**Solutions to Final Exam**

ECON 772001 - Math for Economists — Peter Ireland
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Due Thursday, December 18

## 1. The Community Lake

Given the initial stock of water $w(0)$ in the lake, the community planners choose $c(t)$ for all $t \in [0, \infty)$ and $w(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t}[\ln(c(t)) + \alpha \ln(w(t))]\, \mathrm{d}t,$$

subject to the constraint

$$R - c(t) \geq \dot{w}(t),$$

for all $t \in [0, \infty)$

a. With the maximized Hamiltonian for the planners' problem written in current value form as

$$H(w(t), \theta(t)) = \max_{c(t)} \ln(c(t)) + \alpha \ln(w(t)) + \theta(t)[R - c(t)],$$

the maximum principle implies that the solution to the problem is described by the first-order condition

$$\frac{1}{c(t)} - \theta(t) = 0$$

and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - \frac{\alpha}{w(t)}$$

and

$$\dot{w}(t) = R - c(t).$$

b. Rewrite the first-order condition as

$$1 = \theta(t)c(t)$$

and differentiate with respect to $t$ to get

$$0 = \dot{\theta}(t)c(t) + \theta(t)\dot{c}(t).$$

Now, use the differential equation to substitute out for $\dot{\theta}(t)$; this yields

$$0 = \rho\theta(t)c(t) - \frac{\alpha c(t)}{w(t)} + \theta(t)\dot{c}(t).$$

Finally, use the first-order condition again to replace $\theta(t)$ with $1/c(t)$:

$$0 = \rho - \frac{\alpha c(t)}{w(t)} + \frac{\dot{c}(t)}{c(t)}.$$

Rearranging this last expression yields the Euler equation

$$\dot{c}(t) = \left[\frac{\alpha c(t)}{w(t)} - \rho\right] c(t)$$

that combines with the water accumulation constraint

$$\dot{w}(t) = R - c(t)$$

to describe the behavior of consumption of water and the level of the lake water along their optimal paths.

c. The water accumulation constraint implies that in a steady-state with $\dot{w}(t) = 0$, consumption must be constant and equal to

$$c^* = R.$$

Intuitively, consuming water in line with the flow of rainfall leaves the water level in the lake unchanged. The Euler equation then provides the solution for the steady-state level of lake water:

$$w^* = \frac{\alpha R}{\rho}.$$

Intuitively, the optimal level of lake water depends positively on the weight $\alpha$ on $w(t)$ in utility, positively on the flow of rainfall $R$, and negatively on the discount rate $\rho$ as a measure of impatience.

The phase diagram for this model reveals that if the initial water level $w(0) = w_0$ is less than the steady-state level $w^*$, initial consumption must also be below its steady-state level $c^*$. Along the stable manifold, consumption and the water level both rise as they converge to their steady-state levels. Conversely, if the initial water level $w(0) = w_0$ is above the steady-state level $w^*$, initial consumption must also be above its steady-state level $c^*$. Along the stable manifold, consumption and the water level both fall as they converge to their steady-state levels.

## 2. Investment and the Business Cycle

Using the conjectured form of the value, function the Bellman equation for the social planner's problem is

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F\ln(z_t) + \beta F\ln(k_t^\alpha - c_t) + \beta G\rho\ln(z_t).$$

a. The first-order condition for $c_t$ is

$$\frac{1}{c_t} - \frac{\beta F}{k_t^\alpha - c_t} = 0$$

and the envelope condition for $k_t$ is

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1}}{k_t^\alpha - c_t}.$$

b. The first-order condition for $c_t$ implies

$$c_t = \left(\frac{1}{1+\beta F}\right) k_t^\alpha.$$

Substituting this expression for $c_t$ into the envelope condition for $k_t$ yields the solution for the constant $F$:

$$F = \frac{\alpha}{1-\alpha\beta}.$$

Substituting this solution for $F$ into the first-order condition for $c_t$ and the binding constraint for $k_{t+1}$ then yields

$$c_t = (1-\alpha\beta)k_t^\alpha$$

and

$$k_{t+1} = \alpha\beta z_t k_t^\alpha.$$

c. Finally, substituting the solutions for $F$, $c_t$, and $k_t$ back into the Bellman equation and observing that the equation must hold for all values of $\ln(z_t)$ provides the solutions for the constants $E$ and $G$:

$$E = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta)\right]$$

and

$$G = \left(\frac{1}{1-\beta\rho}\right)\left(\frac{\alpha\beta}{1-\alpha\beta}\right).$$