Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, October 27 at 12noon

1. The Kuhn-Tucker Theorem

The problem is to choose x and y to maximize the objective function

$$F(x,y) = (1-x-y)(x+2y),$$

subject to the constraints

$$1 \geq x+y,$$

$$x \geq 0,$$

and

$$y \geq 0.$$

a. One definition of the Lagrangian treats the nonnegativity constraints symmetrically with the constraint on the sum of x and y by assigning a separate multiplier to each:

$$L(x,y,\lambda,\mu_x,\mu_y) = (1-x-y)(x+2y) + \lambda(1-x-y) + \mu_x x + \mu_y y.$$

b. With the Lagrangian defined as above, the Kuhn-Tucker theorem implies that the values x^* and y^* that solve the problem, together with the associated values λ^*, μ_x^*, and μ_y^* of the Lagrange multipliers, must satisfy the first order conditions

$$1 - 2x^* - 3y^* - \lambda^* + \mu_x^* = 0$$

and

$$2 - 3x^* - 4y^* - \lambda^* + \mu_y^* = 0,$$

the constraints

$$1 \geq x^* + y^*,$$

$$x^* \geq 0,$$

and

$$y^* \geq 0,$$

the non-negativity conditions

$$\lambda^* \geq 0,$$

$$\mu_x^* \geq 0,$$

and

$$\mu_y^* \geq 0,$$

and the complementary slackness conditions

$$\lambda^*(1 - x^* - y^*) = 0,$$

$$\mu_x^* x^* = 0,$$

and

$$\mu_y^* y^* = 0.$$

c. In order to find the numerical values of x^* and y^*, consider first the possibility that none of the three constraints bind. In this case, the complementary slackness conditions require all three multipliers to equal zero and the first-order conditions simplify to

$$1 - 2x^* - 3y^* = 0$$

and

$$2 - 3x^* - 4y^* = 0.$$

But these conditions require $x^* = 2$ and $y^* = -1$, violating the non-negativity constraint on y. We now know that at least one of the constraints must bind at the optimum.

Observe next, that the solution cannot be such that all three constraints are binding. For this would require x^* and y^* to both equal zero but also to sum to one. Notice, in fact, that if the constraint

$$1 = x^* + y^*$$

binds at the optimum, the value of the objective function equals zero. Since it is possible to obtain positive values for the objective function by choosing any small but positive values for x^* and y^*, this first constraint cannot bind at the optimum nor, for that matter, can it be the case that x^* and y^* both equal zero, since the objective function will equal zero in that case, too.

We've now narrowed the set of possibilities down to two: either $x^* = 0$ and $\lambda^* = \mu_y^* = 0$ or $y^* = 0$ and $\lambda^* = \mu_x^* = 0$.

Consider, therefore, the case where $x^* = 0$ and $\lambda^* = \mu_y^* = 0$. In this case, the first-order conditions require

$$1 - 3y^* + \mu_x^* = 0$$

and

$$2 - 4y^* = 0,$$

or $y^* = \mu_x^* = 1/2$. As this configuration of values satisfies all of the Kuhn-Tucker conditions, it is a candidate solution.

Consider, next, the case where $y^* = 0$ and $\lambda^* = \mu_x^* = 0$. In this case, the first-order conditions require

$$1 - 2x^* = 0$$

and

$$2 - 3x^* + \mu_y^* = 0,$$

or $x^* = 1/2$ and $\mu_y^* = -1/2$. As the Kuhn-Tucker conditions require μ_y^* to be non-negative, they rule out this case as a possible solution.

Evidently, the solution has $x^* = 0$ and $y^* = 1/2$.

2. Consumer Optimization

The consumer chooses c_1 and c_2 to maximize the utility function

$$U(c_1, c_2) = c_1^{1/2} + c_2^{1/2},$$

subject to the budget constraint

$$Y \geq p_1 c_1 + p_2 c_2.$$

a. With the Lagrangian for the consumer's defined as

$$L(c_1, c_2, \lambda) = c_1^{1/2} + c_2^{1/2} + \lambda(Y - p_1 c_1 - p_2 c_2),$$

the first-order conditions are

$$\frac{1}{2(c_1^*)^{1/2}} - \lambda^* p_1 = 0$$

and

$$\frac{1}{2(c_2^*)^{1/2}} - \lambda^* p_2 = 0.$$

b. Rearrange the first-order conditions so that they read

$$c_1^* = \left(\frac{1}{2\lambda^* p_1}\right)^2$$

and

$$c_2^* = \left(\frac{1}{2\lambda^* p_2}\right)^2,$$

then substitute these expressions into the binding budget constraint to obtain

$$Y = \left(\frac{1}{2\lambda^*}\right)^2 \left(\frac{1}{p_1} + \frac{1}{p_2}\right)$$

or

$$\left(\frac{1}{2\lambda^*}\right)^2 = \left(\frac{p_1 p_2}{p_1 + p_2}\right) Y.$$

Finally, use this last expression to eliminate λ^* from the previous expressions for c_1^* and c_2^*, thereby obtaining the solutions

$$c_1^* = \left(\frac{p_2}{p_1 + p_2}\right)\left(\frac{Y}{p_1}\right)$$

and

$$c_2^* = \left(\frac{p_1}{p_1 + p_2}\right)\left(\frac{Y}{p_2}\right).$$

c. The solutions for c_1^* and c_2^* just derived imply that the shares of income optimally spent are

$$\frac{p_1 c_1^*}{Y} = \frac{p_2}{p_1 + p_2}$$

and

$$\frac{p_2 c_2^*}{Y} = \frac{p_1}{p_1 + p_2}.$$

3. Symmetry of Marshallian Demands?

The Marshallian demand curves $c_1^* = M_1(p_1, p_2, Y)$ and $c_2^* = M_2(p_1, p_2, Y)$ and indirect utility function $V(p_1, p_2, Y)$ describe the solution to the utility-maximization problem

$$V(p_1, p_2, Y) = \max_{c_1, c_2} \; U(c_1, c_2) \text{ subject to } Y \geq p_1 c_1 + p_2 c_2.$$

The Hicksian demand curves $c_1^* = H_1(p_1, p_2, \bar{U})$ and $c_2^* = H_2(p_1, p_2, \bar{U})$ and expenditure function $E(p_1, p_2, \bar{U})$ describe the solution to the cost minimization problem

$$E(p_1, p_2, \bar{U}) = \min_{c_1, c_2} \; p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U},$$

a. The envelope theorem, applied to the cost minimization problem, implies that

$$\frac{\partial E(p_1, p_2, \bar{U})}{\partial p_i} = H_i(p_1, p_2, \bar{U})$$

for $i = 1$ and $i = 2$ and hence that the symmetry conditions for Hicksian demands

$$\frac{\partial H_1(p_1, p_2, \bar{U})}{\partial p_2} = \frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_1 \partial p_2} = \frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_2 \partial p_1} = \frac{\partial H_2(p_1, p_2, \bar{U})}{\partial p_1}$$

must always hold.

b. Since the Marshallian and Hicksian demands coincide when $Y = E(p_1, p_2, \bar{U})$, the Marshallian and Hicksian demand curves satisfy

$$H_i(p_1, p_2, \bar{U}) = M_i(p_1, p_2, E(p_1, p_2, \bar{U}))$$

for $i = 1$ and $i = 2$. Therefore,

$$\frac{\partial H_i(p_1, p_2, \bar{U})}{\partial p_j} = \frac{\partial M_i(p_1, p_2, E(p_1, p_2, \bar{U}))}{\partial p_j} + \frac{\partial M_i(p_1, p_2, E(p_1, p_2, \bar{U}))}{\partial Y} \frac{\partial E(p_1, p_2, \bar{U})}{\partial p_j}.$$

The envelope theorem then implies that

$$\frac{\partial H_i(p_1, p_2, \bar{U})}{\partial p_j} = \frac{\partial M_i(p_1, p_2, E(p_1, p_2, \bar{U}))}{\partial p_j} + \frac{\partial M_i(p_1, p_2, E(p_1, p_2, \bar{U}))}{\partial Y} H_j(p_1, p_2, \bar{U})$$

or, more simply,

$$\frac{\partial M_i(p_1,p_2,Y)}{\partial p_j} = \frac{\partial H_i(p_1,p_2,\bar{U})}{\partial p_j} - \frac{\partial M_i(p_1,p_2,Y)}{\partial Y} M_j(p_1,p_2,Y),$$

must hold for all $i = 1, 2$ and $j = 1, 2$ when $Y = E(p_1, p_2, \bar{U})$.

c. Set $i = 1$ and $j = 2$, so that the Slutsky equation specializes to

$$\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = \frac{\partial H_1(p_1,p_2,\bar{U})}{\partial p_2} - \frac{\partial M_1(p_1,p_2,Y)}{\partial Y} M_2(p_1,p_2,Y).$$

Now use the symmetry condition for Hicksian demands to rewrite this expression as

$$\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = \frac{\partial H_2(p_1,p_2,\bar{U})}{\partial p_1} - \frac{\partial M_1(p_1,p_2,Y)}{\partial Y} M_2(p_1,p_2,Y),$$

and use the Slutsky equation with $i = 2$ and $j = 1$ to obtain

$$\begin{aligned}\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = {} & \frac{\partial M_2(p_1,p_2,Y)}{\partial p_1} \\ & + \frac{\partial M_2(p_1,p_2,Y)}{\partial Y} M_1(p_1,p_2,Y) - \frac{\partial M_1(p_1,p_2,Y)}{\partial Y} M_2(p_1,p_2,Y)\end{aligned}$$

d. This last result implies that

$$\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = \frac{\partial M_2(p_1,p_2,Y)}{\partial p_1} + \frac{M_1(p_1,p_2,Y) M_2(p_1,p_2,Y)}{Y}(\eta_2 - \eta_1),$$

where

$$\eta_i = \frac{Y}{M_i(p_1,p_2,Y)} \frac{\partial M_i(p_1,p_2,Y)}{\partial Y}$$

denotes the income elasticity of demand for good i. From this last expression, we can see that the symmetry condition

$$\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = \frac{\partial M_2(p_1,p_2,Y)}{\partial p_1}$$

for Marshallian demand will hold only in the special case where the two goods have equal income elasticities.