Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2026

Due Thursday, November 5

This exam has five questions on five pages. Please check to make sure that your copy has all five questions and all five pages. The questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Subsistence Consumption

Consider a utility-maximizing consumer, who solves

$$\max_{c_1,c_2} \ \alpha \ln(c_1 - \gamma) + (1 - \alpha) \ln(c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2,$$

where c_1 and c_2 denote consumption of two goods, I is the consumer's income, and p_1 and p_2 are the prices at which the consumer purchases the two goods in perfectly competitive markets. This problem is similar to the ones you studied when working on problem set 1, but introduces a novel element through the parameter $\gamma \geq 0$, which can be thought of as a subsistence level below which consumption of good one cannot fall. The logarithmic utility function implies that the subsistence constraint $c_1 \geq \gamma$ and the non-negativity constraint $c_2 \geq 0$ do not have to be imposed directly when solving the problem. You can also assume that the parameter restriction $I \geq p_1 \gamma$ holds, so that the problem has a well-defined solution.

a. Define (set up) the Lagrangian for the consumer's problem and write down the first-order conditions that help characterize the consumer's optimal choices c_1^* and c_2^*.

b. Use the first-order conditions you derived in part (a), above, together with the budget constraint, which will bind at the optimum, to find the solutions for c_1^* and c_2^* in terms of income I, prices p_1 and p_2, and the preference parameters α and γ.

c. Use your solutions from part (b), above, to verify that both goods are normal (that is, both c_1^* and c_2^* increase when income I goes up) and ordinary (that is, c_1^* falls when p_1 rises and c_2^* falls when p_2 rises).

2. Leisure Demand and Labor Supply

Now consider a variant of problem 1 that reinterprets consumption good 2 as leisure and introduces a labor supply choice. Letting T denote the consumer's time endowment and ℓ denote leisure, hours worked are $T - \ell$ and total labor income is $w(T - \ell)$, where w is the wage rate. The consumer now solves

$$\max_{c_1,\ell} \ \alpha \ln(c_1 - \gamma) + (1 - \alpha) \ln(\ell) \text{ subject to } I + w(T - \ell) \geq p_1 c_1,$$

where I now has the specific interpretation of non-labor income. Rearranging the budget constraint to read

$$I + wT \geq p_1 c_1 + w\ell$$

makes clear that the wage w also represents the price of leisure. Assume that the parameters satisfy

$$I + wT > p_1\gamma > I - \left(\frac{\alpha}{1-\alpha}\right) wT,$$

as these restrictions will ensure that there is a an interior solution with $c_1^* > \gamma$ and $0 < \ell^* < T$. so that you don't have to incorporate these constraints into the Lagrangian when solving this problem.

a. Define (set up) the Lagrangian for the consumer's problem and write down the first-order conditions that help characterize the consumer's optimal choices c_1^* and ℓ^*.

b. Use the first-order conditions you derived in part (a), above, together with the budget constraint, which will bind at the optimum, to find the solutions for c_1^* and ℓ^* in terms of non-labor income I, the price p_1 of good 1 and the wage rate w, the time endowment T, and the preference parameters α and γ.

c. Use your solutions from part (b), above, to show that consumption c_1^* and leisure ℓ^* both increase when non-labor income I goes up. Use your results to show, as well, that while consumption c_1^* falls when p_1 rises, leisure ℓ^* may rise or fall when the wage rate w rises, depending on whether or not labor income I is sufficient to cover the cost $p_1\gamma$ of the subsistence level of good 1.

3. A Generalized Model with Endowments

That ℓ^* might rise as the wage w increases in problem 2 raises the interesting question: is leisure a Giffen good? If so, this example would seem to be particularly unusual, since a Giffen good is necessarily an inferior good and, in problem 1, the analogous good 2 is a normal good. In fact, the mechanism that allows ℓ^* to rise when w increases is quite different from the mechanism that leads to a Giffen good. An increase in the price of a Giffen good produces a negative wealth effect that, since the good is inferior, leads to an increase in consumption that completely offsets the usual substitution effect. In problem 2, by contrast, an increase in the wage increases the value of the consumer's time endowment wT that, since

leisure is normal, leads to an increase in ℓ^* that offsets the usual substitution effect. There is nothing special about leisure: what generates the unexpected response of consumption of leisure to its price, the wage rate, is the fact that the consumer has an endowment of the good, time, that he or she supplies to the market as well as consuming.

To see this, consider another modification of problem 1 in which the consumer is endowed with y_1 units of good 1 and y_2 units of good 2 as well as with income I. Now the consumer solves

$$\max_{c_1,c_2} \ \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(c_2) \text{ subject to } I + p_1 y_1 + p_2 y_2 \geq p_1 c_1 + p_2 c_2.$$

As in question 1, the parameter restriction

$$I + p_1 y_1 + p_2 y_2 > p_1 \gamma$$

ensures that the problem has a well-defined solution.

a. Define (set up) the Lagrangian for the consumer's problem and write down the first-order conditions that help characterize the consumer's optimal choices c_1^* and c_2^*.

b. Use the first-order conditions you derived in part (a), above, together with the budget constraint, which will bind at the optimum, to find the solutions for c_1^* and c_2^* in terms of income I, the endowments y_1 and y_2, prices p_1 and p_2, and the preference parameters α and γ.

c. Use your solutions from part (b), above, to show that while c_1^* always falls when p_1 rises, c_2^* may either rise or fall when p_2 rises.

4. Hicksian Demands

Now let's return to the simpler model originally considered in question 1, but solve instead the problem of minimizing the expenditure needed to achieve utility level $\bar{U}$:

$$\min_{c_1,c_2} \ p_1 c_1 + p_2 c_2 \text{ subject to } \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(c_2) \geq \bar{U},$$

a. Define (set up) the Lagrangian for the consumer's problem and write down the first-order conditions that help characterize the consumer's optimal choices c_1^* and c_2^*.

b. Use the first-order conditions you derived in part (a), above, together with the constraint, which will bind at the optimum, to find the solutions for c_1^* and c_2^* in terms of the required utility level $\bar{U}$, prices p_1 and p_2, and the preference parameters α and γ.

c. Use your solutions from part (b), above, to verify that these Hicksian demand curves slope downward: c_1^* decreases when p_1 rises and c_2^* decreases when p_2 rises.

5. A Modified Slutsky Equation

The distinction between the case of a Giffen good and the case where a wealth effect stemming from the revaluation of an endowment when the good's price changes can be seen more generally with the help of a modified Slutsky equation. To derive this equation, it will be easier to assume that the consumer's preferences over goods 1 and 2 are described by a general utility function $U(c_1, c_2)$ instead of the specific utility function from questions 1, 3, and 4. The result does not depend, in particular, on whether there is a subsistence point for good 1.

Consider first the expenditure minimization problem

$$\min_{c_1,c_2} \; p_1c_1 + p_2c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

Let

$$c_1^* = h_1(p_1, p_2, \bar{U})$$

and

$$c_2^* = h_2(p_1, p_2, \bar{U})$$

describe the solution to this problem. The functions h_1 and h_2 are, as in question 4, the Hicksian demand curves for the two goods. If we define the minimum expenditure function

$$e(p_1, p_2, \bar{U}) = \min_{c_1,c_2} \; p_1c_1 + p_2c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U},$$

then the envelope theorem will imply that

$$\frac{\partial e(p_1, p_2, \bar{U})}{\partial p_1} = h_1(p_1, p_2, \bar{U})$$

and

$$\frac{\partial e(p_1, p_2, \bar{U})}{\partial p_2} = h_2(p_1, p_2, \bar{U}).$$

Next, consider the utility maximization problem

$$\max_{c_1,c_2} \; U(c_1, c_2) \text{ subject to } I + p_1y_1 + p_2y_2 \geq p_1c_1 + p_2c_2,$$

identical to the one from question 3 but with the more general utility function. Let

$$c_1^* = c_1(p_1, p_2, I, y_1, y_2)$$

and

$$c_2^* = c_2(p_1, p_2, I, y_1, y_2)$$

describe the solution to this problem, where the functions c_1 and c_2 describe how the solution depends on prices p_1 and p_2, income I, and the endowments y_1 and y_2.

In the case where parameters are such that

$$e(p_1, p_2, \bar{U}) = I + p_1 y_1 + p_2 y_2$$

or, equivalently,

$$I = e(p_1, p_2, \bar{U}) - p_1 y_1 - p_2 y_2,$$

the solutions to the expenditure minimization and utility maximization problems will coincide. This coincidence can be expressed via the identities

$$c_1(p_1, p_2, e(p_1, p_2, \bar{U}) - p_1 y_1 - p_2 y_2, y_1, y_2) = h_1(p_1, p_2, \bar{U})$$

and

$$c_2(p_1, p_2, e(p_1, p_2, \bar{U}) - p_1 y_1 - p_2 y_2, y_1, y_2) = h_2(p_1, p_2, \bar{U})$$

which must hold for all possible combinations of p_1, p_2, $\bar{U}$, y_1, and y_2.

Since the two goods enter symmetrically into this generalized problem, consider the second of these two identities:

$$c_2(p_1, p_2, e(p_1, p_2, \bar{U}) - p_1 y_1 - p_2 y_2, y_1, y_2) = h_2(p_1, p_2, \bar{U}).$$

Differentiate both sides of this equation by p_2 to derive the modified Slutsky equation

$$\frac{\partial c_2}{\partial p_2} = \frac{\partial h_2}{\partial p_2} - \frac{\partial c_2}{\partial I}(c_2^* - y_2),$$

where

$$c_2^* = h_2(p_1, p_2, \bar{U}) = c_2(p_1, p_2, e(p_1, p_2, \bar{U}) - p_1 y_1 - p_2 y_2, y_1, y_2).$$

In the case without endowments, the function c_2 is the Marshallian demand curve and this last result corresponds to the Slutsky equation. In that case, the equation makes clear that a Giffen good must be inferior, with $\partial c_2/\partial I < 0$, to offset the substitution effect implied by the Hicksian demand curve and produce an increase in c_2^* when p_2 rises.

In the case with endowments, the function c_2 is not really a Marshallian demand curve, since the consumer's total income depends on the value of his or her endowment, which depends on prices. In this case, c_2^* can increase when p_2 rises even though it is not a Giffen good. In fact, this can happen even when $\partial c_2/\partial I > 0$, provided the consumer is a net supplier of good 2, setting $c_2^* < y_2$.