Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, November 2

1. Utility Maximization

The consumer solves

$$\min_{c_F, c_O} \ \alpha \ln(c_F) + (1-\alpha)\ln(c_O) \text{ subject to } Y \geq p_F c_F + p_0 c_0.$$

The Lagrangian for this problem can be written as

$$L(c_F, c_O, \lambda) = \alpha \ln(c_F) + (1-\alpha)\ln(c_O) + \lambda(Y - p_F c_F - p_0 c_0).$$

The first-order conditions for c_F^* and c_O^* are

$$\frac{\alpha}{c_F^*} - \lambda^* p_F = 0$$

and

$$\frac{1-\alpha}{c_O^*} - \lambda^* p_O = 0,$$

and can be rearranged to read

$$c_F^* = \frac{\alpha}{\lambda^* p_F}$$

and

$$c_O^* = \frac{1-\alpha}{\lambda^* p_O}.$$

Substituting these last expressions into the binding constraint

$$Y = p_F c_F^* + p_O c_O^*$$

provides the solution

$$\lambda^* = \frac{1}{Y},$$

which can then be substituted back into the previous expressions to obtain

$$c_F^* = \frac{\alpha Y}{p_F}$$

and

$$c_O^* = \frac{(1-\alpha)Y}{p_O}.$$

2. Expenditure Minimization

The consumer solves

$$\min_{c_A, c_B} \ p_A c_A + p_B c_B \text{ subject to } \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \geq c_F.$$

a. Letting λ denote the nonnegative multiplier on the constraint, the Lagrangian for this problem can be defined as

$$L(c_A, c_B, \lambda) = p_A c_A + p_B c_B - \lambda \left[\left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} - c_F \right]$$

b. With the Lagrangian defined as above, the Kuhn-Tucker theorem implies that the values c_A^* and c_B^* that solve the problem, together with the associated value λ^* of the Lagrange multiplier, must satisfy the first-order conditions

$$p_A - \lambda^* \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} c_A^{*-\frac{1}{\theta}} = 0$$

and

$$p_B - \lambda^* \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} c_B^{*-\frac{1}{\theta}} = 0$$

c. Together with the binding constraint

$$c_F = \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

the first-order conditions form a system of three equations in the three unknowns c_A^*, c_B^*, and λ^*. Although there are many ways of solving this system, perhaps the easiest is to start by dividing the first-order condition for c_A by the first-order condition for c_B to obtain

$$\frac{p_A}{p_B} = \left(\frac{c_B^*}{c_A^*} \right)^{\frac{1}{\theta}}$$

or

$$c_B^* = \left(\frac{p_A}{p_B} \right)^{\theta} c_A^*,$$

and then substitute this expression for c_B^* into the binding constraint to obtain the solution

$$c_A^* = c_F \left(\frac{p_A^{1-\theta} + p_B^{1-\theta}}{p_A^{1-\theta}} \right)^{\frac{\theta}{1-\theta}}$$

Substituting this solution for c_A^* back into the previous expression for c_B^* then yields the solution

$$c_B^* = c_F \left(\frac{p_A^{1-\theta} + p_B^{1-\theta}}{p_B^{1-\theta}} \right)^{\frac{\theta}{1-\theta}}.$$

Although this approach allows us to find c_A^* and c_B^* without having to solve for λ^*, for future reference it is helpful to note that, by substituting these solutions into either of the two first-order conditions, one can find

$$\lambda^* = \left(p_A^{1-\theta} + p_B^{1-\theta} \right)^{\frac{1}{1-\theta}}.$$

d. With the minimum expenditure function defined as

$$E(c_F, p_A, p_B) = \min_{c_A, c_B} \; p_A c_A + p_B c_B \text{ subject to } \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \geq c_F,$$

it is possible to find the partial derivative of E with respect to c_F in either of two ways. The first is to use the solutions for c_A^* and c_B^* to find

$$E(c_F, p_A, p_B) = p_A c_A^* + p_B c_B^* = c_F \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}},$$

then differentiate to obtain

$$\frac{\partial E(c_F, p_A, p_B)}{\partial c_F} = \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

The second is to use the envelope theorem and the solution for λ^* to obtain

$$\frac{\partial E(c_F, p_A, p_B)}{\partial c_F} = \lambda^* = \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

Since this partial derivative can be interpreted as the marginal cost of producing the quantity aggregate, it suggests defining the price aggregate for food as

$$p_F = \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

3. More Detailed Utility Maximization

The consumer solves

$$\max_{c_A, c_B, c_O} \left(\frac{\alpha\theta}{\theta-1}\right) \ln\left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right) + (1-\alpha)\ln(c_O) \text{ subject to } Y \geq p_A c_A + p_B c_B + p_O c_O.$$

a. Letting λ denote the nonnegative multiplier on the constraint, the Lagrangian for this problem can be defined as

$$L(c_A, c_B, c_O, \lambda) = \left(\frac{\alpha\theta}{\theta-1}\right) \ln\left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right) + (1-\alpha)\ln(c_O) + \lambda(Y - p_A c_A - p_B c_B - p_O c_O).$$

b. With the Lagrangian defined as above, the Kuhn-Tucker theorem implies that the values c_A^*, c_B^*, and c_O^* that solve this problem, together with the associated value λ^* of the Lagrange multiplier, must satisfy the first-order conditions

$$\frac{\alpha c_A^{*-\frac{1}{\theta}}}{c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}} - \lambda^* p_A = 0,$$

$$\frac{\alpha c_B^{*-\frac{1}{\theta}}}{c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}} - \lambda^* p_B = 0,$$

and

$$\frac{1-\alpha}{c_O^*} - \lambda^* p_O = 0.$$

c. Together with the binding constraint

$$Y = p_0 c_O^* + p_A c_A^* + p_B c_B^*,$$

the first-order conditions for a system of four equations in the four unknowns c_A^*, c_B^*, c_O^*, and λ^*. As a first step in solving this system, multiply the first-order condition for c_A by c_A^*, the first-order condition for c_B by c_B^*, and the first-order condition for c_O by c_O^*. Then, after dividing each first-order condition by λ^*, substituting them all into the binding budget constraint provides the solutions

$$\lambda^* = \frac{1}{Y}$$

and hence

$$c_0^* = \frac{(1-\alpha)Y}{p_O},$$

just as in question 1.

d. Next, divide the first-order condition for c_A by the first-order condition for c_B to obtain

$$\left(\frac{c_B^*}{c_A^*}\right)^{\frac{1}{\theta}} = \frac{p_A}{p_B}.$$

or

$$c_B^* = \left(\frac{p_A}{p_B}\right)^{\theta} c_A^*,$$

and substitute this expression into

$$c_F^* = \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

to obtain

$$c_A^* = c_F^* \left(\frac{p_A^{1-\theta} + p_B^{1-\theta}}{p_A^{1-\theta}}\right)^{\frac{\theta}{1-\theta}}$$

Substituting this expression for c_A^* back into the previous expression for c_B^* then yields

$$c_B^* = c_F^* \left(\frac{p_A^{1-\theta} + p_B^{1-\theta}}{p_B^{1-\theta}}\right)^{\frac{\theta}{1-\theta}}.$$

Both of these solutions coincide with those from question 2, with c_F^* in place of c_F.

e. It only remains to find c_F^*. This can be done by substituting the expressions for λ^*, c_A^*, and c_B^* back into the first-order condition for c_A to obtain

$$c_F^* = \frac{\alpha Y}{\left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.},$$

which coincides with the solution to question 1 if we define the price aggregate for food as

$$p_F = \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$