Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, November 1

1. Properties of Hicksian Demands

The consumer solves

$$\min_{c_1,c_2} \; p_1c_1 + p_2c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

a. With the Lagrangian defined as

$$L(c_1, c_2, \lambda) = p_1c_1 + p_2c_2 - \lambda[U(c_1, c_2) - \bar{U}],$$

the first-order conditions for the consumer's problem are

$$p_1 - \lambda^* U_1(c_1^*, c_2^*) = 0$$

and

$$p_2 - \lambda^* U_2(c_1^*, c_2^*) = 0$$

b. The first-order conditions from part (a) indicate that so long as the goods prices p_1 and p_2 are strictly positive and the utility function U is strictly increasing in both its arguments, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices c_1^* and c_2^* and the corresponding value of the Lagrange multiplier. Assuming that the utility function is also such that the consumer's expenditure minimization problem has a unique solution, these three equations define the Hicksian demand functions $c_1^* = h_1(p_1, p_2, \bar{U})$ and $c_2^* = h_2(p_1, p_2, \bar{U})$. In particular, these Hicksian demands must satisfy the efficiency condition

$$\frac{p_1}{p_2} = \frac{U_1[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})]}{U_2[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})]} \tag{1}$$

and the binding constraint

$$U[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})] = \bar{U} \tag{2}$$

for all values of $(p_1, p_2, \bar{U})$. Differentiating (2) with respect to p_1 yields

$$U_1[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})]\frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_1}$$
$$+ U_2[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})]\frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_1} = 0.$$

Moving the first term from the left to the right, dividing through by U_2, and using (1) then shows that

$$\frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} = -\left(\frac{p_1}{p_2}\right)\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1}.$$

Similarly, differentiating (2) with respect to p_2 yields

$$U_1[h_1(p_1,p_2,\bar{U}),h_2(p_1,p_2,\bar{U})]\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2}$$
$$+ U_2[h_1(p_1,p_2,\bar{U}),h_2(p_1,p_2,\bar{U})]\frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} = 0.$$

Moving the first term from the left to the right, dividing through by U_2, and using (1) then shows that

$$\frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} = -\left(\frac{p_1}{p_2}\right)\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2}.$$

c. Now define the minimum expenditure function E as

$$E(p_1,p_2,\bar{U}) = \min_{c_1,c_2}\ p_1c_1 + p_2c_2 \text{ subject to } U(c_1,c_2) \geq \bar{U}.$$

The envelope theorem implies that

$$\frac{\partial E(p_1,p_2,\bar{U})}{\partial p_1} = h_1(p_1,p_2,\bar{U}),$$
$$\frac{\partial E(p_1,p_2,\bar{U})}{\partial p_2} = h_2(p_1,p_2,\bar{U}),$$

and hence

$$\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} = \frac{\partial^2 E(p_1,p_2,\bar{U})}{\partial p_1 \partial p_2} = \frac{\partial^2 E(p_1,p_2,\bar{U})}{\partial p_2 \partial p_1} = \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1}.$$

d. Finally, consider the matrix

$$\begin{bmatrix} \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1} & \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} \\ \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} & \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Given a value for

$$a = \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1},$$

the results from (b) and (c) imply that

$$c = \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} = -\left(\frac{p_1}{p_2}\right)\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1} = -a\left(\frac{p_1}{p_2}\right),$$
$$b = \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} = \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} = -a\left(\frac{p_1}{p_2}\right),$$

and

$$d = \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} = -\left(\frac{p_1}{p_2}\right)\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} = a\left(\frac{p_1}{p_2}\right)^2.$$

2. Cost Minimization and the Production Function

The firm solves

$$\min_{k,l} \; rk + wl \text{ subject to } f(k,l) \geq \bar{y}.$$

a. With the Lagrangian defined as

$$L(k,l,\lambda) = rk + wl - \lambda[f(k,l) - \bar{y}],$$

the first-order conditions for the firm's problem are

$$r - \lambda^* f_1(k^*, l^*) = 0$$

and

$$w - \lambda^* f_2(k^*, l^*) = 0.$$

b. The first-order conditions from part (a) indicate that so long as the factor prices r and w are strictly positive and the production function is strictly increasing in both of its arguments, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices k^* and l^* and the corresponding value of the Lagrange multiplier. Assuming that the production function is also such that the firm's cost minimization problem as a unique solution, these three equations define, in particular, the conditional factor demand curves $k^* = k(r,w,\bar{y})$ and $l^* = l(w,r,\bar{y})$. In particular, these conditional factor demands must satisfy the efficiency condition

$$\frac{r}{w} = \frac{f_1[k(r,w,\bar{y}), l(r,w,\bar{y})]}{f_2[k(r,w,\bar{y}), l(r,w,\bar{y})]} \tag{3}$$

and the binding constraint

$$f[k(r,w,\bar{y}), l(r,w,\bar{y})] = \bar{y} \tag{4}$$

for all values of $(r,w,\bar{y})$. Note that the factor prices r and w enter directly into these optimality conditions only through their ratio r/w. This tells us that the conditional factor demand curves will be homogenous of degree zero in the factor prices: the optimal choices k^* and l^* will not change when r and w both double, with the output requirement $\bar{y}$ left unchanged. It also tells us that the minimum cost function, defined by

$$C(r,w,\bar{y}) = \min_{k,l} \; rk + wl \text{ subject to } f(k,l) \geq \bar{y},$$

will be homogeneous of degree one in the factor prices: the value of $C(r,w,\bar{y})$ will double when r and w both double with the output requirement $\bar{y}$ left unchanged.

c. Suppose that, after recognizing this property of the cost function, we hypothesize further that the cost function takes the specific form

$$C(r,w,\bar{y}) = \bar{y}^b r^a w^{1-a}, \tag{5}$$

where the exponents on the factor prices sum to one. Can we determine what the assumed functional form (5) for the cost function implies for the form of the firm's production function $f(k,l)$? The answer is yes! To see how, start by applying the envelope theorem (in this case, Shephard's lemma) to the firm's cost minimization problem, to obtain two equations that link the conditional factor demands $k(r,w,\bar{y})$ and $l(r,w,\bar{y})$ to the partial derivatives of the cost function. Now we know that

$$k(r,w,\bar{y}) = \frac{\partial C(r,w,\bar{y})}{\partial r} = a\bar{y}^b r^{a-1} w^{1-a}$$

and

$$l(r,w,\bar{y}) = \frac{\partial C(r,w,\bar{y})}{\partial w} = (1-a)\bar{y}^b r^a w^{-a}.$$

d. Rearranging and then combining these two results shows that when $k = k(r,w,\bar{y})$ and $l = l(r,w,\bar{y})$ are chosen by the cost-minimizing firm,

$$\left(\frac{a\bar{y}^b}{k}\right)^{1/(1-a)} = \frac{r}{w} = \left[\frac{l}{(1-a)\bar{y}^b}\right]^{1/a}$$

or, perhaps more simply,

$$\left(\frac{a\bar{y}^b}{k}\right)^{a} = \left[\frac{l}{(1-a)\bar{y}^b}\right]^{1-a},$$

or, even more simply,

$$\bar{y} = Ak^{a/b} l^{(1-a)/b},$$

where

$$A = \left[\frac{1}{a^a(1-a)^{1-a}}\right]^{1/b}.$$

Evidently, the firm's production function is

$$f(k,l) = Ak^{a/b} l^{(1-a)/b}.$$

3. Discretion versus Commitment in Optimal Monetary Policymaking

The central bank chooses U and π to maximize

$$-(1/2)(U - kU^n)^2 - (b/2)\pi^2, \tag{7}$$

subject to the constraint

$$U = U^n - \alpha(\pi - \pi^e). \tag{6}$$

Two cases below are distinguished by whether the central bank takes π^e as given when it makes its decisions under discretion or whether it recognizes that π^e will equal π under commitment.

a. Under discretion, the central bank solves

$$\max_{\pi} = -(1/2)[(1-k)U^n - \alpha(\pi - \pi^e)]^2 - (b/2)\pi^2.$$

The first-order condition

$$\alpha[(1-k)U^n - \alpha(\pi^* - \pi^e)] - b\pi^* = 0$$

implies that

$$\pi^* = \left(\frac{\alpha}{b+\alpha^2}\right)[(1-k)U^n + \alpha\pi^e].$$

b. In equilibrium, private agents with rational expectations will set $\pi^e = \pi^*$. Substituting this condition into the expression for the central bank's optimal choice of π yields

$$\pi^* = \left[\frac{\alpha(1-k)}{b}\right] U^n > 0.$$

Under discretion, the central bank chooses positive inflation in a futile attempt to exploit the Phillips curve.

c. With commitment, the central bank recognizes that it cannot fool the private sector and solves

$$\max_{\pi} = -(1/2)[(1-k)U^n]^2 - (b/2)\pi^2$$

by setting inflation equal to zero

$$\pi^* = 0.$$

d. Under both discretion and commitment, $U = U^n$ holds in equilibrium. Under discretion, however, the central bank ends up choosing a positive inflation rate, whereas with commitment it chooses inflation equal to its target of zero. Thus, in this example, the central bank achieves a higher value of its objective function by choosing commitment instead of discretion.