Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, November 1

This exam has three questions, each with four parts, on six pages. Before you begin, please check to make sure that your copy has all three questions and all six pages. Each question will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Properties of Hicksian Demands

Consider a consumer with preferences over two goods described by the utility function $U(c_1, c_2)$, where c_1 and c_2 are the amounts consumed of goods 1 and 2. Suppose this consumer purchases the goods in perfectly competitive markets at prices p_1 and p_2 in order to minimize the total expenditure needed to reach utility level $\bar{U}$, solving the constrained minimization problem:

$$\min_{c_1, c_2} \; p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

a. Set up (define) the Lagrangian for this problem. Then, write down the first-order conditions for the consumer's optimal choices c_1^* and c_2^*. *Note:* As discussed in class, there are a number of ways to define the Lagrangian for any given constrained optimization problem. Use whichever form you prefer, but make sure your first-order conditions are consistent with your chosen definition of the Lagrangian.

b. Your first-order conditions from part (a) should indicate that so long as the goods prices p_1 and p_2 are strictly positive and the utility function U is strictly increasing in both its arguments, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices c_1^* and c_2^* and the corresponding value of the Lagrange multiplier. Assuming that the utility function is also such that the consumer's expenditure minimization problem has a unique solution, these three equations define the Hicksian demand functions $c_1^* = h_1(p_1, p_2, \bar{U})$ and $c_2^* = h_2(p_1, p_2, \bar{U})$. In particular, these Hicksian demands must satisfy the efficiency condition

$$\frac{p_1}{p_2} = \frac{U_1[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})]}{U_2[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})]}, \tag{1}$$

where U_i, $i = 1, 2$, denote the partial derivatives of the utility function with respect to each of its two arguments, and the binding constraint

$$U[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})] = \bar{U} \tag{2}$$

for all values of $(p_1, p_2, \bar{U})$. Differentiate (2) with respect to p_1, then use the resulting expression together with (1) to obtain a restriction, implied by the consumer's optimizing behavior, that links the partial derivative

$$\frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_1}$$

to the prices p_1 and p_2 and the partial derivative

$$\frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_1}.$$

Similarly, differentiate (2) with respect to p_2 and use (1) to obtain a restriction that links

$$\frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_2}$$

to the prices p_1 and p_2 and

$$\frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_2}.$$

c. Now define the minimum expenditure function E as

$$E(p_1, p_2, \bar{U}) = \min_{c_1, c_2} \; p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

Use the envelope theorem, together with the symmetry condition

$$\frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_1 \partial p_2} = \frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_2 \partial p_1}$$

to obtain yet another restriction implied by consumer optimality that links

$$\frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_2}.$$

to

$$\frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_1}.$$

d. Finally, consider the matrix

$$\begin{bmatrix} \frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_1} & \frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_2} \\ \frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_1} & \frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_2} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Use the restrictions you derived in parts (b) and (c) to show how each of the values b, c, and d in this matrix depends on the value of a and the prices p_1 and p_2. These restrictions imply that, in this two-good case, knowing one partial derivative of one of the Hicksian demand curves allows you to infer both of the partial derivatives of both of the Hicksian demand curves.

2. Cost Minimization and the Production Function

Consider a firm that produces output y with capital k and labor l according to the production function

$$y = f(k, l).$$

Suppose this firm rents capital at the rental rate r and hires labor at the wage rate w in perfectly competitive markets in order to minimize the costs of producing at least $\bar{y}$ units of output, solving the constrained minimization problem

$$\min_{k,l}\ rk + wl \text{ subject to } f(k, l) \geq \bar{y}.$$

a. Set up (define) the Lagrangian for this problem. Then, write down the first-order conditions for the firm's optimal choices k^* and l^*. *Note:* As discussed in class, there are a number of ways to define the Lagrangian for any given constrained optimization problem. Use whichever form you prefer, but make sure your first-order conditions are consistent with your chosen definition of the Lagrangian.

b. Your first-order conditions from part (a) should indicate that so long as the factor prices r and w are strictly positive and the production function is strictly increasing in both of its arguments, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices k^* and l^* and the corresponding value of the Lagrange multiplier. Assuming that the production function is also such that the firm's cost minimization problem has a unique solution, these three equations define the conditional factor demand curves $k^* = k(r, w, \bar{y})$ and $l^* = l(w, r, \bar{y})$. In particular, these conditional factor demands must satisfy the efficiency condition

$$\frac{r}{w} = \frac{f_1[k(r, w, \bar{y}), l(r, w, \bar{y})]}{f_2[k(r, w, \bar{y}), l(r, w, \bar{y})]}, \tag{3}$$

where f_i, $i = 1, 2$, denote the partial derivatives of the production function with respect to each of its two arguments, and the binding constraint

$$f[k(r, w, \bar{y}), l(r, w, \bar{y})] = \bar{y} \tag{4}$$

for all values of $(r, w, \bar{y})$. Note that the factor prices r and w enter directly into these optimality conditions only through their ratio r/w. What does this fact tell you about the conditional factor demand curves: what, in particular, will happen to the optimal choices k^* and l^* when r and w both double, with the output requirement $\bar{y}$ left unchanged? Now define the minimum cost function by

$$C(r, w, \bar{y}) = \min_{k,l}\ rk + wl \text{ subject to } f(k, l) \geq \bar{y}.$$

What happens to the value of $C(r, w, \bar{y})$ when r and w both double with the output requirement $\bar{y}$ left unchanged?

c. Your answers to the questions in part (b) should imply that the minimum cost function is homogenous of degree one in the factor prices r and w. Suppose that, after recognizing this property of the cost function, we hypothesize further that the cost function takes the specific form

$$C(r, w, \bar{y}) = \bar{y}^b r^a w^{1-a}, \tag{5}$$

where the exponents on the factor prices sum to one. Can we determine what the assumed functional form (5) for the cost function implies for the form of the firm's production function $f(k, l)$? The answer is yes! To see how, start by applying the envelope theorem (in this case, Shephard's lemma) to the firm's cost minimization problem, to obtain two equations that link the conditional factor demands $k(r, w, \bar{y})$ and $l(r, w, \bar{y})$ to the partial derivatives of the cost function.

d. Your answers from part (b), together with the assumed functional form from (5), should indicate that

$$k(r, w, \bar{y}) = \frac{\partial C(r, w, \bar{y})}{\partial r} = a\bar{y}^b r^{a-1} w^{1-a}$$

and

$$l(r, w, \bar{y}) = \frac{\partial C(r, w, \bar{y})}{\partial w} = (1-a)\bar{y}^b r^a w^{-a}.$$

Rearranging and then combining these two results shows that when $k = k(r, w, \bar{y})$ and $l = l(r, w, \bar{y})$ are chosen by the cost-minimizing firm,

$$\left(\frac{a\bar{y}^b}{k}\right)^{1/(1-a)} = \frac{r}{w} = \left[\frac{l}{(1-a)\bar{y}^b}\right]^{1/a}$$

or, perhaps more simply,

$$\left(\frac{a\bar{y}^b}{k}\right)^{a} = \left[\frac{l}{(1-a)\bar{y}^b}\right]^{1-a}.$$

To complete the derivation, use this last equality to solve for $\bar{y}$ in terms of k, l, a, and b. This solution should reveal the form of the production function $f(k, l)$!

3. Discretion versus Commitment in Optimal Monetary Policymaking

At the Federal Reserve in the United States and at other central banks around the world, monetary policymakers prefer to maintain a large degree of discretion, making their policy decisions meeting-by-meeting instead of committing to a pre-announced plan ahead of time. But does this approach help or hurt, when it comes to achieving their stabilization goals for unemployment and inflation? Finn E. Kydland and Edward C. Prescott ("Rules Rather than Discretion: The Inconsistency of Optimal Plans," *Journal of Political Economy*, June 1977) and Robert J. Barro and David B. Gordon ("A Positive Theory of Monetary Policy in a Natural Rate Model," *Journal of Political Economy*, August 1983) present examples to show that better outcomes can be achieved under commitment.

This question asks you to work through a version of their model that highlights three key assumptions needed to make their argument work. First, a strong form of the "natural rate hypothesis" must hold: inflationary monetary policy can only lower the unemployment rate by surprising private agents. Second, private agents have "rational expectations:" they know the central bank's objectives and can therefore anticipate how monetary policy decisions will be made. Third, the central bank must wish to push unemployment below the "natural rate," that is, the rate prevailing when there are no monetary surprises.

Setting up the example requires two equations. The first is a Phillips curve

$$U = U^n - \alpha(\pi - \pi^e), \tag{6}$$

with $\alpha > 0$, that reflects the natural rate hypothesis: the central bank can push unemployment U below the natural rate $U^n > 0$ only by creating inflation π that exceeds the rate π^e expected by private agents. The second is an objective function for the central bank

$$-(1/2)(U - kU^n)^2 - (b/2)\pi^2, \tag{7}$$

with $0 < k < 1$ and $b > 0$, that penalizes deviations of unemployment U from a target kU^n that, in light of the assumption that $0 < k < 1$, lies below the natural rate U^n and deviations of inflation π from a target of zero. In (7), $b > 0$ is the weight that the central bank places on its objective for inflation relative to unemployment.

a. Suppose first that the central bank operates with "discretion," meaning that it chooses the rates of unemployment U and inflation π after private agents have formed their expectation π^e. In this case, taking U^n and π^e as given, the central bank maximizes the objective function (7) subject to the constraint in (6). You can treat this as constrained optimization problem and set up the Lagrangian before taking first-order conditions for U and π, or you can just substitute the constraint (6) into the objective function (7) and take the first-order condition for π. Either way, use your results to find an expression that shows how the central bank's choice of π depends on U^n, π^e, and the parameters α, b, and k.

b. Assume next that private agents have rational expectations and set their expected rate of inflation π^e equal to the value of π implied by the optimality condition you just derived in part (a). Note that this rational expectation assumption implies, through (6), that despite the central bank's efforts to use surprise inflation to lower the unemployment rate, U will always equal U^n in equilibrium. Use your equation from part (a) to show that, nevertheless, the discretionary central bank will choose a positive rate of inflation π so long as $U^n > 0$, $\alpha > 0$, $b > 0$, and $0 < k < 1$.

c. Now suppose instead that the central bank commits to a choice of the inflation rate π before private agents have fixed their expectation π^e. Suppose also that the central bank anticipates that, once it announces its choice of π, rational expectations in the private sector will imply that $\pi^e = \pi$. In light of (1), the central bank under commitment recognizes that it is futile to try creating surprise inflation and realizes that $U = U^n$ will prevail no matter what. In this case with commitment, therefore, the central bank solves the simpler problem

$$\max_{\pi} \; -(1/2)[(1-k)U^n]^2 - (b/2)\pi^2.$$

What is the central bank's optimal choice of π in this case?

d. Finally, use your results from parts (b) and (c) to confirm that, in this example, the central bank achieves a higher value of its objective function by choosing commitment instead of discretion.