pi(T)[y(T) - y^*] = 0 \tag{18}$$

$$\lim_{T\to\infty} e^{-\rho T}\theta(T)[y(T) - y^*] = 0 \tag{18$'$}$$

Thus, when the maximum principle in continuous time is stated in terms of the current-value Hamiltonian instead of the present-value Hamiltonian, (22)-(24) and (17) or (18) are replaced by (22′)-(24′) and (17′) or (18′).

6 The Calculus of Variations

It is also possible to use the calculus of variations to characterize the solution to some continuous time dynamic optimization problems.

The standard calculus of variations problem takes the general form: choose the function $y(t)$ for $t \in [0, T]$ to maximize

$$F(y, \dot{y}, t) = \int_0^T f(y(t), \dot{y}(t), t)\, \mathrm{d}t,$$

and therefore allows the single-period objective function f to depend not just on the period-t value of $y(t)$ but also in its rate of change $\dot{y}(t)$ and on the time period t itself. The method does not allow any constraints to be imposed on $y(t)$, except for boundary conditions that specify values for $y(0)$ and $y(T)$. The calculus of variations is therefore less flexible than the maximum principle in the sense that there are problems that can be solved using the maximum principle but not using the calculus of variations.

Still, the Ramsey model is simple enough to be expressed as a calculus of variations problem, and the associated arguments give us yet another way of deriving the same pair of differential equations that we obtained via the method of Lagrange multipliers and the maximum principle.

The social planner's problem is to choose functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t,$$

subject to the constraints $k(0)$ given and

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. Since the logarithmic utility function is strictly increasing, we know that this capital accumulation constraint will always hold as an equality at the optimum. Thus, we can substitute it into the utility function to reduce the problem to one of choosing $k(t)$ for $t \in (0, \infty)$ to maximize

$$F(k, \dot{k}, t) = \int_0^\infty e^{-\rho t} \ln(k(t)^\alpha - \delta k(t) - \dot{k}(t))\, \mathrm{d}t,$$

which now has exactly the same form as the calculus of variations problem.

To characterize the solution starting from this formulation, we must once again use a variational argument. Suppose that $k^*(t)$ describes the optimal trajectory for the capital stock, and let

$$k(t) = k^*(t) + \varepsilon \eta(t),$$

be an admissible variation, where ε is any real number and $\eta(t)$ is a continuously differentiable function that satisfies $\eta(0) = 0$, so that $k(0)$ equals the same given value of $k^*(0)$. It follows from the definition of $k(t)$ that

$$\dot{k}(t) = \dot{k}^*(t) + \varepsilon \dot{\eta}(t).$$

Our variational argument says that, since $k^*(t)$ is optimal, it should not be possible, starting from $\varepsilon = 0$, to choose a small but nonzero value of ε and get a higher value for the objective function. That is, a necessary condition for $k^*(t)$ is

$$\left.\frac{dF(k^* + \varepsilon\eta, \dot{k}^* + \varepsilon\dot{\eta}, t)}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

To derive the more specific implications of this necessary condition, note that

$$F(k^* + \varepsilon\eta, \dot{k}^* + \varepsilon\dot{\eta}, t) = \int_0^\infty e^{-\rho t} \ln[(k^*(t) + \varepsilon\eta(t))^\alpha - \delta(k^*(t) + \varepsilon\eta(t)) - \dot{k}^*(t) - \varepsilon\dot{\eta}(t)]\, \mathrm{d}t$$

and therefore

$$\frac{dF(k^* + \varepsilon\eta, \dot{k}^* + \varepsilon\dot{\eta}, t)}{d\varepsilon}$$
$$= \int_0^\infty e^{-\rho t} \left[\frac{\alpha(k^*(t) + \varepsilon\eta(t))^{\alpha-1}\eta(t) - \delta\eta(t) - \dot{\eta}(t)}{(k^*(t) + \varepsilon\eta(t))^\alpha - \delta(k^*(t) + \varepsilon\eta(t)) - \dot{k}^*(t) - \varepsilon\dot{\eta}(t)}\right] \mathrm{d}t$$

and, in particular,

$$\left.\frac{dF(k^* + \varepsilon\eta, \dot{k}^* + \varepsilon\dot{\eta}, t)}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^\infty e^{-\rho t} \left[\frac{\alpha k^*(t)^{\alpha-1}\eta(t) - \delta\eta(t) - \dot{\eta}(t)}{k^*(t)^\alpha - \delta k^*(t) - \dot{k}^*(t)}\right] \mathrm{d}t$$

or, using the binding constraint to simplify the expression in the denominator,

$$\left.\frac{dF(k^* + \varepsilon\eta, \dot{k}^* + \varepsilon\dot{\eta}, t)}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^\infty e^{-\rho t} \left[\frac{\alpha k^*(t)^{\alpha-1}\eta(t) - \delta\eta(t) - \dot{\eta}(t)}{c^*(t)}\right] \mathrm{d}t.$$

To make further progress, we need to use another integration by parts argument to eliminate the term involving $\dot{\eta}(t)$ from the numerator of this last expression. Consider first that

$$\frac{d}{dt}\left[\frac{e^{-\rho t}\eta(t)}{c^*(t)}\right] = -\frac{\rho e^{-\rho t}\eta(t)}{c^*(t)} + \frac{e^{-\rho t}\dot{\eta}(t)}{c^*(t)} - \frac{e^{-\rho t}\eta(t)\dot{c}^*(t)}{[c^*(t)]^2}.$$

Integrating both sides from $t = 0$ to $t = \infty$ yields

$$\int_0^\infty \left\{\frac{d}{dt}\left[\frac{e^{-\rho t}\eta(t)}{c^*(t)}\right]\right\} \mathrm{d}t = -\int_0^\infty e^{-\rho t}\left[\frac{\rho + \dot{c}^*(t)/c^*(t)}{c^*(t)}\right]\eta(t)\, \mathrm{d}t + \int_0^\infty \left[\frac{e^{-\rho t}\dot{\eta}(t)}{c^*(t)}\right] \mathrm{d}t.$$

Evaluating the integral on the left-hand side yields

$$\int_0^\infty \left\{\frac{d}{dt}\left[\frac{e^{-\rho t}\eta(t)}{c^*(t)}\right]\right\} \mathrm{d}t = \lim_{T\to\infty} \frac{e^{-\rho T}\eta(T)}{c^*(T)} - \frac{\eta(0)}{c^*(0)}.$$

We've already observed that $\eta(0) = 0$, since the variation $k(t)$, like $k^*(t)$, must satisfy the initial condition $k(0)$ given. In addition, we already know that the transversality condition in the Ramsey model requires that

$$0 = \lim_{T\to\infty} \frac{e^{-\rho T}k(T)}{c^*(T)} = \lim_{T\to\infty} \frac{e^{-\rho T}[k^*(T) + \varepsilon\eta(T)]}{c^*(T)},$$

so that, evidently,

$$0 = \lim_{T\to\infty} \frac{e^{-\rho T}\eta(T)}{c^*(T)}$$

as well. Thus, any admissible variation must be such that

$$\int_0^\infty \left\{ \frac{d}{dt} \left[\frac{e^{-\rho t}\eta(t)}{c^*(t)} \right] \right\} \mathrm{d}t = \lim_{T\to\infty} \frac{e^{-\rho T}\eta(T)}{c^*(T)} - \frac{\eta(0)}{c^*(0)} = 0 - 0 = 0.$$

It therefore follows from our integration by parts argument that

$$\int_0^\infty \left[\frac{e^{-\rho t}\dot{\eta}(t)}{c^*(t)} \right] \mathrm{d}t = \int_0^\infty e^{-\rho t} \left[\frac{\rho + \dot{c}^*(t)/c^*(t)}{c^*(t)} \right] \eta(t)\,\mathrm{d}t.$$

Thus, optimality of $c^*(t)$ and $k^*(t)$ requires that these functions satisfy

$$\begin{aligned} 0 &= \left. \frac{dF(k^* + \varepsilon\eta, \dot{k}^* + \varepsilon\dot{\eta}, t)}{d\varepsilon} \right|_{\varepsilon=0} \\ &= \int_0^\infty e^{-\rho t} \left[\frac{\alpha k^*(t)^{\alpha-1}\eta(t) - \delta\eta(t) - \dot{\eta}(t)}{c^*(t)} \right] \mathrm{d}t \\ &= \int_0^\infty e^{-\rho t} \left[\frac{\alpha k^*(t)^{\alpha-1} - \delta - \rho - \dot{c}^*(t)/c^*(t)}{c^*(t)} \right] \eta(t)\,\mathrm{d}t, \end{aligned}$$

This optimality condition must hold for *any* continuously differentiable function $\eta(t)$. And since $c^*(t)$ must always be positive, the possibility of choosing $\eta(t)$ to always have the same sign as

$$\alpha k^*(t)^{\alpha-1} - \delta - \rho - \dot{c}^*(t)/c^*(t)$$

reveals that the optimality condition can only hold if

$$\dot{c}^*(t) = [\alpha k^*(t)^{\alpha-1} - \delta - \rho]c^*(t).$$

for all $t \in [0, \infty)$. And, as we already know, this differential equation, together with the binding constraint

$$\dot{k}^*(t) = k^*(t)^\alpha - \delta k^*(t) - c^*(t),$$

the initial condition $k^*(0)$ given, and the transversality condition

$$\lim_{T\to\infty} \frac{e^{-\rho T}k^*(T)}{c^*(T)} = 0$$

describe the solution to the Ramsey model.