Solutions to Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
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Tuesday, October 29, 1:30 - 2:45pm

1. Linear Expenditure System

The consumer solves

$$\max_{c_1,c_2,c_3} a_1 \ln(c_1 - x_1) + a_2 \ln(c_2 - x_2) + a_3 \ln(c_3 - x_3)$$

subject to

$$Y \geq p_1 c_1 + p_2 c_2 + p_3 c_3.$$

where the positive parameters a_1, a_2, and a_3 satisfy

$$a_1 + a_2 + a_3 = 1$$

and the positive parameters x_1, x_2, and x_3 are such that

$$Y - p_1 x_1 - p_2 x_2 - p_3 x_3 > 0.$$

a. With the Lagrangian for the consumer's problem defined as

$$L(c_1, c_2, c_3, \lambda) = a_1 \ln(c_1 - x_1) + a_2 \ln(c_2 - x_2) + a_3 \ln(c_3 - x_3) + \lambda(Y - p_1 c_1 - p_2 c_2 - p_3 c_3),$$

the Kuhn-Tucker condition implies that there exists a value λ^* that, together with the values c_1^*, c_2^*, and c_3^* that solve this problem, satisfy the first-order conditions

$$\frac{a_1}{c_1^* - x_1} - \lambda^* p_1 = 0,$$

$$\frac{a_2}{c_2^* - x_2} - \lambda^* p_2 = 0,$$

and

$$\frac{a_3}{c_3^* - x_3} - \lambda^* p_3 = 0.$$

b. Rearrange the first-order conditions to obtain

$$c_1^* = x_1 + \frac{a_1}{\lambda^* p_1},$$

$$c_2^* = x_2 + \frac{a_2}{\lambda^* p_2},$$

and

$$c_3^* = x_3 + \frac{a_3}{\lambda^* p_3}.$$

then substitute these expressions into the binding budget constraint to obtain

$$\frac{1}{\lambda^*} = Y - p_1 x_1 - p_2 x_2 - p_3 x_3,$$

using the condition that $a_1 + a_2 + a_3 = 1$. Substituting this result back into the expressions for optimal consumptions yields solutions

$$c_1^* = x_1 + \left(\frac{a_1}{p_1}\right)(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

$$c_2^* = x_2 + \left(\frac{a_2}{p_2}\right)(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

and

$$c_3^* = x_3 + \left(\frac{a_3}{p_3}\right)(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

which can be interpreted as saying: the consumer starts by purchasing the "essential" amounts x_1, x_2, and x_3 of each good, then allocates his or her "discretionary" income $Y - p_1 x_1 - p_2 x_2 - p_3 x_3$ to additional spending on each good in shares a_1, a_2, and a_3.

c. Multiplying the solutions from part (b) for consumption of each good by its price highlights another implication: optimal expenditures depend linearly on income and prices. Specifically,

$$p_1 c_1^* = p_1 x_1 + a_1(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

$$p_2 c_2^* = p_2 x_2 + a_2(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

and

$$p_3 c_3^* = p_3 x_3 + a_3(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

For more about the linear expenditure system and its econometric application, see Richard Stone, "Linear Expenditure Systems and Demand Analysis: An Application to the Pattern of British Demand," *Economic Journal*, September 1954, pp.511-527.

2. Le Chatelier's Principle

a. The firm's long-run problem is

$$\max_{k,l,y} yp - rk - wl \text{ subject to } 4k^{1/4}l^{1/4} \geq y.$$

Associated with this problem, define the Lagrangian as

$$L(k, l, y, \lambda) = yp - rk - wl + \lambda(4k^{1/4}l^{1/4} - y).$$

The first-order conditions

$$r = \lambda^*(k^*)^{-3/4}(l^*)^{1/4},$$

$$w = \lambda^*(k^*)^{1/4}(l^*)^{-3/4},$$

and

$$p = \lambda^*,$$

and can used to find the long-run factor demand functions

$$l^*(p, r, w) = p^2 r^{-1/2} w^{-3/2}$$

and

$$k^*(p, r, w) = p^2 r^{-3/2} w^{-1/2}.$$

Substituting these solutions into the binding constraint yields the long-run output supply function

$$y^*(p, r, w) = 4pr^{-1/2}w^{-1/2},$$

with

$$\frac{\partial y^*(p, r, w)}{\partial p} = 4r^{-1/2}w^{-1/2}.$$

b. The firm's short-run problem is

$$\max_{l,y} yp - r\bar{k} - wl \text{ subject to } 4\bar{k}^{1/4}l^{1/4} \geq y.$$

Associated with this problem, define the Lagrangian as

$$L(l, y, \lambda) = yp - r\bar{k} - wl + \lambda(4\bar{k}^{1/4}l^{1/4} - y).$$

The first-order conditions

$$w = \lambda^s \bar{k}^{1/4}(l^s)^{-3/4},$$

and

$$p = \lambda^s,$$

combine to yield the short-run labor demand function

$$l^s(p, w, \bar{k}) = p^{4/3} w^{-4/3} \bar{k}^{1/3}.$$

Substituting this solution into the binding constraint yields the short-run output supply function

$$y^s(p, w, \bar{k}) = 4p^{1/3} w^{-1/3} \bar{k}^{1/3},$$

with

$$\frac{\partial y^s(p, w, \bar{k})}{\partial p} = (4/3)p^{-2/3} w^{-1/3} \bar{k}^{1/3}.$$

c. Substituting

$$\bar{k} = k^*(p, r, w) = p^2 r^{-3/2} w^{-1/2}$$

into the expression for $\partial y^s(p, w, \bar{k})/\partial p$ just derived yields

$$\left.\frac{\partial y^s(p, w, \bar{k})}{\partial p}\right|_{\bar{k}=k^*} = (4/3)r^{-1/2}w^{-1/2} < 4r^{-1/2}w^{-1/2} = \frac{\partial y^*(p, r, w)}{\partial p},$$

illustrating that a version of Le Chatelier's principle holds: adjustment of output to a change in price is larger on the long run, when capital input can adjust, than in the short-run, while capital is fixed.

3. Optimal and Equilibrium Allocations

a. Taking the initial capital stock $k(0) > 0$ as given, a social planner chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize a representative consumer's utility,

$$\int_0^\infty e^{-\rho t} \ln(c(t))\,\mathrm{d}t,$$

subject to the capital accumulation constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. For this problem, the maximized current-value Hamiltonian takes the form

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

The first-order condition

$$\frac{1}{c(t)} - \theta(t) = 0$$

and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t)) = \rho\theta - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\theta(k(t), \theta(t)) = k(t)^\alpha - \delta k(t) - c(t),$$

characterize the Pareto optimal allocations in this economy.

b. Taking the initial capital stock $k(0) > 0$ as given, the representative consumer chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t))\,\mathrm{d}t,$$

subject to the capital accumulation constraint

$$r(t)k(t) + w(t) - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. For this problem, the maximized current-value Hamiltonian takes the form

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[r(t)k(t) + w(t) - \delta k(t) - c(t)].$$

The first-order condition

$$\frac{1}{c(t)} - \theta(t) = 0$$

and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t)) = \rho\theta - \theta(t)[r(t) - \delta]$$

and

$$\dot{k}(t) = H_\theta(k(t), \theta(t)) = r(t)k(t) + w(t) - \delta k(t) - c(t),$$

characterize the consumer's optimal choices.

c. The representative firm chooses $k(t)$ and $n(t)$ to maximize profits

$$k(t)^\alpha n(t)^{1-\alpha} - r(t)k(t) - w(t)n(t),$$

taking $r(t)$ and $w(t)$ as given. The first-order conditions

$$\alpha k(t)^{\alpha-1} n(t)^{1-\alpha} - r(t) = 0$$

and

$$(1-\alpha)k(t)^\alpha n(t)^{-\alpha} - w(t) = 0$$

characterize the firm's optimal choices.

d. In equilibrium, labor supply must equal labor demand, so that $n(t) = 1$ and the firm must earn zero profits, so that

$$k(t)^\alpha n(t)^{1-\alpha} = r(t)k(t) + w(t)n(t)$$

Substituting $n(t) = 1$ into the firm's first-order condition for $k(t)$ reveals that in equilibrium, the rental rate on capital is related to the marginal product of capital according to

$$r(t) = \alpha k^{\alpha-1}.$$

Substituting $n(t) = 1$ into the zero-profit condition reveals that, as well, income earned by the consumer equals output produced by the firm:

$$r(t)k(t) + w(t) = k(t)^\alpha.$$

Substituting these equilibrium conditions into the differential equations describing the solution to the consumer's problem yields the same pair of differential equations

$$\dot{\theta}(t) = \rho\theta - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t),$$

that characterize the Pareto optimal allocations for this economy. The consumer's first-order condition coincides with the social planner's as well. These observations confirm that the two welfare theorems apply. The equilibrium allocation is Pareto optimal and the optimal allocation can be supported in a competitive equilibrium.