Berge's Maximum Theorem

References:

Acemoglu, Appendix A.6

Stokey-Lucas-Prescott, Section 3.3

Ok, Sections E.1-E.3

Claude Berge, *Topological Spaces* (1963), Chapter 6

Berge's Maximum Theorem

So far, we've simply assumed that each of our constrained optimization problems has a solution that varies smoothly with the parameters.

Weierstrass' extreme value theorem says that a continuous function attains its maximum (and minimum) on a compact set.

Berge's maximum theorem imposes additional restrictions on the objective function and constraint set to guarantee that the problem's solution varies smoothly with the parameters.

The Problem

n choice variables:

$$x \in X \subseteq \mathbb{R}^n$$

m parameters:

$$\theta \in \Theta \subseteq \mathbb{R}^m$$

objective function:

$$F : X \times \Theta \to \mathbb{R}$$

set of feasible values for x given θ:

$$G : \Theta \to X \quad (G : \Theta \rightrightarrows X)$$

The problem:

$$\sup_{x \in G(\theta)} F(x, \theta)$$

The Problem

Because G is a multi-valued *correspondence*, not a function, we need to generalize more familiar notions of continuity.

Start by assuming that G is *compact valued*: for all $\theta \in \Theta$, $G(\theta) \subseteq X \subseteq \mathbb{R}^n$ is compact.

Since X is a subset of $\mathbb{R}^n$, this just means that $G(\theta)$ is closed and bounded (Heine-Borel theorem).

The Problem

Notions of continuity for (compact valued) correspondences can be expressed in terms of sets or sequences.

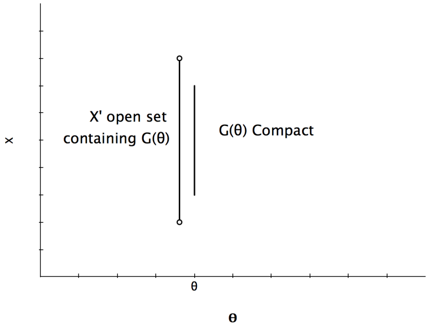
The set definitions will help us visualize what the definitions require.

The sequence definitions will help us complete our proofs.

Continuity Concepts for Correspondences

The correspondence G is *upper hemicontinuous* at $\theta \in \Theta$ if $G(\theta)$ is nonempty and if, for every open set $X' \subseteq X$ with $G(\theta) \subseteq X'$ (every open subset X' of X containing $G(\theta)$), there exists a $\delta > 0$ such that for every $\theta' \in N_\delta(\theta)$ (every θ' in some δ-neighborhood of θ), $G(\theta') \subseteq X'$ ($G(\theta')$ is contained in X' as well).

Continuity Concepts for Correspondences



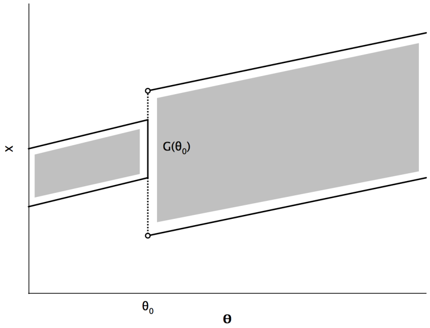
Take any θ' near θ. $G(\theta')$ must also be in X'. G can't suddenly "explode."

Continuity Concepts for Correspondences

The *compact valued* correspondence G is *upper hemicontinuous* at $\theta \in \Theta$ if $G(\theta)$ is nonempty and if, for every sequence $\{\theta_j\}$ with $\theta_j \to \theta$ and every sequence $\{x_j\}$ with $x_j \in G(\theta_j)$ for all j, there exists a convergent subsequence $\{x_{j_k}\}$ such that $x_{j_k} \to x \in G(\theta)$.

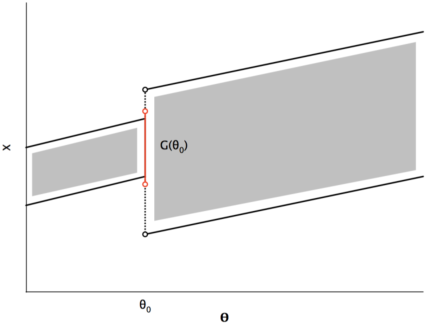
If G is not compact valued, then these conditions are sufficient, but not necessary, for G to be upper hemicontinuous.

Continuity Concepts for Correspondences



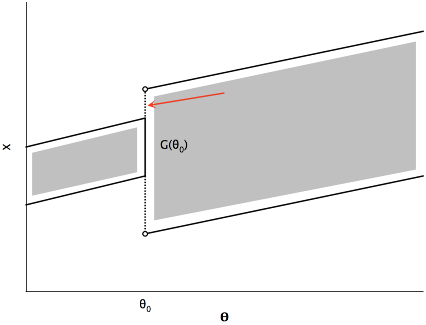
G is not upper hemicontinuous at θ_0.

Continuity Concepts for Correspondences



G is not upper hemicontinuous at θ_0. The red open set contains $G(\theta_0)$ but not $G(\theta')$ immediately to the right of θ_0.

Continuity Concepts for Correspondences

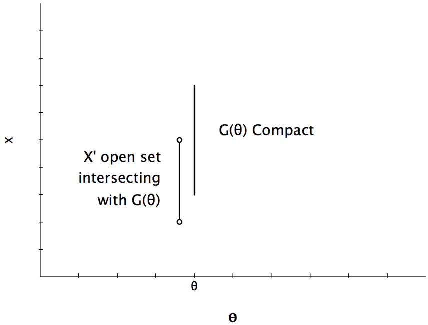


G is not upper hemicontinuous at θ_0. There is no convergent subsequence along the red arrow that converges to a point $x_0 \in G(\theta_0)$.

Continuity Concepts for Correspondences

The correspondence G is *lower hemicontinuous* at $\theta \in \Theta$ if $G(\theta)$ is nonempty and if, for every open set $X' \subseteq X$ with $G(\theta) \cap X' \neq \emptyset$ (every open subset X' of X intersecting with $G(\theta)$), there exists a $\delta > 0$ such that for every $\theta' \in N_\delta(\theta)$, $G(\theta') \cap X' \neq \emptyset$ (some δ-neighborhood of θ such that $G(\theta')$ also intersects with X' for every θ' in that neighborhood).

Continuity Concepts for Correspondences



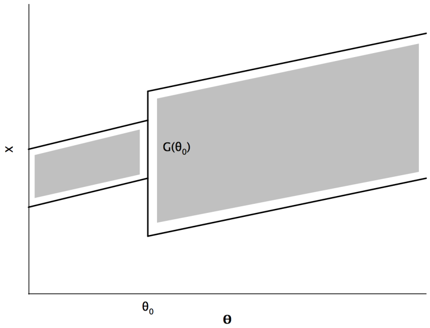
Take any θ' near θ. $G(\theta')$ must also intersect with X'. G can't suddenly "implode."

Continuity Concepts for Correspondences

The correspondence G is *lower hemicontinuous* at $\theta \in \Theta$ if $G(\theta)$ is nonempty and if, for every $x \in G(\theta)$ and every sequence $\{\theta_j\}$ such that $\theta_j \to \theta$, there is a $J \geq 1$ and a sequence $\{x_j\}$ such that $x_j \in G(\theta_j)$ for all $j \geq J$ and $x_j \to x$.

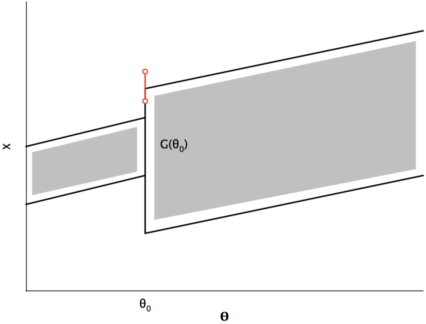
The correspondence G is *continuous* at $\theta \in \Theta$ if it is both upper and lower hemicontinuous at θ. The correspondence G is *continuous* if it is continuous at every $\theta \in \Theta$.

Continuity Concepts for Correspondences



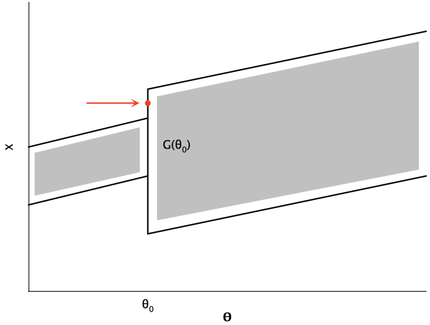
G is not lower hemicontinuous at θ_0.

Continuity Concepts for Correspondences



G is not lower hemicontinuous at θ_0. The red open set intersects with $G(\theta_0)$ but not with $G(\theta')$ immediately to the left of θ_0.

Continuity Concepts for Correspondences



G is not lower hemicontinuous at θ_0. There is no sequence with $x_j \in G(\theta_j)$ converging from the left to the red point $x \in G(\theta_0)$.

Example: Perfect Substitutes

$$\max_{c_1, c_2} U(c_1 + c_2) \text{ subject to } 1 \geq c_1 + pc_2,\ c_1 \geq 0,\ c_2 \geq 0,$$

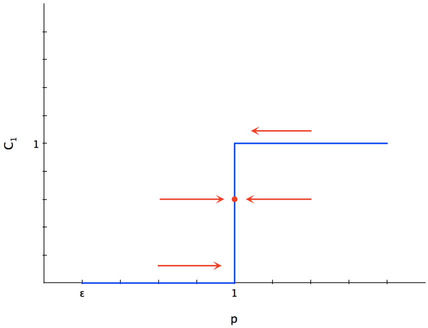
where $U' > 0$ and $p \geq \varepsilon > 0$.

$$c_1^* = \begin{cases} 1 & \text{for} \quad p > 1 \\ [0, 1] & \text{for} \quad p = 1 \\ 0 & \text{for} \quad 1 > p \geq \varepsilon \end{cases}$$

$$c_2^* = \begin{cases} 0 & \text{for} \quad p > 1 \\ [0, 1] & \text{for} \quad p = 1 \\ 1/p & \text{for} \quad 1 > p \geq \varepsilon \end{cases}$$

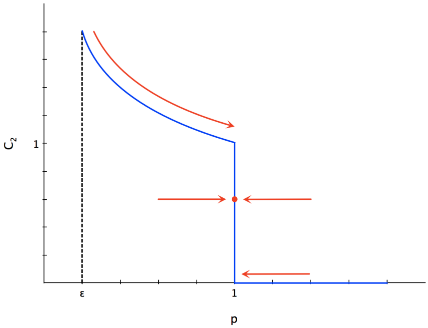
Both demand correspondences are upper but not lower hemicontinuous.

Example: Perfect Substitutes



c_1^* is upper but not lower hemicontinuous.

Example: Perfect Substitutes



c_2^* is upper but not lower hemicontinuous.

Continuity Concepts for Correspondences

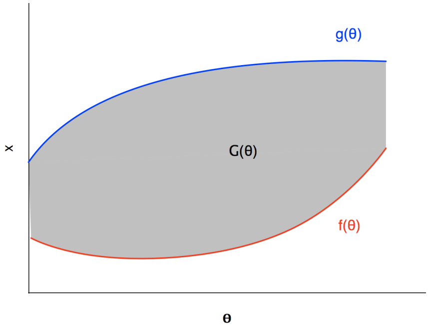
To build up more intuition, let's construct a correspondence that is both upper and lower hemicontinuous.

Let $f : \Theta \to \mathbb{R}$ and $g : \Theta \to \mathbb{R}$ be continuous functions satisfying $f(\theta) \leq g(\theta)$ for all $\theta \in \Theta$.

$$G(\theta) = \{x \in \mathbb{R} | f(\theta) \leq x \leq g(\theta)\}$$

is nonempty for all $\theta \in \Theta$. Let's show that it meets the other requirements for upper and lower hemicontinuity.

Continuity Concepts for Correspondences



$G(\theta)$ is both upper and lower hemicontinuous. Hence $G(\theta)$ is continuous.

Upper Hemicontinuity

Fix $\theta \in \Theta$ and let $\{\theta_j\}$ be a sequence with $\theta_j \to \theta$.

Since G is nonempty, we can find a sequence $\{x_j\}$ with $x_j \in G(\theta_j)$ for all j.

Does this sequence have a convergent subsequence with limit $x \in G(\theta)$?

Upper Hemicontinuity

Yes!

Since $\theta_j \to \theta$, there exists a closed and bounded set $\hat{\Theta} \in \Theta \in \mathbb{R}$ such that, for some $J \geq 1$, all θ_j with $j \geq J$ and θ will be contained in $\hat{\Theta}$.

Moreover, since f and g are continuous, once θ_j, $j \geq J$ are contained in the closed and bounded set $\hat{\Theta}$, all of the x_j, $j \geq J$, will also lie in some closed and bounded subset of $\mathbb{R}$.

Upper Hemicontinuity

By the Bolzano-Weierstrass theorem, the sequence $\{\theta_j, x_j\}$ in $\mathbb{R}^2$ has a convergent subsequence with limit point (θ, x).

And since each element of this subsequence satisfies

$$f(\theta_{j_k}) \leq x_{j_k} \leq g(\theta_{j_k})$$

and f and g are continuous, the limit point x must satisfy $x \in G(\theta)$ as well.

Lower Hemicontinuity

Fix $\theta \in \Theta$ and $x \in G(\theta)$. Let $\{\theta_j\}$ be a sequence with $\theta_j \to \theta$.

Is there a sequence $\{x_j\}$ with $x_j \in G(\theta_j)$ for all j and $x_j \to x$?

Yes! If $f(\theta) = g(\theta)$, then just take $x_j = g(\theta_j)$ for all j. Clearly, $x_j \in G(\theta_j)$ for all j and, since g is continuous, $x_j \to g(\theta) = x$.

Lower Hemicontinuity

If, on the other hand, $f(\theta) < g(\theta)$, note that

$$\frac{x - f(\theta)}{g(\theta) - f(\theta)}$$

is well-defined and, since $x \in G(\theta)$, lies between zero and one.

Lower Hemicontinuity

Therefore, the sequence $\{x_j\}$ with

$$x_j = \left[1 - \frac{x - f(\theta)}{g(\theta) - f(\theta)}\right] f(\theta_j) + \left[\frac{x - f(\theta)}{g(\theta) - f(\theta)}\right] g(\theta_j)$$

satisfies $x_j \in G(\theta_j)$ and, since f and g are continuous,

$$\begin{aligned} x_j &\to \left[1 - \frac{x - f(\theta)}{g(\theta) - f(\theta)}\right] f(\theta) + \left[\frac{x - f(\theta)}{g(\theta) - f(\theta)}\right] g(\theta) \\ &= f(\theta) + x - f(\theta) = x. \end{aligned}$$

Continuity Concepts for Correspondences

In the special case where $f(\theta) = g(\theta)$ for all θ, so that $G(\theta)$ is single-valued, then the assumption that g is a continuous function implies that G is continuous correspondence.

In fact the converse is also true. Assume that $G(\theta)$ is single-valued, with $G(\theta) = g(\theta)$ for some function g. If G is either an upper or lower hemicontinuous correspondence, then g is a continuous function. See the notes for a proof.

Berge's Maximum Theorem

Let $X \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^m$. Let $F : X \times \Theta \to \mathbb{R}$ be a continuous function, and let $G : \Theta \to X$ be a compact valued and continuous correspondence. Then the maximum value function

$$V(\theta) = \max_{x \in G(\theta)} F(x, \theta)$$

is well-defined and continuous, and the optimal policy correspondence

$$x^*(\theta) = \{x \in G(\theta) \mid F(x, \theta) = V(\theta)\}$$

is nonempty, compact valued, and upper hemicontinuous.

Berge's Maximum Theorem

To prove the theorem, fix $\theta \in \Theta$.

Note first that since $G(\theta)$ is nonempty and compact, and since $F(\cdot, \theta)$ is continuous, Weierstrass' extreme value theorem implies that $V(\theta)$ is well-defined and that $x^*(\theta)$ is nonempty.

Next, we will show that $x^*(\theta)$ is compact valued.

Berge's Maximum Theorem

Since $x^*(\theta) \subseteq G(\theta)$ and $G(\theta)$ is compact, it follows that $x^*(\theta)$ is bounded.

Let $\{x_j\}$ be a sequence with $x_j \in x^*(\theta)$ for all j and $x_j \to x$. Since $G(\theta)$ is closed, it must be that $x \in G(\theta)$. And since $V(\theta) = F(x_j, \theta)$ for all j and F is continuous, it follows that $F(x, \theta) = V(\theta)$. Hence, $x \in x^*(\theta)$, so $x^*(\theta)$ is closed.

Therefore, by the Heine-Borel theorem, $x^*(\theta)$ is compact valued.

Berge's Maximum Theorem

Now we will show that $x^*(\theta)$ is upper hemicontinuous.

Fix $\theta \in \Theta$ and let $\{\theta_j\}$ be any sequence with $\theta_j \to \theta$. Then let $\{x_j\}$ be a sequence with $x_j \in x^*(\theta_j)$ for all j.

Is there a convergent subsequence with $x_{j_k} \to x \in x^*(\theta)$?

Berge's Maximum Theorem

Yes! Since G is upper hemicontinuous, there exists a subsequence $\{x_{j_k}\}$ converging to $x \in G(\theta)$.

Now, let $x' \in G(\theta)$ as well. Since G is lower hemicontinuous, there exists a sequence $\{x'_{j_k}\}$ with $x'_{j_k} \in G(\theta_{j_k})$ and $x'_{j_k} \to x'$. And since $F(x_{j_k}, \theta_{j_k}) \geq F(x'_{j_k}, \theta_{j_k})$ for all k and F is continuous, it follows that $F(x, \theta) \geq F(x', \theta)$. Moreover, this condition holds for *any* $x' \in G(\theta)$.

It follows that $x \in x^*(\theta)$, so that x^* is upper hemicontinuous.

Berge's Maximum Theorem

Finally, we will show that $V(\theta)$ is continuous.

Fix $\theta \in \Theta$, and let $\{\theta_j\}$ be any sequence with $\theta_j \to \theta$. Choose $x_j \in x^*(\theta_j)$ for all j and let

$$\bar{v} = \limsup_j V(\theta_j) = \lim_{j\to\infty} \left[\sup_{k \geq j} V(\theta_k) \right]$$

and

$$\underline{v} = \liminf_j V(\theta_j) = \lim_{j\to\infty} \left[\inf_{k \geq j} V(\theta_k) \right]$$

Berge's Maximum Theorem

Then there exists a subsequence $\{x_{j_k}\}$ such that

$$\bar{v} = \lim_k F(x_{j_k}, \theta_{j_k}).$$

But since x^* is upper hemicontinuous, these exists a subsequence of $\{x_{j_{k_l}}\}$ of $\{x_{j_k}\}$ converging to $x \in x^*(\theta)$

Hence

$$\bar{v} = \lim_l F(x_{j_{k_l}}, \theta_{j_{k_l}}) = F(x, \theta) = V(\theta).$$

Berge's Maximum Theorem

There also exists a subsequence $\{x_{j_k}\}$ such that

$$\underline{v} = \lim_k F(x_{j_k}, \theta_{j_k}).$$

But since x^* is upper hemicontinuous, these exists a subsequence of $\{x_{j_{k_l}}\}$ of $\{x_{j_k}\}$ converging to $x \in x^*(\theta)$

Hence

$$\underline{v} = \lim_l F(x_{j_{k_l}}, \theta_{j_{k_l}}) = F(x, \theta) = V(\theta).$$

Therefore, $V(\theta_j) \to V(\theta)$, completing the proof.

Example: Utility Maximization

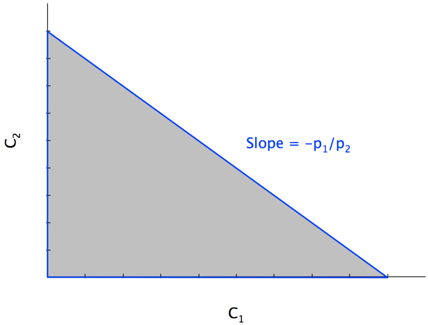
$$\max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2, c_1 \geq 0, c_2 \geq 0.$$

To apply Berge's theorem, we need U to be continuous and

$$G(I, p_1, p_2) = \{(c_1, c_2) \in \mathbb{R}^2 \mid I \geq p_1 c_1 + p_2 c_2, c_1 \geq 0, c_2 \geq 0\}$$

to be compact valued and continuous.

Example: Utility Maximization



Clearly, for any $I > 0$, $p_1 > 0$, and $p_2 > 0$, G is nonempty and compact valued. We only need to show that G satisfies the remaining requirements for upper and lower hemicontinuity.

Upper Hemicontinuity

Fix $(I, p_1, p_2) \in \mathbb{R}^3_{++}$, and let $\{I_j, p_{1j}, p_{2j}\}$ be a sequence in $\mathbb{R}^3_{++}$ with $(I_j, p_{1j}, p_{2j}) \to (I, p_1, p_2)$.

Since G is non-empty, there is a sequence $\{c_{1j}, c_{2j}\}$ with $(c_{1j}, c_{2j}) \in G(I_j, p_{1j}, p_{2j})$ for all j.

Is there a convergent subsequence $\{c_{1j_k}, c_{2j_k}\}$ with limit point $(c_1, c_2) \in G(I, p_1, p_2)$?

Upper Hemicontinuity

Yes! Since $(I_j, p_{1j}, p_{2j}) \to (I, p_1, p_2)$, there is a closed and bounded set $\hat{\Theta} \subseteq \mathbb{R}^3_{++} \subseteq \mathbb{R}^3$, such that, for some $J \geq 1$, all of the (I_j, p_{1j}, p_{2j}), $j \geq J$, and (I, p_1, p_2) are contained in $\hat{\Theta}$.

Moreover, the structure of G in this case implies that all of the $(c_{1j}, c_{2j}) \in G(I_j, p_{1j}, p_{2j})$ for $j \geq J$ will lie in a closed and bounded subset of $\mathbb{R}^2$.

Upper Hemicontinuity

Thus, for all $j \geq J$, all elements of the sequence $\{I_j, p_{1j}, p_{2j}, c_{1j}, c_{2j}\}$ lie in a closed and bounded subset of $\mathbb{R}^5$.

By the Bolzano-Weierstrass theorem, this sequence has a convergent subsequence $\{I_{j_k}, p_{1j_k}, p_{2j_k}, c_{1j_k}, c_{2k_k}\}$ with limit point (I, p_1, p_2, c_1, c_2). And since each element of this convergent subsequence satisfies

$$I_{j_k} \geq p_{1j_k} c_{1j_k} + p_{2j_k} c_{2j_k},\ c_{1j_k} \geq 0,\ \text{and}\ c_{2j_k} \geq 0,$$

it is easy to see that the limit point will also have to satisfy $(c_1, c_2) \in G(I, p_1, p_2)$.

Lower Hemicontinuity

Fix $(I, p_1, p_2) \in \mathbb{R}^3_{++}$ and $(c_1, c_2) \in G(I, p_1, p_2)$.

Then let $\{I_j, p_{1j}, p_{2j}\}$ be a sequence in $\mathbb{R}^3_{++}$ with $(I_j, p_{1j}, p_{2j}) \to (I, p_1, p_2)$.

Is there a sequence $\{c_{1j}, c_{2j}\}$ with $(c_{1j}, c_{2j}) \in G(I_j, p_{1j}, p_{2j})$ for all j and $(c_{1j}, c_{2j}) \to (c_1, c_2)$?

Yes! If $c_1 = c_2 = 0$, then $\{c_{1j}, c_{2j}\} = \{0, 0\}$ works.

Lower Hemicontinuity

So suppose that $c_1 > 0$ and/or $c_2 > 0$. Construct $\{c_{1j}, c_{2j}\}$ with

$$c_{1j} = \left(\frac{I_j}{I}\right)\left(\frac{p_1 c_1 + p_2 c_2}{p_{1j} c_1 + p_{2j} c_2}\right) c_1$$

and

$$c_{2j} = \left(\frac{I_j}{I}\right)\left(\frac{p_1 c_1 + p_2 c_2}{p_{1j} c_1 + p_{2j} c_2}\right) c_2.$$

Clearly, $c_{1j} \geq 0$ and $c_{2j} \geq 0$ for all j.

Lower Hemicontinuity

In addition

$$\begin{aligned} p_{1j}c_{1j} + p_{2j}c_{2j} &= \left(\frac{I_j}{I}\right)\left(\frac{p_1c_1 + p_2c_2}{p_{1j}c_1 + p_{2j}c_2}\right)(p_{1j}c_1 + p_{2j}c_2) \\ &= I_j\left(\frac{p_1c_1 + p_2c_2}{I}\right) \leq I_j, \end{aligned}$$

so that $(c_{1j}, c_{2j}) \in G(I_j, p_{1j}, p_{2j})$ for all j.

Moreover, $(c_{1j}, c_{2j}) \to (c_1, c_2)$, completing the proof.

Example: Utility Maximization

For the problem

$$\max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2, c_1 \geq 0, c_2 \geq 0,$$

we now know that if U is continuous:

1. $V(I, p_1, p_2)$ is well-defined and continuous.
2. $c_1^*(I, p_1, p_2)$ and $c_2^*(I, p_1, p_2)$ are nonempty, compact valued, and upper hemicontinuous correspondences.

And if we assume as well that U is strictly concave, the Marshallian (Walrasian) demand functions $c_1^*(I, p_1, p_2)$ and $c_2^*(I, p_1, p_2)$ are single-valued and continuous.