Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
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Thursday, October 25, 1:30 - 2:45pm

This exam has four questions on four pages; before you begin, please check to make sure that your copy has all four questions and all four pages. Each question has two parts, and each part of each question is worth five points, for a total of 10 points per question and 40 points overall.

1. Utility Maximization

Consider the problem solved by a consumer who uses his or her income I to purchase in perfectly competitive markets c_1 units of good 1 at the price of p_1 per unit and c_2 units of good 2 at the price of p_2 per unit to maximize utility

$$U(c_1, c_2) = \ln(c_1) - (1/2)(c_2 - \bar{c})^2,$$

with $\bar{c} > 0$, subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2.$$

In characterizing the solution to this problem, below, assume that the optimal choices for consumption will turn out to be strictly positive so that it is not necessary to explicitly impose the nonnegativity constraints $c_1 \geq 0$ and $c_2 \geq 0$ when setting up the Lagrangian or taking the first-order conditions.

a. Write down the Lagrangian for the consumer's utility maximization problem. Then, write down the first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve this problem along with the corresponding value of the Lagrange multiplier.

b. Next, specialize the problem by setting $I = 10$, $p_1 = 2$, $p_2 = 1$, and $\bar{c} = 10$, and use your results from part (a), above, to find numerical values for the optimal choices c_1^* and c_2^*.

2. Roy's Identity

Consider the problem faced by another consumer who uses her or her income I to purchase in perfectly competitive markets c_1 units of good 1 at the price of p_1 per unit and c_2 units of good 2 at the price of p_2 per unit to maximize the utility function

$$U(c_1, c_2) = \left(\frac{\sigma}{\sigma - 1}\right) \ln \left[\left(\frac{1}{2}\right) \left(c_1^{\frac{\sigma-1}{\sigma}}\right) + \left(\frac{1}{2}\right) \left(c_2^{\frac{\sigma-1}{\sigma}}\right) \right]$$

subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2,$$

where σ, a positive parameter not equal to one ($\sigma > 0$ and $\sigma \neq 1$), measures the consumer's constant elasticity of substitution between the two goods.

a. It turns out that the indirect utility function for this problem, defined as the maximized level of utility for any given parameter configuration (p_1, p_2, I), takes the form

$$V(p_1, p_2, I) = \ln(I) - \left(\frac{1}{1 - \sigma}\right) \ln \left(p_1^{1-\sigma} + p_2^{1-\sigma}\right) + \left(\frac{\sigma}{1 - \sigma}\right) \ln(2)$$

Use this expression for the indirect utility function, together with Roy's identity (equvialently, the envelope theorem), to derive formulas for the Marshallian demand curves $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$, describing how the consumer's optimal choices of c_1 and c_2 depend on the prices p_1 and p_2 as well as his or her income I. *Note:* You can also find the Marshallian demand curves by solving the consumer's utility maximization problem, but given the indirect utility function, using Roy's identity should be easier.

b. In the limiting case where $\sigma \to 1$, so that the elasticity of substitution equals one, the utility and indirect utility functions from the problem above simplify to

$$U(c_1, c_2) = (1/2)\ln(c_1) + (1/2)\ln(c_2)$$

and

$$V(p_1, p_2, I) = \ln(I) - (1/2)\ln(p_1) - (1/2)\ln(p_2) - \ln(2).$$

What are the Marshallian demand curves $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$ now? Again, you can answer this question by applying Roy's identity to the indirect utility function, by solving the consumer's utility maximization problem, or just by remembering what the solution looks like from problem set 1.

3. The Ramsey Model

Consider a version of the Ramsey model in which the representative consumer's preferences over consumption $c(t)$ at each date $t \in [0, \infty)$ are described by a general utility function $u(c(t))$ that is strictly increasing and strictly concave, instead of taking on the specific logarithmic form as we assumed in class. For this version of the model, the social planner's problem becomes one of choosing functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize utility over the infinite horizon

$$\int_0^\infty e^{-\rho t} u(c(t))\, \mathrm{d}t,$$

where $\rho > 0$ is the consumer's discount rate, subject to the capital accumulation constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t),$$

for all $t \in [0, \infty)$, where the parameter from the aggregate production function satisfies $0 < \alpha < 1$ and the depreciation rate for capital satisfies $\delta > 0$, taking the initial capital stock $k(0)$ as given.

a. Write down an expression for the maximized current value Hamiltonian for the social planner's problem, using $\theta(t)$ to denote the multiplier on the capital accumulation constraint. Then, write down the first-order condition for $c(t)$ and the differential equations for $\theta(t)$ and $k(t)$ that, according to the maximum principle, must be satisfied by the solution to the social planner's problem.

b. Just as in the case with log utility, this version of the model has a unique nontrivial steady state, in which $c(t) = c^*$, $k(t) = k^*$, and $\theta(t) = \theta^*$ are all equal to constants, so that $\dot{k}(t) = \dot{\theta}(t) = 0$. Use the optimality conditions you derived in part (a) above, to solve for the steady-state values c^* and k^* for consumption and the capital stock in terms of the model's parameters: ρ, α, and δ.

4. Natural Resource Depletion

Let c_t denote society's consumption of an exhaustible natural resource at each date $t = 0, 1, 2, \ldots$, and suppose that a representative consumer gets utility from this resource as described by

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1 - \sigma} \right),$$

where the discount factor lies between zero and one, with $0 < \beta < 1$, and the parameter describing the curvature of the single-period utility function is strictly positive, with $\sigma > 0$.

Let s_t denote the stock of the resource that remains at the beginning of period $t = 0, 1, 2, \ldots$, and consider a social planner who takes the initial stock s_0 of the exhaustible resource as given and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize the representative consumer's utility subject to the constraints

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, \ldots$, which indicate that since the resource is nonrenewable, consumption c_t at each date t subtracts from the stock s_t that remains at the beginning of t to determine the stock s_{t+1} that remains at the beginning of $t + 1$.

a. As we discussed in class, discrete-time dynamic optimization problems like this one can be solved using either the method of Lagrange multipliers or the maximum principle. Use whichever method you find easiest or most convenient to derive a set of optimality conditions that, together with the initial condition s_0 given and the transversality condition, which for this problem is

$$\lim_{T \to \infty} \beta^T c_T^{-\sigma} s_{T+1} = 0,$$

describe the evolution of the optimal choices for c_t and s_t.

b. Use your optimality conditions from part (a), above, to derive an expression that shows how the optimal growth rate of consumption, c_{t+1}/c_t, depends on the preference parameters β and σ.