Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, November 2

This exam has three questions on five pages; before you begin, please check to make sure that your copy has all three questions and all five pages. Each part of each question is worth ten points. Therefore, question 1 with one part is worth 10 points, question 2 with four parts is worth 40 points, and question 3 with five parts is worth 50 points, for a total of 100 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Utility Maximization

Consider a consumer with preferences over food c_F and all other goods c_O as described by the utility function

$$\alpha \ln(c_F) + (1 - \alpha) \ln(c_O),$$

where the parameter α lies between zero and one, with $0 < \alpha < 1$. If the consumer has income Y and purchases food and all other goods at prices p_F and p_O in perfectly competitive markets, he or she solves

$$\max_{c_F, c_O} \ \alpha \ln(c_F) + (1 - \alpha) \ln(c_O) \text{ subject to } Y \geq p_F c_F + p_O c_O.$$

Write down the expressions that show how the solutions c_F^* and c_O^* to this problem depend on Y, p_F, p_O, and α.

Note: You can derive these expressions by setting up the Lagrangian, taking the first-order conditions, and using those first-order conditions together with the binding constraint to solve for c_F^* and c_O^*. Or you can just recall that you've solved a version of this problem before, in problem set 1, and simply adapt the solution you derived there to apply to "food" and "all other goods" instead of "good 1" and "good 2."

2. Expenditure Minimization

Next, let's acknowledge that "food" c_F from question 1 is an aggregate of individual food items. Suppose, in particular, that the consumer's total food consumption is determined by his or her consumption c_A of apples and c_B of bananas according to

$$c_F = \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

were $\theta > 0$ is a parameter measuring the elasticity of substitution between apples and bananas in forming or "producing" the food aggregate c_F. While this specification allows for imperfect substitutability in preferences between apples and bananas, it also has the property of being homogeneous of degree one: if the consumer buys twice as many apples and bananas, the food aggregate doubles as well.

The question we want to answer next is: If we know the prices p_A of apples and p_B of bananas, can we also define a corresponding price aggregate to measure p_F to go along with the quantity aggregate c_F that appears in question 1?

To answer this question, suppose that the consumer minimizes the cost of producing at least c_F units of the food aggregate by solving

$$\min_{c_A, c_B} \; p_A c_A + p_B c_B \text{ subject to } \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \geq c_F.$$

a. Define (write down) the Lagrangian for this problem. *Note:* There are a variety of ways to do this. You can choose whatever formulation you find most convenient; just make sure your answers to part (b), below, are consistent with your preferred definition here.

b. Next, write down the two first-order conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values of c_A^* and c_B^* that solve this problem.

c. Together with the binding constraint,

$$c_F = \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

your two first-order conditions from part (b) form a system of three equations in three unknowns: the values c_A^* and c_B^* that solve the problem and the associated value of the Lagrange multiplier. Use these equations to solve for the optimal choices c_A^* and c_B^* in terms of the parameters: c_F, p_A, p_B, and θ. *Note:* There are a number of ways to do this – use whichever you find most convenient. But perhaps the easiest is to divide one first-order condition by the other to eliminate the multiplier, then substitute the resulting expression into the binding constraint.

d. With the minimum expenditure function defined as

$$E(c_F, p_A, p_B) = \min_{c_A, c_B} \; p_A c_A + p_B c_B \text{ subject to } \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \geq c_F,$$

find an expression for the partial derivative of E with respect to the parameter c_F,

$$\frac{\partial E(c_F, p_A, p_B)}{\partial c_F}$$

in terms of p_A and p_B. *Note:* There are at least two ways to do this – again, use whichever you find most convenient. One is to substitute your solutions for c_A^* and c_B^* from part (c) directly into the expenditure function. The other is to find the solution for the Lagrange multiplier implied by the solutions for c_A^* and c_B^* from part (c), and use the envelope theorem instead.

This partial derivative has an interpretation as the "marginal cost" to the consumer of obtaining an additional unit of the food aggregate c_F. Thus, in in comparing the solution to question 3 below to the solution to question 1 above, it will be our candidate for measuring the price aggregate p_F corresponding to the food aggregate c_F.

3. More Detailed Utility Maximization

Now let's substitute the formula for the food aggregate

$$c_F = \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

from question 2 into the utility function

$$\alpha \ln(c_F) + (1-\alpha)\ln(c_O),$$

from question 1 to see what happens when we consider the optimal choices of individual good items as well as all other goods in a more detailed utility maximization problem. Our general aim is to verify that the solutions to this more detailed problem are consistent with the solutions to questions 1 and 2.

If the consumer has income Y, and purchases apples, bananas, and all other goods at prices p_A, p_B, and p_O in perfectly competitive markets, he or she solves

$$\max_{c_A, c_B, c_O} \left(\frac{\alpha\theta}{\theta-1} \right) \ln \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right) + (1-\alpha)\ln(c_O) \text{ subject to } Y \geq p_A c_A + p_B c_B + p_O c_O.$$

a. Define (write down) the Lagrangian for this problem. *Note:* Once again, there are a variety of ways to do this. You can choose whatever formulation you find most convenient; just make sure your answers to part (b), below, are consistent with your preferred definition here.

b. Next, write down the first-order conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values of c_A^*, c_B^*, and c_O^* that solve this problem.

c. Together with the binding constraint

$$Y = p_A c_A^* + p_B c_B^* + p_O c_O^*,$$

your three first-order conditions form a system of four equations in four unknowns: the values c_A^*, c_B^*, and c_O^* that solve the problem and the associated value of the Lagrange multiplier. As a first step in solving this system, multiply the first-order condition for c_A by c_A^*, the first-order condition for c_B by c_B^*, and the first-order condition for c_O by c_O^*. Then, after dividing each first-order condition by the Lagrange multiplier, substitute them into the binding budget constraint to find a simple solution for the multiplier in terms of Y. Then, substitute this solution for the multiplier back into the first-order condition for c_O to obtain a solution for c_O^* that coincides exactly with the one you wrote down previously in answering question 1.

d. Next, divide the first-order condition for c_A by the first-order condition for c_B, and use the resulting expression together with the definition of optimal food consumption

$$c_F^* = \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

to find expressions for c_A^* and c_B^* in terms of c_F^*, p_A, p_B, and θ. These expressions should coincide with your solutions from question 2, with c_F^* in place of c_F.

e. It only remains to find c_F^*. This can be done by substituting the expressions for the multiplier, c_A^*, and c_B^* that you just derived in parts (c) and (d) into either the first-order condition for c_A or the first-order condition for c_B. Verify that when the price aggregate p_F for food is defined as suggested by the solution to part (d) of question 2, the solution for c_F^* derived here coincides with the one you wrote down in answering question 1.

Summary and Generalization

Questions 1-3 illustrate that under certain circumstances, it is possible for a consumer to decide first on how much to spend on food versus all other goods and then, only later, how to break spending on food down into components spent on individual food items. Likewise, under the same circumstances, it is possible for an economist who simply wants to study a consumer's total spending on food to solve an aggregated problem like that in question 1, moving on to study more complicated problems like those in questions 2 and 3 only if spending on individual food items is of additional interest.

The general conditions under which this approach is possible are identified by Charles Blackorby, Daniel Primont, and R. Robert Russell in "Separability: A Survey," Chapter 2 in the *Handbook of Utility Theory, Volume 1: Principles*, edited by Salvador Barberà, Peter J. Hammond, and Christian Seidel (Kluwer Academic Publishers, 1998). First, preferences over individual food items must be weakly separable from those over all other goods, meaning that they can be described by a utility function that takes the form

$$\mathcal{U}(c_A, c_B, c_O) = U(f(c_A, c_B), c_O)$$

for some functions U and f. In this general specification as in the example above,

$$c_F = f(c_A, c_B)$$

defines the quantity aggregate for food. Second, the aggregator function f must be homothetic; this assumption guarantees that there exists a corresponding price aggregate,

$$p_F = g(p_A, p_B),$$

defining p_F in terms of p_A and p_B. Together, these assumptions hold if and only if the utility function over the three goods takes the form

$$\mathcal{U}(c_A, c_B, c_O) = \tilde{U}(\tilde{f}(c_A, c_B), c_O),$$

where $\tilde{f}$ is homogeneous of degree one.

The CES (constant elasticity of substitution) aggregator introduced here in question 2 can be a bit tedious to work with, but usefully allows for different degrees of substitutability between components of the aggregate and is still more tractable than most alternatives. Therefore, it gets used a lot in macroeconomics, international trade, and international finance.