Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2024

Due Thursday, November 7

This exam has three questions on five pages. Please check to make sure that your copy has all three questions and all five pages. Each question will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Profit Maximization

Consider a firm that hires n workers to produce y units of output according to the production function

$$y = f(n) = 10n - n^2.$$

Normalize the price of output to equal one, and let $w > 0$ denote the real wage. Then the profit-maximizing firm solves

$$\max_n 10n - n^2 - wn \text{ subject to } n \geq 0.$$

There are a variety of ways to find the solution to this problem; use whichever you find most convenient to answer the questions below.

a. Find the value of $\bar{w}$ such that, for all $w \geq \bar{w}$, the firm hires no workers, choosing $n^* = 0$.

b. Assuming instead that $0 < w < \bar{w}$, find the firm's labor demand curve, that is, the function $n(w)$ linking the firm's optimal choice of $n^* \geq 0$ of n to the wage rate w.

c. For any value of $w > 0$, define the firm's profit function π as its maximized profit for that value of w, that is:

$$\pi(w) = \max_n 10n - n^2 - wn \text{ subject to } n \geq 0.$$

Using your results from parts (a) and (b), above, find the expressions that show how $\pi(w)$ depends on w for the two cases, where $w \geq \bar{w}$ and $0 < w < \bar{w}$.

d. Finally, find an expression for $\pi'(w)$, the derivative of the profit function with respect to the wage, assuming that $0 < w < \bar{w}$.

2. Optimal Risk Sharing

Consider an economy consisting of a finite number of consumers, indexed by $j = 1, 2, \dots, J$. There are two periods: a planning period ($t = 0$) and a production and consumption period ($t = 1$). Looking ahead from the planning period $t = 0$, there are a finite number of possible states of the world in the production-consumption period $t = 1$, indexed by $s = 1, 2, \dots, S$; each occurs with probability $\pi(s)$, with $0 < \pi(s) < 1$ for all $s = 1, 2, \dots, S$ and, of course,

$$\sum_{s=1}^{S} \pi(s) = 1.$$

Consumers have identical preferences, as described by the expected utility function

$$\sum_{s=1}^{S} \pi(s) u[c_j(s)],$$

with

$$u[c_j(s)] = -\frac{1}{\sigma} \exp[-\sigma c_j(s)]$$

and where $\sigma > 0$ and $c_j(s)$ is consumer j's consumption in state s at $t = 1$. This utility function implies that the coefficient of absolute risk aversion $-u''(c)/u'(c)$ is constant and equal to σ.

In each state $s = 1, 2, \dots, S$ during period $t = 1$, consumer j supplies one unit of labor inelastically to produce $y_j(s)$ units of output. Hence, looking ahead from $t = 0$ the fundamental source of uncertainty is over each worker's productivity at $t = 1$. In the analysis to follow, it will be helpful to let

$$y_a(s) = \frac{1}{J} \sum_{j=1}^{J} y_j(s)$$

denote average output per worker in each state $s = 1, 2, \dots, S$ at $t = 1$.

Suppose now that at $t = 0$, a social planner takes outputs $y_j(s)$, $j = 1, 2, \dots, J$ and $s = 1, 2, \dots, S$ as given, and chooses consumptions $c_j(s)$, $j = 1, 2, \dots, J$ and $s = 1, 2, \dots, S$, for all consumers in all states of the world to maximize an equally-weighted sum of their utilities

$$\sum_{j=1}^{J} \sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\},$$

subject to the aggregate resource constraints

$$\sum_{j=1}^{J} y_j(s) \geq \sum_{j=1}^{J} c_j(s)$$

for all $s = 1, 2, \ldots, S$. This is a constrained maximization problem with a finite number $J \times S$ of choice variables and a finite number S of constraints. Hence, the Kuhn-Tucker theorem can be applied by defining the Lagrangian

$$\sum_{j=1}^{J} \sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\} + \sum_{s=1}^{S} \lambda(s) \left[\sum_{j=1}^{J} y_j(s) - \sum_{j=1}^{J} c_j(s) \right],$$

where for each $s = 1, 2, \ldots, S$, $\lambda(s)$ is the Lagrange multiplier on the aggregate resource constraint for state s.

a. Write down the first-order conditions for the social planner's problem. *Note:* Given the symmetry of this problem, it is possible to derive all of these first-order conditions at once by fixing a particular value of j and a particular value of s, differentiating the Lagrangian by $c_j(s)$ for those values of j, and s, and setting the result equal to zero. The same first-order condition will then have to hold for all $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$.

b. Next, use the first-order conditions you derived above to solve for the optimal $c_j^*(s)$ in terms of the coefficient of absolute risk aversion σ, the probability $\pi(s)$, and the associated value of the Lagrange multiplier $\lambda^*(s)$.

c. Now substitute your solutions for $c_j^*(s)$ into the binding constraint for state s,

$$J y_a(s) = \sum_{j=1}^{J} y_j(s) = \sum_{j=1}^{J} c_j^*(s),$$

to find a solution for the multiplier $\lambda^*(s)$ in terms of the coefficient of absolute risk aversion σ, the probability $\pi(s)$, and the average output per worker $y_a(s)$.

d. Finally, substitute your solution for $\lambda^*(s)$ back into the expression for $c_j^*(s)$ you derived in part (b) to find the solution for $c_j^*(s)$ in terms of $y_a(s)$. Note that this solution implies that the social planner optimally chooses to fully insure each individual consumer against "idiosyncratic risk," since each consumer's consumption depends only on aggregate output and not on his or her own level of productivity. At the same time, however, the solution indicates that the social planner cannot insure consumers against "aggregate risk," since their consumptions must rise or fall as the average output per worker $y_a(s)$ rises or falls across states.

3. Equilibrium Risk Sharing

The second welfare theorem of economics implies that the Pareto optimal allocation that solves the social planner's problem in question 2 can be supported in a competitive equilibrium. But how?

Suppose that the initial planning period $t = 0$ in the economy from question 2 is replaced by an initial trading period $t = 0$, in which consumers buy and sell claims for state-contingent delivery of goods during the production and consumption period $t = 1$. A consumer who sells a contingent claim for any state $s = 1, 2, \dots, S$ receives $p(s)$ dollars at $t = 0$, but is required to deliver one unit of consumption in that state s at $t = 1$. A consumer who buys a contingent claim for any state $s = 1, 2, \dots, S$ must pay $p(s)$ dollars at $t = 0$, but will receive one unit of consumption in that state s at $t = 1$.

Now, in the trading period $t = 0$, each consumer j takes current prices $p(s)$, $s = 1, 2, \dots, S$, and future outputs $y_j(s)$, $s = 1, 2, \dots, S$ as given, and chooses future consumptions $c_j(s)$, $s = 1, 2, \dots, S$, to maximize his or her own utility

$$\sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\},$$

subject to the budget constraint

$$\sum_{s=1}^{S} p(s) y_j(s) \geq \sum_{s=1}^{S} p(s) c_j(s).$$

This is a constrained maximization problem with a finite number S of choice variables and a single constraint. Hence, the Kuhn-Tucker theorem can be applied by defining the Lagrangian

$$\sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\} + \lambda_j \left[\sum_{s=1}^{S} p(s) y_j(s) - \sum_{s=1}^{S} p(s) c_j(s) \right],$$

where λ_j is the single Lagrange multiplier on consumer j's budget constraint.

a. Write down the first-order conditions for the consumer's problem. *Note:* Once again, given the symmetry of this problem, it is possible to derive all of these first-order conditions at once by fixing a particular value of s, differentiating the Lagrangian by $c_j(s)$ for that value of s, and setting the result equal to zero. The same first-order condition will then have to hold for all $s = 1, 2, \dots, S$.

b. Next, use the first-order conditions you derived above to solve for the optimal $c_j^*(s)$ in terms of the coefficient of absolute risk aversion σ, the probability $\pi(s)$, the contingent claim price $p(s)$, and the associated value of the Lagrange multiplier λ_j^*.

c. Now let's guess that the equilibrium prices that support the optimal allocation from question 2 are

$$p(s) = \pi(s) \exp[-\sigma y_a(s)]$$

for all $s = 1, 2, \ldots, S$. Where does this conjecture come from? To see, take a look back at your solution for $\lambda^*(s)$ from question 2, part (c), and recall the interpretation of Lagrange multipliers as "shadow prices." Recall, also, that the social planner from question 2 weighted all individual agent's utility equally when solving for the optimal allocations. Thus, it makes sense to guess that the prices from above will prevail in a competitive equilibrium when

$$\sum_{s=1}^{S} p(s) y_j(s) = \sum_{s=1}^{S} p(s) y_a(s)$$

for all $j = 1, 2, \ldots, J$. Note that this symmetry condition does not require agents to have equal productivities state-by-state; it simply requires that agents have equal endowments of wealth when valued in the time $t = 0$ trading session. Using this symmetry condition for wealth, substitute your solutions for $c_j^*(s)$, together with the conjectured equilibrium prices, into consumer j's binding budget constraint,

$$\sum_{s=1}^{S} p(s) y_a(s) = \sum_{s=1}^{S} p(s) y_j(s) = \sum_{s=1}^{S} p(s) c_j^*(s),$$

to find a solution for the multiplier λ_j^*.

d. Finally, substitute the conjectured prices $p(s)$ and your solution for λ_j^* back into the expression for $c_j^*(s)$ you derived in part (b) to find the solution for $c_j^*(s)$ in terms of $y_a(s)$. To verify that these are equilibrium allocations, it only remains to show that the market clearing conditions,

$$\sum_{j=1}^{J} c_j^*(s) = \sum_{j=1}^{J} y_j(s) = J y_a(s)$$

hold for all $s = 1, 2, \ldots, S$, but this is easy. Your solutions for $c_j^*(s)$, $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$, here should coincide those that you derived from the social planner's problem in question 2. This confirms that the welfare theorems hold and, more specifically, that the time $t = 0$ markets for contingent claims allow each consumer to insure against idiosyncratic but not aggregate risk.