Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2020

Due Tuesday, October 27 at 12noon

This exam has three questions on four pages; before you begin, please check to make sure that your copy has all three questions and all four pages. Each part of each question is worth ten points. Therefore, questions 1 and 2 with three parts are worth 30 points each and question 3 with four parts is worth 40 points, for a total of 100 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. The Kuhn-Tucker Theorem

Consider the constrained optimization problem of choosing values for x and y to maximize the objective function

$$F(x, y) = (1 - x - y)(x + 2y),$$

subject to the constraints

$$1 \geq x + y,$$

$$x \geq 0,$$

and

$$y \geq 0.$$

a. Define (write down) the Lagrangian for this problem. *Note:* There are a variety of ways to do this. You can choose whatever formulation you find most convenient; just make sure your answers to part (b), below, are consistent with your preferred definition here.

b. Next, write down the first-order conditions, constraints, non-negativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values x^* and y^* of x and y that solve the problem, together with the corresponding values of the Lagrange multiplier or multipliers, depending on how you chose to handle the nonnegativity constraints on x and y in defining your Lagrangian in part (a).

c. Finally, use your results from part (b) to find the numerical values of x^* and y^* the solve the problem.

2. Consumer Optimization

Consider a consumer who uses his or her income Y to purchase c_1 units of good 1 at the perfectly competitive price of p_1 per unit and c_2 units of good 2 at the perfectly competitive price p_2 to maximize the utility function

$$U(c_1, c_2) = c_1^{1/2} + c_2^{1/2},$$

subject to the budget constraint

$$Y \geq p_1 c_1 + p_2 c_2.$$

a. Write down the first-order conditions for the consumer's optimal choices c_1^* and c_2^* of c_1 and c_2.

b. Use your first-order conditions from part (a), together with the consumer's budget constraint, which will bind at the optimum, to find the solutions for c_1^* and c_2^* in terms of the parameters p_1, p_2, and Y.

c. Use your solutions from part (b) to find expressions for the optimal shares $p_1 c_1 / Y$ and $p_2 c_2 / Y$ of income spent on the two goods.

3. Symmetry of Marshallian Demands?

Consider two closely-related problems faced by a consumer whose preferences over two goods are described by the utility function $U(c_1, c_2)$, where c_1 is consumption of good 1 and c_2 is consumption of good 2. Both problems assume that markets are perfectly competitive: the consumer can purchase as many or as few units as he or she wishes at the price of p_1 per unit of good 1 and p_2 per unit of good 2.

In the utility-maximization problem, the consumer takes his or her income Y as given, and maximizes utility subject to a budget constraint:

$$\max_{c_1,c_2} U(c_1, c_2) \text{ subject to } Y \geq p_1c_1 + p_2c_2.$$

Assume that, for all possible values for the parameters $Y > 0$, $p_1 > 0$, and $p_2 > 0$, this problem has a unique solution. Then, as the parameter values vary, the solutions for consumption of the two goods are described by the Marshallian demand curves $c_1^* = M_1(p_1, p_2, Y)$ and $c_2^* = M_2(p_1, p_2, Y)$ and the maximized value of utility is described by the indirect utility function

$$V(p_1, p_2, Y) = \max_{c_1,c_2} U(c_1, c_2) \text{ subject to } Y \geq p_1c_1 + p_2c_2.$$

In the expenditure-minimization problem, the consumer minimizes the cost of achieving a given level of utility $\bar{U}$:

$$\min_{c_1,c_2} p_1c_1 + p_2c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U},$$

Assume again that, for all possible values for the parameters $\bar{U}$, $p_1 > 0$, and $p_2 > 0$, this problem has a unique solution. Then, as the parameter values vary, the solutions for consumption of the two goods are described by the Hicksian demand curves $c_1^* = H_1(p_1, p_2, \bar{U})$ and $c_2^* = H_2(p_1, p_2, \bar{U})$ and the minimized cost is described by the expenditure function

$$E(p_1, p_2, \bar{U}) = \min_{c_1,c_2} p_1c_1 + p_2c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U},$$

a. Assuming that the expenditure function has continuous second partial derivatives, those derivatives must satisfy the symmetry condition

$$\frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_1 \partial p_2} = \frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_2 \partial p_1}.$$

Use the envelope theorem, applied to the expenditure-minimization problem, to show that under these conditions, the Hicksian demands must satisfy the symmetry condition

$$\frac{\partial H_1(p_1, p_2, \bar{U})}{\partial p_2} = \frac{\partial H_2(p_1, p_2, \bar{U})}{\partial p_1}.$$

The question we'll address next is: under what circumstance do the Marshallian demand curves display a similar symmetry condition? It turns out that this question can be answered with repeated use of the envelope theorem.

b. As a first step, it is useful to re-derive the Slutsky equations linking the Marshallian and Hicksian demands. To begin, note that the Marshallian and Hicksian demands coincide at the point where $Y = E(p_1, p_2, \bar{U})$, so that income in the utility maximization problem equals the minimum cost achieved in the expenditure minimization problem. For all parameter combination $\bar{U}$, $p_1 > 0$, and $p_2 > 0$, therefore, the Marshallian and Hicksian demand curves satisfy

$$H_i(p_1, p_2, \bar{U}) = M_i(p_1, p_2, E(p_1, p_2, \bar{U}))$$

for $i = 1$ and $i = 2$. Use this condition, together with the implications of the envelope theorem applied to the expenditure minimization problem, to show that the Slutsky equation,

$$\frac{\partial M_i(p_1, p_2, Y)}{\partial p_j} = \frac{\partial H_i(p_1, p_2, \bar{U})}{\partial p_j} - \frac{\partial M_i(p_1, p_2, Y)}{\partial Y} M_j(p_1, p_2, Y),$$

must hold for all $i = 1, 2$ and $j = 1, 2$ when $Y = E(p_1, p_2, \bar{U})$.

c. Set $i = 1$ and $j = 2$, so that the Slutsky equation specializes to

$$\frac{\partial M_1(p_1, p_2, Y)}{\partial p_2} = \frac{\partial H_1(p_1, p_2, \bar{U})}{\partial p_2} - \frac{\partial M_1(p_1, p_2, Y)}{\partial Y} M_2(p_1, p_2, Y).$$

Now use the symmetry condition for Hicksian demands to rewrite this expression as

$$\frac{\partial M_1(p_1, p_2, Y)}{\partial p_2} = \frac{\partial H_2(p_1, p_2, \bar{U})}{\partial p_1} - \frac{\partial M_1(p_1, p_2, Y)}{\partial Y} M_2(p_1, p_2, Y),$$

and use the Slutsky equation with $i = 2$ and $j = 1$ to obtain

$$\frac{\partial M_1(p_1, p_2, Y)}{\partial p_2} = \frac{\partial M_2(p_1, p_2, Y)}{\partial p_1} + \frac{\partial M_2(p_1, p_2, Y)}{\partial Y} M_1(p_1, p_2, Y) - \frac{\partial M_1(p_1, p_2, Y)}{\partial Y} M_2(p_1, p_2, Y),$$

d. From this last expression, we can see that the symmetry condition

$$\frac{\partial M_1(p_1, p_2, Y)}{\partial p_2} = \frac{\partial M_2(p_1, p_2, Y)}{\partial p_1}$$

for Marshallian demand curves will not hold in general. By multiplying the second term on the right-hand side by

$$\left[\frac{M_2(p_1, p_2, Y)}{Y}\right] \left[\frac{Y}{M_2(p_1, p_2, Y)}\right] = 1$$

and the third term on the right-hand side by

$$\left[\frac{M_1(p_1, p_2, Y)}{Y}\right] \left[\frac{Y}{M_1(p_1, p_2, Y)}\right] = 1$$

however, you should be able to answer the following question: if the symmetry condition for Marshallian demands *does* hold, what must be true about the income elasticities of demand for goods 1 and 2?