Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
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1. Subsistence Consumption

The utility-maximizing consumer solves

$$\max_{c_1,c_2} \ \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

a. The Lagrangian for the consumer's problem is

$$L(c_1, c_2, \lambda) = \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(c_2) + \lambda(I - p_1 c_1 + p_2 c_2),$$

and the first-order conditions for the values c_1^* and c_2^* that solve this problem together with the associated value λ^* are

$$\frac{\alpha}{c_1^* - \gamma} - \lambda^* p_1 = 0$$

and

$$\frac{1-\alpha}{c_2^*} - \lambda^* p_2 = 0.$$

b. Rearrange the first-order conditions to read

$$c_1^* = \frac{\alpha}{\lambda^* p_1} + \gamma$$

and

$$c_2^* = \frac{1-\alpha}{\lambda^* p_2},$$

Then substitute these expressions into the binding budget constraint to solve for λ^*:

$$I = p_1 c_1^* + p_2 c_2^* = \frac{\alpha}{\lambda^*} + p_1\gamma + \frac{1-\alpha}{\lambda^*} = \frac{1}{\lambda^*} + p_1\gamma$$

and hence

$$\lambda^* = \frac{1}{I - p_1\gamma}.$$

Now substitute this solution for λ^* back into the previous expressions for c_1^* and c_2^* to obtain

$$c_1^* = \frac{\alpha(I - p_1\gamma)}{p_1} + \gamma$$

and

$$c_2^* = \frac{(1-\alpha)(I - p_1\gamma)}{p_2}.$$

Intuitively, the consumer uses the first $p_1\gamma$ units of income to achieve the subsistence level γ for good 1, then divides the remaining income $I - p_1\gamma$ into shares α and $1-\alpha$ spent on goods 1 and 2.

c. Taking the partial derivatives

$$\frac{\partial c_1^*}{\partial I} = \frac{\alpha}{p_1} > 0,$$
$$\frac{\partial c_2^*}{\partial I} = \frac{1-\alpha}{p_2} > 0,$$
$$\frac{\partial c_1^*}{\partial p_1} = -\frac{\alpha I}{p_1^2} < 0,$$

and

$$\frac{\partial c_2^*}{\partial p_2} = -\frac{(1-\alpha)(I - p_1\gamma)}{p_2^2} < 0,$$

confirms that both goods are normal and ordinary.

2. Leisure Demand and Labor Supply

Now the consumer solves

$$\max_{c_1,\ell} \ \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(\ell) \text{ subject to } I + w(T-\ell) \geq p_1 c_1.$$

a. The Lagrangian for the problem is

$$L(c_1, \ell, \lambda) = \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(\ell) + \lambda(I + wT - p_1 c_1 - w\ell),$$

and the first-order conditions are

$$\frac{\alpha}{c_1^* - \gamma} - \lambda^* p_1 = 0$$

and

$$\frac{1-\alpha}{\ell^*} - \lambda^* w = 0.$$

b. Rearrange the first-order conditions to read

$$c_1^* = \frac{\alpha}{\lambda^* p_1} + \gamma$$

and

$$\ell^* = \frac{1-\alpha}{\lambda^* w}.$$

Then substitute these expressions into the binding budget constraint to solve for λ^*:

$$I + wT = \frac{\alpha}{\lambda^*} + p_1\gamma + \frac{1-\alpha}{\lambda^*} = \frac{1}{\lambda^*} + p_1\gamma$$

and hence

$$\lambda^* = \frac{1}{I + wT - p_1\gamma}.$$

Now substitute this solution for λ^* back into the previous expressions for c_1^* and ℓ^* to obtain

$$c_1^* = \frac{\alpha(I + wT - p_1\gamma)}{p_1} + \gamma$$

and

$$\ell^* = \frac{(1-\alpha)(I + wT - p_1\gamma)}{w}.$$

c. Taking the partial derivatives

$$\frac{\partial c_1^*}{\partial I} = \frac{\alpha}{p_1} > 0,$$

and

$$\frac{\partial \ell^*}{\partial I} = \frac{1-\alpha}{w} > 0$$

shows that consumption and leisure both increase when I rises. Likewise, taking the partial derivative

$$\frac{\partial c_1^*}{\partial p_1} = -\frac{\alpha(I + wT)}{p_1^2} < 0$$

shows that c_1^* decreases when p_1 rises. However, the partial derivative

$$\frac{\partial \ell^*}{\partial w} = -\frac{(1-\alpha)(I - p_1\gamma)}{w^2}$$

reveals that now, the possibility that $I < p_1\gamma$ or $I > p_1\gamma$ allows leisure to either rise or fall when the wage increases.

3. A Generalized Model with Endowments

The utility-maximizing consumer solves

$$\max_{c_1, c_2} \ \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(c_2) \text{ subject to } I + p_1 y_1 + p_2 y_2 \geq p_1 c_1 + p_2 c_2.$$

a. The Lagrangian for the consumer's problem is

$$L(c_1, c_2, \lambda) = \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(c_2) + \lambda(I + p_1 y_1 + p_2 y_2 - p_1 c_1 + p_2 c_2),$$

and the first-order conditions are

$$\frac{\alpha}{c_1^* - \gamma} - \lambda^* p_1 = 0$$

and

$$\frac{1-\alpha}{c_2^*} - \lambda^* p_2 = 0.$$

b. Rearrange the first-order conditions to read

$$c_1^* = \frac{\alpha}{\lambda^* p_1} + \gamma$$

and

$$c_2^* = \frac{1-\alpha}{\lambda^* p_2},$$

Then substitute these expressions into the binding budget constraint to solve for λ^*:

$$I + p_1 y_1 + p_2 y_2 = p_1 c_1^* + p_2 c_2^* = \frac{\alpha}{\lambda^*} + p_1\gamma + \frac{1-\alpha}{\lambda^*} = \frac{1}{\lambda^*} + p_1\gamma$$

and hence

$$\lambda^* = \frac{1}{I + p_1 y_1 + p_2 y_2 - p_1\gamma}.$$

Now substitute this solution for λ^* back into the previous expressions for c_1^* and c_2^* to obtain

$$c_1^* = \frac{\alpha(I + p_1 y_1 + p_2 y_2 - p_1\gamma)}{p_1} + \gamma$$

and

$$c_2^* = \frac{(1-\alpha)(I + p_1 y_1 + p_2 y_2 - p_1\gamma)}{p_2}.$$

c. Taking the partial derivative

$$\frac{\partial c_1^*}{\partial p_1} = -\frac{\alpha(I + p_2 y_2)}{p_1^2} < 0$$

confirms that c_1^* decreases when p_1 rises. However, the partial derivative

$$\frac{\partial c_2^*}{\partial p_2} = -\frac{(1-\alpha)(I + p_1 y_1 - p_1\gamma)}{p_2^2}$$

shows that c_2^* can rise or fall as p_2 rises, depending on whether $I + p_1 y_1 - p_1\gamma < 0$ or $I + p_1 y_1 - p_1\gamma > 0$.

4. Hicksian Demands

The expenditure minimization problem is

$$\min_{c_1, c_2} \; p_1 c_1 + p_2 c_2 \text{ subject to } \alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(c_2) \geq \bar{U}.$$

a. The Lagrangian for the problem is

$$L(c_1, c_2, \lambda) = p_1 c_1 + p_2 c_2 - \lambda[\alpha \ln(c_1 - \gamma) + (1-\alpha)\ln(c_2) - \bar{U}],$$

and the first-order conditions are

$$p_1 - \frac{\lambda^* \alpha}{c_1^* - \gamma} = 0$$

and

$$p_2 - \frac{\lambda^* (1-\alpha)}{c_2^*}.$$

b. Rearrange the first-order conditions to read

$$c_1^* - \gamma = \frac{\lambda^* \alpha}{p_1}$$

and

$$c_2^* = \frac{\lambda^* (1-\alpha)}{p_2}.$$

Then substitute these expressions into the binding constraint to solve for λ^*:

$$\bar{U} = \alpha[\ln(\lambda^*) + \ln(\alpha) - \ln(p_1)] + (1-\alpha)[\ln(\lambda^*) + \ln(1-\alpha) - \ln(p_2)]$$

and hence

$$\ln(\lambda^*) = \bar{U} + \alpha \ln(p_1) + (1-\alpha)\ln(p_2) - \alpha \ln(\alpha) - (1-\alpha)\ln(1-\alpha)$$

or

$$\lambda^* = \frac{p_1^\alpha p_2^{1-\alpha} \exp(\bar{U})}{\alpha^\alpha (1-\alpha)^{1-\alpha}}.$$

Now substitute this solution for λ^* back into the previous expressions for c_1^* and c_2^* to obtain

$$c_1^* = \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \left(\frac{p_2}{p_1}\right)^{1-\alpha} \exp(\bar{U}) + \gamma$$

and

$$c_2^* = \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} \left(\frac{p_1}{p_2}\right)^{\alpha} \exp(\bar{U}).$$

c. Taking the partial derivatives

$$\frac{\partial c_1^*}{\partial p_1} = (\alpha - 1) \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \left(\frac{p_2}{p_1}\right)^{1-\alpha} \frac{\exp(\bar{U})}{p_1} < 0$$

and

$$\frac{\partial c_2^*}{\partial p_2} = -\alpha \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} \left(\frac{p_1}{p_2}\right)^{\alpha} \frac{\exp(\bar{U})}{p_2} < 0$$

confirms that both Hicksian demand curves slope downward.

5. A Modified Slutsky Equation

Start with the identity

$$c_2(p_1, p_2, e(p_1, p_2, \bar{U}) - p_1 y_1 - p_2 y_2, y_1, y_2) = h_2(p_1, p_2, \bar{U}).$$

Since this equality must hold for all possible combinations of p_1, p_2, $\bar{U}$, y_1, and y_2, it will continue to hold after differentiating both sides by p_2 to obtain

$$\frac{\partial c_2}{\partial p_2} + \frac{\partial c_2}{\partial I}\left[\frac{\partial e(p_1, p_2, \bar{U})}{\partial p_1} - y_2\right] = \frac{\partial h_2}{\partial p_2}.$$

The envelope theorem applied to the expenditure minimization problem implies that

$$\frac{\partial e(p_1, p_2, \bar{U})}{\partial p_2} = h_2(p_1, p_2, \bar{U}) = c_2(p_1, p_2, e(p_1, p_2, \bar{U}) - p_1 y_1 - p_2 y_2, y_1, y_2) = c_2^*.$$

Therefore,

$$\frac{\partial c_2}{\partial p_2} = \frac{\partial h_2}{\partial p_2} - \frac{\partial c_2}{\partial I}(c_2^* - y_2).$$

In the case without endowments, the function c_2 is the Marshallian demand curve and this last result corresponds to the Slutsky equation. In that case, the equation makes clear that a Giffen good must be inferior, with $\partial c_2/\partial I < 0$, to offset the substitution effect implied by the Hicksian demand curve and produce an increase in c_2^* when p_2 rises. In the case with endowments, c_2^* can rise when p_2 goes up even when $\partial c_2/\partial I > 0$, provided the consumer is a net supplier of good 2, setting $c_2^* < y_2$.