Solutions to Midterm Exam

ECON 772001 - Math for Economists
Boston College, Department of Economics

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1. Utility Maximization

The consumer chooses c_1 and c_2 to maximize the utility function

$$U(c_1, c_2) = \ln(c_1) - (1/2)(c_2 - \bar{c})^2,$$

subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2.$$

a. With the Lagrangian for this problem defined as

$$L(c_1, c_2, \lambda) = \ln(c_1) - (1/2)(c_2 - \bar{c})^2 + \lambda(I - p_1 c_1 - p_2 c_2),$$

the Kuhn-Tucker theorem implies that the optimal choices c_1^* and c_2^* and the associated value λ^* of the Lagrange multiplier must satisfy the first-order conditions

$$\frac{1}{c_1^*} - \lambda^* p_1 = 0$$

and

$$-(c_2^* - \bar{c}) - \lambda^* p_2 = 0,$$

the constraint

$$I \geq p_1 c_1^* + p_2 c_2^*,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0.$$

b. Since the utility function is strictly increasing in c_1, we know in advance that $\lambda^* > 0$ and the budget constraint will always bind. Based on these observations, we can substitute the parameter settings $I = 10$, $p_1 = 2$, $p_2 = 1$, and $\bar{c} = 10$ into the first-order conditions and constraint to obtain the system of three equations in the three unknowns c_1^*, c_2^*, and λ^*:

$$\frac{1}{c_1^*} - 2\lambda^* = 0$$

$$-(c_2^* - 10) - \lambda^* = 0,$$

and

$$10 = 2c_1^* + c_2^*.$$

Rearrange the first-order conditions to read

$$c_1^* = \frac{1}{2\lambda^*}$$

and

$$c_2^* = 10 - \lambda^*,$$

and substitute these expressions into the constraint to obtain

$$10 = \frac{1}{\lambda^*} + 10 - \lambda^*$$

or, more simply

$$(\lambda^*)^2 = 1.$$

Since $\lambda^* > 0$, this last equation requires $\lambda^* = 1$. The optimal consumptions are therefore

$$c_1^* = \frac{1}{2}$$

and

$$c_2^* = 9.$$

2. Roy's Identity

To answer this question, it helps to consider first the general utility maximization problem

$$\max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2.$$

By defining the indirect utility function

$$V(p_1, p_2, I) = \max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2,$$

we can use the envelope theorem (in this case, Roy's identity) to recover the Marshallian demand curves as

$$c_1^*(p_1, p_2, I) = -\frac{V_1(p_1, p_2, I)}{V_3(p_1, p_2, I)}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{V_2(p_1, p_2, I)}{V_3(p_1, p_2, I)}.$$

a. For the specific case of the constant elasticity utility function, the indirect utility function is given by

$$V(p_1, p_2, I) = \ln(I) - \left(\frac{1}{1-\sigma}\right) \ln\left(p_1^{1-\sigma} + p_2^{1-\sigma}\right) + \left(\frac{\sigma}{1-\sigma}\right) \ln(2).$$

Therefore, applying Roy's identity in this case yields

$$c_1^*(p_1, p_2, I) = -\frac{V_1(p_1, p_2, I)}{V_3(p_1, p_2, I)} = \frac{I p_1^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{V_2(p_1, p_2, I)}{V_3(p_1, p_2, I)} = \frac{I p_2^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}.$$

b. In the limiting case where $\sigma \to 1$, so that the elasticity of substitution equals one, the indirect utility function simplifies to

$$V(p_1, p_2, I) = \ln(I) - (1/2)\ln(p_1) - (1/2)\ln(p_2) - \ln(2),$$

and Roy's identity implies

$$c_1^*(p_1, p_2, I) = -\frac{V_1(p_1, p_2, I)}{V_3(p_1, p_2, I)} = \frac{I}{2p_1}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{V_2(p_1, p_2, I)}{V_3(p_1, p_2, I)} = \frac{I}{2p_2}.$$

3. The Ramsey Model

The social planner takes the initial capital stock $k(0)$ as given and chooses functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize utility

$$\int_0^\infty e^{-\rho t} u(c(t))\, \mathrm{d}t$$

subject to the capital accumulation constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t),$$

for all $t \in [0, \infty)$.

a. The current value Hamiltonian for the social planner's problem is

$$\bar{H}(c(t), k(t), \theta(t)) = u(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

With the maximized Hamiltonian defined by extension as

$$H(k(t), \theta(t)) = \max_{c(t)} \; u(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)],$$

the maximum principle implies that the solution to the dynamic problem must satisfy the first-order condition

$$u'(c(t)) - \theta(t) = 0$$

and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t)) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\theta(k(t), \theta(t)) = k(t)^\alpha - \delta k(t) - c(t).$$

b. In the unique non-trivial steady state, where $c(t) = c^*$, $k(t) = k^*$, and $\theta(t) = \theta^*$ are all equal to constants, so that $\dot{k}(t) = \dot{\theta}(t) = 0$, the first differential equation, for $\dot{\theta}(t)$, implies that

$$k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

and the second differential equation, for $\dot{k}(t)$, implies that

$$c^* = \left(\frac{\delta + \rho}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\delta + \rho}{\alpha}\right)^{\frac{1}{\alpha-1}}.$$

4. Natural Resource Depletion

The social planner takes s_0 as given and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize utility

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma}\right)$$

subject to the constraints

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

a. Although this problem can also be solved using the method of Lagrange multipliers, let's use the maximum principle instead. Start by defining the maximized present value Hamiltonian as

$$H(s_t, \pi_{t+1}; t) = \max_{c_t} \ \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma}\right) - \pi_{t+1} c_t.$$

Then the maximum principle implies that the sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ that solve the dynamic problem must satisfy the first-order condition

$$\beta^t c_t^{-\sigma} - \pi_{t+1} = 0$$

and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_s(s_t, \pi_{t+1}; t) = 0$$

and

$$s_{t+1} - s_t = H_\pi(s_t, \pi_{t+1}; t) = -c_t,$$

as well as the initial condition s_0 given and the transversality condition

$$\lim_{T\to\infty} \beta^T c_T^{-\sigma} s_{T+1} = 0.$$

b. Since the first difference equation implies that $\pi_{t+1} = \pi_t$ for all $t = 1, 2, 3, \ldots$, the first-order condition implies that

$$\beta^{t+1} c_{t+1}^{-\sigma} = \beta^t c_t^{-\sigma}$$

for all $t = 0, 1, 2, \ldots$. This result shows that optimal consumption growth is constant, with

$$c_{t+1}/c_t = \beta^{1/\sigma}$$

for all $t = 0, 1, 2, \ldots$. Since the right-hand side of this last expression is less than one, consumption of the natural resource declines over time for all values of σ, generalizing the result that you derived previously for the special case of log utility, with $\sigma = 1$.