Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2024

Due Thursday, November 7

1. Profit Maximization

The profit-maximizing firm solves

$$\max_n 10n - n^2 - wn \text{ subject to } n \geq 0.$$

a. With the Lagrangian defined as

$$L(n, \lambda) = 10n - n^2 - wn + \lambda n,$$

the Kuhn-Tucker conditions describing the solution to the firm's problem are the first-order condition

$$10 - 2n^* - w + \lambda^* = 0.$$

the constraint

$$n^* \geq 0,$$

the nonnegativity conditiion

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^* n^* = 0.$$

Suppose first that $n^* = 0$. Then the first-order condition requires

$$\lambda^* = w - 10,$$

which is greater than or equal to zero as required by the nonnegativity condition only when $w \geq 10$. Suppose on the other hand that $n^* > 0$. Then the complementary slackness condition requires $\lambda^* = 0$ and the first-order conditions implies that

$$n^* = \frac{10 - w}{2},$$

which is greater than or equal to zero as required by the constraint only when $0 < w \leq 10$. Combining these observations shows that $n^* = 0$ if $w \geq \bar{w}$, where $\bar{w} = 10$.

b. The analysis from above also shows that when $0 < w < \bar{w} = 10$, the firm's labor demand curve is

$$n(w) = \frac{10 - w}{2}.$$

c. With the profit function defined as

$$\pi(w) = \max_{n} 10n - n^2 - wn \text{ subject to } n \geq 0,$$

the results derived above imply that

$$\pi(w) = 0$$

for $w \geq \bar{w}$ and

$$\pi(w) = (10 - w)\left(\frac{10 - w}{2}\right) - \left(\frac{10 - w}{2}\right)^2 = \frac{1}{4}(10 - w)^2,$$

for $0 < w < \bar{w}$.

d. Differentiating the expression for $\pi(w)$ derived above with respect to w yields

$$\pi'(w) = -\frac{10 - w}{2}.$$

for all $w < \bar{w} = 10$. Note that this same result follows from the envelope theorem:

$$\pi'(w) = -n(w) = -\frac{10 - w}{2}.$$

2. Optimal Risk Sharing

At $t = 0$, the social planner chooses consumptions $c_j(s)$, $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$, for all consumers in all states of the world to maximize

$$\sum_{j=1}^{J} \sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\},$$

subject to the aggregate resource constraints

$$\sum_{j=1}^{J} y_j(s) \geq \sum_{j=1}^{J} c_j(s)$$

for all $s = 1, 2, \ldots, S$. The Lagrangian for this problem can be written as

$$\sum_{j=1}^{J} \sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\} + \sum_{s=1}^{S} \lambda(s) \left[\sum_{j=1}^{J} y_j(s) - \sum_{j=1}^{J} c_j(s) \right].$$

a. Fixing values of j and s, differentiating the Lagrangian by $c_j(s)$ for those values of j, and s, and setting the result equal to zero to yields the first-order conditions

$$\pi(s) \exp[-\sigma c_j^*(s)] - \lambda^*(s) = 0$$

for all $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$.

b. Rearranging each first-order condition yields

$$c_j^*(s) = \frac{\ln[\pi(s)] - \ln[\lambda^*(s)]}{\sigma}$$

for all $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$.

c. Substituting these solutions for $c_j^*(s)$ into the binding constraint for state s yields

$$Jy_a(s) = \sum_{j=1}^{J} y_j(s) = \sum_{j=1}^{J} c_j^*(s) = J\left\{\frac{\ln[\pi(s)] - \ln[\lambda^*(s)]}{\sigma}\right\},$$

which implies

$$\lambda^*(s) = \pi(s)\exp[-\sigma y_a(s)]$$

for all $s = 1, 2, \ldots, S$.

d. Substituting this solution for $\lambda^*(s)$ back into the previous expression for $c_j^*(s)$ provides the solutions

$$c_j^*(s) = y_a(s)$$

for all $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$. This optimal allocation insures each consumer against the shock to his or her own individual level of productivity, but requires that all consumers reduce their consumption when average productivity falls.

3. Equilibrium Risk Sharing

At $t = 0$, each consumer chooses consumptions $c_j(s)$, $s = 1, 2, \ldots, S$, to maximize

$$\sum_{s=1}^{S} \pi(s)\left\{-\frac{1}{\sigma}\exp[-\sigma c_j(s)]\right\},$$

subject to the budget constraint

$$\sum_{s=1}^{S} p(s)y_j(s) \geq \sum_{s=1}^{S} p(s)c_j(s).$$

The Lagrangian for this problem can be written as

$$\sum_{s=1}^{S} \pi(s)\left\{-\frac{1}{\sigma}\exp[-\sigma c_j(s)]\right\} + \lambda_j\left[\sum_{s=1}^{S} p(s)y_j(s) - \sum_{s=1}^{S} p(s)c_j(s)\right].$$

a. Fixing a value of s, differentiating the Lagrangian by $c_j(s)$ for that value of s, and setting the result equal to zero yields the first-order conditions

$$\pi(s)\exp[-\sigma c_j^*(s)] - \lambda_j^* p(s) = 0$$

for all $s = 1, 2, \ldots, S$.

b. Rearranging each first-order condition yields

$$c_j^*(s) = \frac{\ln[\pi(s)] - \ln(\lambda_j^*) - \ln[p(s)]}{\sigma}$$

for all $s = 1, 2, \ldots, S$.

c. If, as conjectured,

$$p(s) = \pi(s)\exp[-\sigma y_a(s)],$$

for all $s = 1, 2, \ldots, S$, then the expression for $c_j^*(s)$ derived above simplifies to

$$c_j^*(s) = y_a(s) - \frac{\ln(\lambda_j^*)}{\sigma}.$$

Using this expression for $c_j^*(s)$, as well as the symmetry condition for initial wealth,

$$\sum_{s=1}^{S} p(s)y_j(s) = \sum_{s=1}^{S} p(s)y_a(s)$$

for all $j = 1, 2, \ldots, J$, consumer j's binding budget constraint implies

$$\sum_{s=1}^{S} p(s)y_a(s) = \sum_{s=1}^{S} p(s)y_j(s) = \sum_{s=1}^{S} p(s)c_j^*(s) = \sum_{s=1}^{S} p(s)y_a(s) - \sum_{s=1}^{S} p(s)\left[\frac{\ln(\lambda_j^*)}{\sigma}\right]$$

or, more simply,

$$0 = \left[\frac{\ln(\lambda_j^*)}{\sigma}\right]\sum_{s=1}^{S} p(s).$$

Since $\sigma > 0$ and $p(s) > 0$ for all $s = 1, 2, \ldots, S$, this condition only holds if $\ln(\lambda_j^*) = 0$ or $\lambda_j^* = 1$.

d. Substituting this solution for λ_j^* back into the expression for $c_j^*(s)$ confirms that

$$c_j^*(s) = y_a(s)$$

for all $s = 1, 2, \ldots, S$. In equilibrium, as under the optimal allocation, each consumer is able fully insure against the shock to his or her own individual level of productivity but not against aggregate fluctuations in reflected in average productivity.