Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2019

Tuesday, October 29, 1:30 - 2:45pm

This exam has three questions on four pages; before you begin, please check to make sure that your copy has all three questions and all four pages. Each part of each question is worth ten points. Therefore, questions 1 and 2 with three parts are worth 30 points each and question 3 with four parts is worth 40 points, for a total of 100 points overall.

1. Linear Expenditure System

Consider the problem solved by a consumer who uses his or her income Y to purchase in perfectly competitive markets c_1 units of good 1 at the price of p_1 per unit, c_2 units of good 2 at the price of p_2 per unit, and c units of good 1 at the price of p_3 per unit to maximize utility

$$U(c_1, c_2, c_3) = a_1 \ln(c_1 - x_1) + a_2 \ln(c_2 - x_2) + a_3 \ln(c_3 - x_3),$$

subject to the budget constraint

$$Y \geq p_1 c_1 + p_2 c_2 + p_3 c_3,$$

where the positive parameters a_1, a_2, and a_3 satisfy

$$a_1 + a_2 + a_3 = 1$$

and the positive parameters x_1, x_2, and x_3 are such that

$$Y - p_1 x_1 - p_2 x_2 - p_3 x_3 > 0.$$

a. Write down the Lagrangian for the consumer's utility maximization problem. Then, write down the first-order conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^*, c_2^*, and c_3^* that solve this problem along with the corresponding value of the Lagrange multiplier.

b. Next, use the first-order conditions from part (a) together with the consumer's budget constraint to find the optimal values for c_1^*, c_2^*, and c_3^* in terms of the parameters Y, p_1, p_2, p_3, a_1, a_2, a_3, x_1, x_2, and x_3. Here, you can assume that the budget constraint binds at the optimum; this will be true because utility is strictly increasing in the consumption of all three goods.

c. Finally, use your solutions from part (b) to find expressions that show how optimal expenditures $p_1 c_1^*$, $p_2 c_2^*$, and $p_3 c_3^*$ on each of the three goods depend linearly on income Y and the prices p_1, p_2, and p_3, given the parameters a_1, a_2, a_3, x_1, x_2, and x_3.

2. Le Chatelier's Principle

Consider a firm that rents capital at the competitive rate r and hires labor at the competitive wage w to produce output y according to the technology described by

$$4k^{1/4}l^{1/4} \geq y,$$

which it then sells at the competitive price p.

a. Suppose that, in the "long run," the firm freely chooses both capital and labor inputs and output to maximize profits. This long-run problem can be stated as

$$\max_{k,l,y} yp - rk - wl \text{ subject to } 4k^{1/4}l^{1/4} \geq y.$$

Use the first-order conditions and binding constraint for this problem to find the firm's optimal long-run supply function $y^*(p, r, w)$ and optimal long-run factor demand functions $k^*(p, r, w)$ and $l^*(p, r, w)$. Then, differentiate the supply function by p to obtain an expression for $\partial y^*(p, r, w)/\partial p$, describing how optimal output supply responds to a change in output price over the long run.

b. Next, suppose that in the "short run," the firm takes its capital input $k = \bar{k}$ as fixed, and simply chooses labor input and output to maximize profits. This short-run problem can be stated as

$$\max_{l,y} yp - r\bar{k} - wl \text{ subject to } 4\bar{k}^{1/4}l^{1/4} \geq y.$$

Use the first-order conditions and binding constraint for this problem to find the firm's optimal short-run supply function $y^s(p, w, \bar{k})$ and optimal short-run labor demand function $l^s(p, w, \bar{k})$. Then, differentiate the supply function by p, still holding capital fixed at $\bar{k}$, to obtain an expression for $\partial y^s(p, w, \bar{k})/\partial p$, describing how optimal output supply responds to a change in output price in the short run.

c. In general, it will not be possible to tell which is larger: the adjustment of output to a change in price in the long run or the short run. Suppose, however, that the short-run fixed capital input just happens to equal the long-run optimal capital input. By substituting $\bar{k} = k^*(p, r, w)$ into the expression for $\partial y^s(p, w, \bar{k})/\partial p$ you derived in part (b) and comparing it to the expression for $\partial y^*(p, r, w)/\partial p$ you derived in part (a), show that a version of Le Chatelier's principle holds: adjustment of output to a change in price is larger in the long run.

3. Optimal and Equilibrium Allocations

As we discussed in class, the Ramsey, or neoclassical growth, model describes an economic environment in which the two welfare theorems apply: the competitive equilibrium allocation is Pareto optimal and the Pareto optimal allocation can be supported in a competitive equilibrium.

a. To verify this, start by characterizing optimal allocations in the same way we did in class. Taking the initial capital stock $k(0) > 0$ as given, a social planner chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize a representative consumer's utility,

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t,$$

subject to the capital accumulation constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$, where $0 < \alpha < 1$ is capital's share in a Cobb-Douglas production function when labor input is normalized to equal one in every period, $\delta > 0$ is the depreciation rate for capital, and $\rho > 0$ is a constant discount rate. For this problem, the maximized current-value Hamiltonian takes the form

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

Using this form for the maximized Hamiltonian, write down the first-order condition for $c(t)$ and the pair of differential equations for $\dot{\theta}(t)$ and $\dot{k}(t)$ that characterize Pareto optimal allocations.

b. Next, consider the activities of a representative consumer in a decentralized version of the same economy. At each instant $t \in [0, \infty)$, the consumer rents $k(t)$ units of capital and supplies one unit of labor inelastically to a representative firm, earning total income $r(t)k(t) + w(t)$. Thus, taking the initial capital stock $k(0) > 0$ as given, the consumer chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t,$$

subject to the capital accumulation constraint

$$r(t)k(t) + w(t) - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. For this problem, the maximized current-value Hamiltonian takes the form

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[r(t)k(t) + w(t) - \delta k(t) - c(t)].$$

Using this form for the maximized Hamiltonian, write down the first-order condition for $c(t)$ and the pair of differential equations for $\dot{\theta}(t)$ and $\dot{k}(t)$ that characterize the consumer's optimal choices. *Note:* Strictly speaking, the maximized current-value Hamiltonian for this problem depends on $r(t)$ and $w(t)$ as well as $k(t)$ and $\theta(t)$; because the consumer takes these factor prices as given, however, you'll only need to differentiate H with respect to $k(t)$ and $\theta(t)$ to derive the differential equations describing the consumer's optimal choices.

c. Next, consider the representative firm, which rents $k(t)$ units of capital and hires $n(t)$ units of labor at each instant $t \in [0, \infty)$ to maximize profits

$$k(t)^{\alpha} n(t)^{1-\alpha} - r(t)k(t) - w(t)n(t),$$

taking $r(t)$ and $w(t)$ as given, where $0 < \alpha < 1$ denotes capital's share in the Cobb-Douglas production function. Write down the first-order conditions for the values of $k(t)$ and $n(t)$ that solve this static, unconstrained optimization problem.

d. In equilibrium, labor demand must equal labor supply, so that $n(t) = 1$, and the representative firm must earn zero profits, so that

$$k(t)^{\alpha} n(t)^{1-\alpha} = r(t)k(t) + w(t)n(t)$$

or, more simply,

$$k(t)^{\alpha} = r(t)k(t) + w(t).$$

Combine these equilibrium conditions with the optimality conditions you derived in parts (b) and (c), in order to show that the first-order condition for $c(t)$ and the differential equations describing the evolution of $\dot{\theta}(t)$ and $\dot{k}(t)$ in the decentralized economy are exactly the same as those you derived in part (a) to describe the Pareto optimal allocations.