Solutions to Midterm Exam

ECON 772001 - Math for Economists
Boston College, Department of Economics

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1. Hicksian Demands

The Hicksian demand curves can be derived from the expenditure minimization problem

$$\min_{c_a, c_b} \; p_a c_a + p_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}.$$

a. With the Lagrangian defined as

$$L(c_a, c_b, \lambda) = p_a c_a + p_b c_b - \lambda[\alpha \ln(c_a) + (1-\alpha)\ln(c_b) - \bar{U}],$$

the first-order conditions for the consumer's optimal choices c_a^* and c_b^* are

$$p_a - \frac{\lambda^* \alpha}{c_a^*} = 0$$

and

$$p_a - \frac{\lambda^*(1-\alpha)}{c_b^*} = 0.$$

b. Rewrite the first-order conditions from part (a) as

$$c_a^* = \frac{\lambda^* \alpha}{p_a}$$

and

$$c_b^* = \frac{\lambda^*(1-\alpha)}{p_b},$$

then substitute these expressions into the binding constraint to obtain

$$\ln(\lambda^*) + \alpha \ln(\alpha) + (1-\alpha)\ln(1-\alpha) - \alpha \ln(p_a) - (1-\alpha)\ln(p_b) = \bar{U}$$

or

$$\lambda^* = \exp\{\bar{U} - \alpha \ln(\alpha) - (1-\alpha)\ln(1-\alpha) + \alpha \ln(p_a) + (1-\alpha)\ln(p_b)\} = \frac{\exp(\bar{U}) p_a^\alpha p_b^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}}.$$

Finally, substitute this solution for λ^* back into the first-order conditions to find the Hicksian demand functions

$$c_a^* = h_a(p_a, p_b, \bar{U}) = \exp(\bar{U}) \left[\frac{\alpha p_b}{(1-\alpha)p_a}\right]^{1-\alpha}$$

and

$$c_b^* = h_b(p_a, p_b, \bar{U}) = \exp(\bar{U}) \left[\frac{(1-\alpha)p_a}{\alpha p_b}\right]^{\alpha}.$$

c. To check the answers from part (b), note that when

$$\bar{U} = V(p_a, p_b, Y) = \alpha \ln(\alpha) + (1-\alpha)\ln(1-\alpha) - \alpha \ln(p_a) - (1-\alpha)\ln(p_b) + \ln(Y)$$

and therefore

$$\exp(\bar{U}) = \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} Y}{p_a^\alpha p_b^{1-\alpha}},$$

the Hicksian demands satisfy

$$h_a[p_a, p_b, V(p_a, p_b, Y)] = \left[\frac{\alpha^\alpha (1-\alpha)^{1-\alpha} Y}{p_a^\alpha p_b^{1-\alpha}}\right] \left[\frac{\alpha p_b}{(1-\alpha)p_a}\right]^{1-\alpha} = \frac{\alpha Y}{p_a} = c_a(p_a, p_b, Y)$$

and

$$h_b[p_a, p_b, V(p_a, p_b, Y)]) = \left[\frac{\alpha^\alpha (1-\alpha)^{1-\alpha} Y}{p_a^\alpha p_b^{1-\alpha}}\right] \left[\frac{(1-\alpha)p_a}{\alpha p_b}\right]^{\alpha} = \frac{(1-\alpha)Y}{p_b} = c_b(p_a, p_b, Y).$$

2. The Cost of Living

Let p_a and p_b denote the prices of apples and bananas in a base year, labeled $t = 0$, and let q_a and q_b denote the prices of apples and bananas in a subsequent year, labeled $t = 1$. Suppose the consumer's preferences are once again described by the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b).$$

a. Define the minimum cost function with reference to the expenditure minimization problem studied above, in question 1, so that for any fixed utility level $\bar{U}$,

$$C(p_a, p_b, \bar{U}) = \min_{c_a, c_b} \; p_a c_a + p_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}$$

and

$$C(q_a, q_b, \bar{U}) = \min_{c_a, c_b} \; q_a c_a + q_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}.$$

The Hicksian demand curves derived in answering question 1 then imply that

$$\begin{aligned} C(p_a, p_b, \bar{U}) &= p_a \exp(\bar{U}) \left[\frac{\alpha p_b}{(1-\alpha)p_a}\right]^{1-\alpha} + p_b \exp(\bar{U}) \left[\frac{(1-\alpha)p_a}{\alpha p_b}\right]^{\alpha} \\ &= \exp(\bar{U}) p_a^\alpha p_b^{1-\alpha} \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right] \end{aligned}$$

and similarly

$$C(q_a, q_b, \bar{U}) = \exp(\bar{U}) q_a^\alpha q_b^{1-\alpha} \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right].$$

It follows from these expressions that

$$P(q_a, q_b, p_a, p_b, \bar{U}) = \frac{C(q_a, q_b, \bar{U})}{C(p_a, p_b, \bar{U})} = \left(\frac{q_a}{p_a}\right)^{\alpha} \left(\frac{q_b}{p_b}\right)^{1-\alpha}.$$

In particular, because the utility function is homothetic, this "true" cost of living index does not depend on the utility level $\bar{U}$.

b. Using the Marshallian demands curves

$$c_a(p_a, p_b, Y) = \frac{\alpha Y}{p_a}$$

and

$$c_b(p_a, p_b, Y) = \frac{(1-\alpha)Y}{p_b},$$

the Laspeyres price index is

$$L(q_a, q_b, p_a, p_b, Y)] = \frac{c_a(p_a, p_b, Y)q_a + c_b(p_a, p_b, Y)q_b}{c_a(p_a, p_b, Y)p_a + c_b(p_a, p_b, Y)p_b} = \alpha \left(\frac{q_a}{p_a}\right) + (1-\alpha)\left(\frac{q_b}{p_b}\right).$$

Again because the utility function is homothetic, the Laspeyres price index does not depend on the level of income Y. But it does not coincide exactly with the true cost of living index from part (a).

c. Since the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b)$$

implies that the budget shares of apples and bananas are constant and equal to α and $1-\alpha$, the Törnqvist-Thiel price index

$$T(q_a, q_b, p_a, p_b, Y) = \left(\frac{q_a}{p_a}\right)^{\alpha} \left(\frac{q_b}{p_b}\right)^{1-\alpha}.$$

coincides with exactly the true price index from part (a).

3. Portfolio Allocation: Part 1

The individual investor solves the constrained optimization problem

$$\max_{w,\mu_p,\sigma_p} U(\mu_p, \sigma_p) \text{ subject to } w\mu_r + (1-w)r_f \geq \mu_p \text{ and } \sigma_p \geq w\sigma_r.$$

a. With the Lagrangian for the investor's problem defined as

$$L(w, \mu_p, \sigma_p, \lambda_1, \lambda_2) = U(\mu_p, \sigma_p) + \lambda_1[w\mu_r + (1-w)r_f - \mu_p] + \lambda_2(\sigma_p - w\sigma_r),$$

the first-order conditions can be written as

$$\lambda_1^*(\mu_r - r_f) - \lambda_2^*\sigma_r,$$

$$U_1(\mu_p^*, \sigma_p^*) - \lambda_1^* = 0,$$

and

$$U_2(\mu_p^*, \sigma_p)^* + \lambda_2^* = 0.$$

b. Use the first-order conditions for μ_p^* and σ_p^* to solve for λ_1^* and λ_2^*, and substitute these solutions into the first-order condition for w^* to get the desired result that the investor sets the marginal rate of substitution between risk and return equal to the "Sharpe ratio" of the risky mutual fund:

$$-\frac{U_2(\mu_p^*, \sigma_p^*)}{U_1(\mu_p^*, \sigma_p^*)} = \frac{\mu_r - r_f}{\sigma_r}.$$

4. Portfolio Management

The professional funds manager solves the constrained optimization problem

$$\max_{v,\mu_r,\sigma_r} \frac{\mu_r - r_f}{\sigma_r} \text{ subject to } v\mu_1 + (1-v)\mu_2 \geq \mu_r$$
$$\text{and } \sigma_r \geq [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2}.$$

a. With the Lagrangian for the investor's problem defined as

$$L(v, \mu_r, \sigma_r, \lambda_1, \lambda_2) = \frac{\mu_r - r_f}{\sigma_r} + \lambda_1[v\mu_1 + (1-v)\mu_2 - \mu_r]$$
$$+ \lambda_2\{\sigma_r - [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2}\},$$

the first-order conditions can be written as

$$\lambda_1^*(\mu_1 - \mu_2) - \lambda_2^* \left\{ \frac{v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}}{[v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}]^{1/2}} \right\} = 0$$

$$\frac{1}{\sigma_r^*} - \lambda_1^* = 0,$$

and

$$-\frac{\mu_r^* - r_f}{\sigma_r^{*2}} + \lambda_2^* = 0.$$

b. Use the first-order conditions for μ_r^* and σ_r^* to solve for λ_1^* and λ_2^*, and substitute these solutions together with the binding constrains for μ_r^* and σ_r^* into the first-order condition for v^* to get the desired result that v^* must satisfy

$$\mu_1 - \mu_2 = \frac{[v^*\mu_1 + (1-v^*)\mu_2 - r_f][v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}]}{v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}}.$$

5. Portfolio Allocation: Part 2

When the individual investor chooses between individual stocks, he or she solves the constrained optimization problem

$$\max_{w,v,\mu_r,\sigma_r,\mu_p,\sigma_p} U(\mu_p,\sigma_p) \text{ subject to } v\mu_1 + (1-v)\mu_2 \geq \mu_r,$$
$$\sigma_r \geq [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2},$$
$$w\mu_r + (1-w)r_f \geq \mu_p,$$
$$\text{and } \sigma_p \geq w\sigma_r.$$

a. With the Lagrangian for the investor's problem defined as

$$\begin{aligned} L(w,v,\mu_r,\sigma_r,\mu_p,\sigma_p,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = {} & U(\mu_p,\sigma_p) \\ & + \lambda_1[v\mu_1 + (1-v)\mu_2 - \mu_r] \\ & + \lambda_2\{\sigma_r - [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2}\}, \\ & + \lambda_3[w\mu_r + (1-w)r_f - \mu_p] \\ & + \lambda_4(\sigma_p - w\sigma_r), \end{aligned}$$

the first-order conditions can be written as

$$\lambda_3^*(\mu_r^* - r_f) - \lambda_4^*\sigma_r^* = 0,$$
$$\lambda_1^*(\mu_1 - \mu_2) - \lambda_2^*\left\{\frac{v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}}{[v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}]^{1/2}}\right\} = 0,$$
$$-\lambda_1^* + \lambda_3^* w^* = 0,$$
$$\lambda_2^* - \lambda_4^* w^* = 0,$$
$$U_1(\mu_p^*,\sigma_p^*) - \lambda_3^* = 0,$$

and

$$U_2(\mu_p^*,\sigma_p)^* + \lambda_4^* = 0.$$

b. Use the first-order conditions for μ_p^* and σ_p^* to solve for λ_3^* and λ_4^*, and substitute these solutions into the first-order condition for w^* to get the same result as in question 3:

$$-\frac{U_2(\mu_p^*,\sigma_p^*)}{U_1(\mu_p^*,\sigma_p^*)} = \frac{\mu_r^* - r_f}{\sigma_r^*}.$$

Then use the first-order conditions for w^*, μ_r^*, and σ_r^* to rewrite the first-order condition for v^* as

$$\mu_1 - \mu_2 = \frac{[v^*\mu_1 + (1-v^*)\mu_2 - r_f][v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}]}{v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}},$$

which replicates the condition from question 4 that dictates the professional funds manager's choice of v^*.