Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
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Due Thursday, November 2

This exam has five questions on seven pages. Before you begin, please check to make sure that your copy has all five questions and all seven pages. Each question will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Hicksian Demands

We've already seen that when a consumer's preferences over apples and bananas are described by the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b),$$

the consumer's Marshallian demand curves for apples and bananas are

$$c_a^* = c_a(p_a, p_b, Y) = \frac{\alpha Y}{p_a}$$

and

$$c_b^* = c_b(p_a, p_b, Y) = \frac{(1-\alpha)Y}{p_b},$$

where Y is income and p_a and p_b are the prices of apples and bananas. Substituting these Marshallian demand curves back into the utility function yields the indirect utility function

$$V(p_a, p_b, Y) = \alpha\ln(\alpha) + (1-\alpha)\ln(1-\alpha) - \alpha\ln(p_a) - (1-\alpha)\ln(p_b) + \ln(Y).$$

This problem asks you to derive, instead, the Hicksian demand curves from the expenditure minimization problem

$$\min_{c_a, c_b}\ p_a c_a + p_b c_b \text{ subject to } \alpha\ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}.$$

a. Set up (define) the Lagrangian for expenditure minimization problem. Then, write down the first-order conditions for the consumer's optimal choices c_a^* and c_b^*. *Note:* As discussed in class, there are a number of ways to define the Lagrangian for any given constrained optimization problem. Use whichever form you prefer, but make sure your first-order conditions are consistent with your chosen definition of the Lagrangian.

b. Your first-order conditions from part (a) should indicate that so long both of the goods prices p_a or p_b are strictly positive, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices c_a^* and c_b^* and the corresponding value of the Lagrange multiplier. Use these three equations to find the Hicksian demand functions $c_a^* = h_a(p_a, p_b, \bar{U})$ and $c_b^* = h_b(p_a, p_b, \bar{U})$.

c. Finally, to check your answers from part (b), show that when $\bar{U} = V(p_a, p_b, Y)$, the Hicksian demands satisfy

$$h_a[p_a, p_b, V(p_a, p_b, Y)] = \frac{\alpha Y}{p_a} = c_a(p_a, p_b, Y)$$

and

$$h_b[p_a, p_b, V(p_a, p_b, Y)] = \frac{(1-\alpha)Y}{p_b} = c_b(p_a, p_b, Y).$$

2. The Cost of Living

If the prices of all goods and services always increased at exactly the same rate, it would be easy to measure the rising cost of living by their common rate of change. But in the more general case where different prices change at different rates, how should those differential rates of increase be aggregated to summarize the rising cost of living? This question asks you to consider and compare three alternatives.

To begin by fixing notation, let p_a and p_b denote the prices of apples and bananas in a base year, labeled $t = 0$, and let q_a and q_b denote the prices of apples and bananas in a subsequent year, labeled $t = 1$. Assume throughout that the consumer's preferences are described by the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b).$$

a. The first index of the cost of living is grounded most firmly in microeconomic theory, and will therefore be called the "true" price index P. Define the minimum cost function with reference to the expenditure minimization problem you studied above, in question 1, so that for any fixed utility level $\bar{U}$,

$$C(p_a, p_b, \bar{U}) = \min_{c_a, c_b} \; p_a c_a + p_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}$$

and

$$C(q_a, q_b, \bar{U}) = \min_{c_a, c_b} \; q_a c_a + q_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}.$$

Now consider using the increase in the cost of achieving the given utility level $\bar{U}$, computed as

$$P(q_a, q_b, p_a, p_b, \bar{U}) = \frac{C(q_a, q_b, \bar{U})}{C(p_a, p_b, \bar{U})},$$

to measure the rise in the cost of living between $t = 0$ and $t = 1$. In general, a drawback to this measure is that it will depend on the utility level $\bar{U}$. Use the Hicksian demand curves that you derived in answering question 1 to show that in this case, however, the aggregate price index depends only on prices and the preference parameter α and to find the specific functional form for

$$P(q_a, q_b, p_a, p_b, \bar{U}) = P(q_a, q_b, p_a, p_b).$$

b. The second index of the cost of living, used more commonly in practice, simply compares the cost of purchasing a fixed basket of c_a^* apples and c_b^* bananas when prices are q_a and q_b to the cost of purchasing the same basket of apples and bananas when prices are p_a and p_b. This is how, for example, the consumer price index (CPI) is computed in the United States. Suppose c_a^* apples and c_b^* are taken to be the quantities consumed in the base year, $t = 0$. Then the Marshallian demand curves imply that

$$c_a^* = c_a(p_a, p_b, Y) = \frac{\alpha Y}{p_a}$$

and

$$c_b^* = c_b(p_a, p_b, Y) = \frac{(1-\alpha)Y}{p_b}.$$

In this case, this second measure of the cost of living,

$$L(q_a, q_b, p_a, p_b, Y) = \frac{q_a c_a(p_a, p_b, Y) + q_b c_b(p_a, p_b, Y)}{p_a c_a(p_a, p_b, Y) + p_b c_b(p_a, p_b, Y)},$$

corresponds to the Laspeyres price index. Use of the Laspeyres price index can be justified using a first-order approximation to the true minimum cost function and an appeal to the envelope theorem, as follows:

$$\begin{aligned} C(q_a, q_b, \bar{U}) &\approx C(p_a, p_b, \bar{U}) + \frac{\partial C(p_a, p_b, \bar{U})}{\partial p_a}(q_a - p_a) + \frac{\partial C(p_a, p_b, \bar{U})}{\partial p_b}(q_b - p_b) \\ &= C(p_a, p_b, \bar{U}) + h_a(p_a, p_b, \bar{U})(q_a - p_a) + h_b(p_a, p_b, \bar{U})(q_b - p_b) \\ &= h_a(p_a, p_b, \bar{U})q_a + h_b(p_a, p_b, \bar{U})q_b, \end{aligned}$$

where the first equality follows from the envelope theorem and the second from the definitions of the cost function $C(p_a, p_b, \bar{U})$ and the Hicksian demands $h_a(p_a, p_b, \bar{U})$ and $h_b(p_a, p_b, \bar{U})$. Therefore

$$P(q_a, q_b, p_a, p_b, \bar{U}) = \frac{C(q_a, q_b, \bar{U})}{C(p_a, p_b, \bar{U})} \approx \frac{h_a(p_a, p_b, \bar{U})q_a + h_b(p_a, p_b, \bar{U})q_b}{h_a(p_a, p_b, \bar{U})p_a + h_b(p_a, p_b, \bar{U})p_b}.$$

Finally, setting $\bar{U} = V(p_a, p_b, Y)$ and using

$$h_a[p_a, p_b, V(p_a, p_b, Y)] = c_a(p_a, p_b, Y) = c_a^*$$

and

$$h_b[p_a, p_b, V(p_a, p_b, Y)] = c_b(p_a, p_b, Y) = c_b^*,$$

it follows that

$$P(q_a, q_b, p_a, p_b, V(p_a, p_b, Y)) \approx \frac{c_a(p_a, p_b, Y)q_a + c_b(p_a, p_b, Y)q_b}{c_a(p_a, p_b, Y)p_a + c_b(p_a, p_b Y)p_b} = L(q_a, q_b, p_a, p_b, Y).$$

In general, a drawback to the Laspeyres price index if that it will depend on the level of income Y. Use the Marshallian demand curves $c_a(p_a, p_b, Y)$ and $c_b(p_a, p_b, Y)$ to show that in this case, however, the Laspeyres price index depends only on prices and the preference parameter α and to find the specific functional form for

$$L(q_a, q_b, p_a, p_b, Y) = L(q_a, q_b, p_a, p_b).$$

c. Although the argument outlined above shows that while the Laspeyres index provides a first-order approximation to the true microeconomic cost of living index, the functional form for $L(q_a, q_b, p_a, p_b)$ does not coincide exactly with the functional form for $P(q_a, q_b, p_a, p_b)$. As a preferable alternative to the Laspeyres index, W.E. Diewert ("Exact and Suplerative Index Numbers," *Journal of Econometrics*, Vol.4, 1976, pp.115-145) suggests the Törnqvist-Theil index, defined by

$$T(q_a, q_b, p_a, p_b, Y) = \left(\frac{q_a}{p_a}\right)^{\bar{s}_a(q_a, q_b, p_a, p_b, Y)} \left(\frac{q_b}{p_b}\right)^{\bar{s}_b(q_a, q_b, p_a, p_b, Y)},$$

where

$$\bar{s}_a(q_a, q_b, p_a, p_b, Y) = \frac{1}{2}\left[\frac{p_a c_a^*(p_a, p_b, Y)}{Y} + \frac{q_a c_a^*(q_a, q_b, Y)}{Y}\right]$$

and

$$\bar{s}_b(q_a, q_b, p_a, p_b, Y) = \frac{1}{2}\left[\frac{p_b c_b^*(p_a, p_b, Y)}{Y} + \frac{q_b c_b^*(q_a, q_b, Y)}{Y}\right]$$

are the average budget shares of apples and bananas from $t = 0$ and $t = 1$. Recalling from problem set 1 that these budget shares are constant, so that

$$\bar{s}_a(q_a, q_b, p_a, p_b, Y) = \alpha$$

and

$$\bar{s}_a(q_a, q_b, p_a, p_b, Y) = 1 - \alpha,$$

show that when preferences are described by the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1 - \alpha) \ln(c_b),$$

the Törnqvist-Theil index depends only on prices and the preference parameter α and that the functional form of

$$T(q_a, q_b, p_a, p_b, Y) = T(q_a, q_b, p_a, p_b)$$

coincides exactly with the form of the true microeconomic price index $P(q_a, q_b, p_a, p_b)$ that you derived in part (a).

3. Portfolio Allocation: Part 1

Consider an individual investor who must allocate the savings in his or her retirement plan either to a perfectly safe asset that pays a known rate of return r_f or to a risky stock mutual fund with random return having expected value μ_r and standard deviation σ_r. Then if w is the fraction of total savings allocated to the risky mutual fund and $1 - w$ the fraction allocated to the risk-free alternative, the investor's portfolio will have random return with expected value

$$\mu_p = w\mu_r + (1 - w)r_f$$

and standard deviation

$$\sigma_p = w\sigma_r.$$

Suppose the investor prefers to earn a higher expected return on his or her portfolio but dislikes volatility in the portfolio's random return, as described by a utility function $U(\mu_p, \sigma_p)$ that is strictly increasing in its first argument but strictly decreasing in its second argument. Then the investor can be depicted as solving the constrained optimization problem

$$\max_{w,\mu_p,\sigma_p} U(\mu_p, \sigma_p) \text{ subject to } w\mu_r + (1 - w)r_f \geq \mu_p \text{ and } \sigma_p \geq w\sigma_r.$$

a. Write down the first-order conditions for the individual investor's problem. *Note:* For this problem and the two that follow, there are a number of ways of deriving the first-order conditions. In particular, you can substitute some or all of the constraints into the objective function or set up the Lagrangian for the constrained optimization problem. Use whichever approach you find most convenient.

b. Use the first-order conditions that you derived in part (a), together with the binding constraints, to show that the optimizing investor sets the marginal rate of substitution between risk and return,

$$-\frac{U_2(\mu_p^*, \sigma_p^*)}{U_1(\mu_p^*, \sigma_p^*)}$$

equal to the "Sharpe ratio" of the risky mutual fund, defined as

$$\frac{\mu_r - r_f}{\sigma_r}.$$

4. Portfolio Management

Now consider a professional funds manager, who makes investment decisions for the risky mutual fund offered to the individual investor from question 3. To retain analytic tractability while, at the same time, highlighting the basic economic ideas, suppose that the professional manager is able to allocate funds across two individual stocks, with random returns that have expected values μ_1 and μ_2, standard deviations σ_1 and σ_2, and covariance σ_{12}. Then if v is the fraction of total funds allocated to stock 1 and $1 - v$ is the fraction of total funds allocated to stock 2, the mutual fund will have a random return with expected value

$$\mu_r = v\mu_1 + (1 - v)\mu_2$$

and standard deviation

$$\sigma_r = [v^2\sigma_1^2 + (1 - v)^2\sigma_2^2 + 2v(1 - v)\sigma_{12}]^{1/2}.$$

Suppose that the manager is instructed to maximize the Sharpe ratio offered by the mutual fund, as a measure of ”expected excess return” (as measured by $\mu_r - r_f$ in the numerator) “per unit of risk” (as measure by σ_r in the denominator). Then the manager can be depicted as solving the constrained optimization problem

$$\max_{v,\mu_r,\sigma_r} \frac{\mu_r - r_f}{\sigma_r} \text{ subject to } v\mu_1 + (1 - v)\mu_2 \geq \mu_r$$
$$\text{and } \sigma_r \geq [v^2\sigma_1^2 + (1 - v)^2\sigma_2^2 + 2v(1 - v)\sigma_{12}]^{1/2}.$$

a. Write down the first-order conditions for the manager’s problem.

b. Use the first-order conditions that you derived in part (a), together with the binding constraints, to show that the manager’s optimal choice of v^* must satisfy

$$\mu_1 - \mu_2 = \frac{[v^*\mu_1 + (1 - v^*)\mu_2 - r_f][v^*\sigma_1^2 - (1 - v^*)\sigma_2^2 + (1 - 2v^*)\sigma_{12}]}{v^{*2}\sigma_1^2 + (1 - v^*)^2\sigma_2^2 + 2v^*(1 - v^*)\sigma_{12}}.$$

5. Portfolio Allocation: Part 2

Suppose that the individual investor from question 3, worried perhaps that the professional funds manager's incentives do not align perfectly with his or her own preferences over risk versus return, decides to allocate funds directly to the two individual stocks as well as the risk-free alternative. Although there are numerous ways of setting up the investor's expanded problem, perhaps the easiest is to think of the individual investor as deciding, first, on the shares of total savings w and $1-w$ allocated to the stock market as a whole versus the risk-free alternative and deciding, second, on the shares of funds allocated to the stock market as a whole v and $1-v$ allocated to individual stocks 1 and 2. Now the investor can be depicted as solving the constrained optimization problem

$$\max_{w,v,\mu_r,\sigma_r,\mu_p,\sigma_p} U(\mu_p,\sigma_p) \text{ subject to } v\mu_1 + (1-v)\mu_2 \geq \mu_r,$$

$$\sigma_r \geq [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2},$$

$$w\mu_r + (1-w)r_f \geq \mu_p,$$

$$\text{and } \sigma_p \geq w\sigma_r.$$

a. Write down the first-order conditions for the individual investor's expanded problem.

b. Use the first-order conditions that you derived in part (a) to show that the optimizing investor still sets the marginal rate of substitution between risk and return equal to the Sharpe ratio of his or her optimally-chosen risky portfolio,

$$-\frac{U_2(\mu_p^*,\sigma_p^*)}{U_1(\mu_p^*,\sigma_p^*)} = \frac{\mu_r^* - r_f}{\sigma_r^*},$$

and that his or her optimally-chosen v^* satisfies exactly the same condition

$$\mu_1 - \mu_2 = \frac{[v^*\mu_1 + (1-v^*)\mu_2 - r_f][v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}]}{v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}}$$

that dictates the professional funds manager's choice of v^*. This comparison illustrates the famous "separation" or "two-fund" theorem from financial economics. This theorem says that individual investors with preferences that can be described by "mean-variance" utility functions don't need to worry about choosing individual stocks. Instead, they can focus on allocating their savings across the professionally-managed stock mutual fund with the highest feasible Sharpe ratio and a risk-free alternative.