Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, December 17

1. Technical Knowledge Spillovers and Long-Run Economic Growth

The representative consumer takes $k(0)$ as given and chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t, \tag{1}$$

subject to the constraint

$$k(t)^\alpha x(t)^{1-\alpha} - \delta k(t) - c(t) \geq \dot{k}(t) \tag{2}$$

for $t \in [0, \infty)$.

a. In present-value form, the maximized Hamiltonian for the consumer's problem is

$$H(k(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)[k(t)^\alpha x(t)^{1-\alpha} - \delta k(t) - c(t)].$$

The maximum principle implies that the solution to the problem is described by the first-order condition

$$\frac{e^{-\rho t}}{c(t)} - \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi(t)} = -H_k(k(t), \pi(t); t) = -\pi(t)[\alpha k(t)^{\alpha-1} x(t)^{1-\alpha} - \delta]$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = k(t)^\alpha x(t)^{1-\alpha} - \delta k(t) - c(t).$$

b. Substituting the equilibrium condition

$$x(t) = k(t) \tag{3}$$

into the differential equations yields

$$\dot{\pi(t)} = -\pi(t)(\alpha - \delta)$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = (1 - \delta)k(t) - c(t).$$

The first-order condition for $c(t)$, when rewritten

$$\pi(t)c(t) = e^{-\rho t}$$

and differentiated with respect to t, implies that

$$\dot{\pi}(t)c(t) + \pi(t)\dot{c}(t) = -\rho\pi(t)c(t).$$

Using the differential equation to eliminate $\dot{\pi}(t)$,

$$-\pi(t)(\alpha - \delta)c(t) + \pi(t)\dot{c}(t) = -\rho\pi(t)c(t).$$

Dividing through by $\pi(t)$ and rearranging then leads to the Euler equation

$$\dot{c}(t) = (\alpha - \delta - \rho)c(t),$$

which combines with the capital accumulation equation

$$\dot{k}(t) = (1 - \delta)k(t) - c(t)$$

to describe the equilibrium paths for consumption and the capital stock.

c. The Euler equation reveals that

$$\frac{\dot{c}(t)}{c(t)} = g \tag{4}$$

for all $t \in [0, \infty)$, where the constant consumption growth rate is

$$g = \alpha - \delta - \rho.$$

d. Using the solution for g just derived, the inequality

$$1 - \delta - g - \rho > 0 \tag{12}$$

must hold since

$$1 - \delta - g - \rho = 1 - \delta - \alpha + \delta + \rho - \rho = 1 - \alpha > 0.$$

We can now conclude from (11) and (12) that the transversality condition requires $A = 0$. And for $t = 0$, (7) then requires

$$c(0) = (1 - \delta - g)k(0). \tag{13}$$

To summarize: with $k(0)$ given and $c(0)$ determined by (13), consumption and the capital stock both grow at the constant rate g is equilibrium, according to (5) and (7) with $A = 0$.

2. Stochastic Growth with Labor Supply and Serially Correlated Shocks

The representative consumer chooses contingency plans for consumption c_t and labor supply n_t for all $t = 0, 1, 2, \dots$ and physical capital k_t for all $t = 1, 2, 3, \dots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln(c_t) - \gamma n_t], \tag{14}$$

subject to the constraints k_0 given and

$$z_t k_t^{\alpha} n_t^{1-\alpha} \geq c_t + k_{t+1}, \tag{15}$$

for all $t = 0, 1, 2, \dots$ and all possible realizations of z_t.

Using the conjectured form of the value function, the Bellman equation for this problem becomes

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t, n_t} \ln(c_t) - \gamma h_t + \beta E + \beta F \ln(z_t k_t^{\alpha} n_t^{1-\alpha} - c_t) + \beta G \rho \ln(z_t). \tag{17}$$

a. Starting from (17), the first-order condition for c_t is

$$\frac{1}{c_t} - \frac{\beta F}{z_t k_t^{\alpha} n_t^{1-\alpha} - c_t} = 0,$$

the first-order condition for n_t is

$$-\gamma + \frac{(1-\alpha)\beta F z_t k_t^{\alpha} n_t^{-\alpha}}{z_t k_t^{\alpha} n_t^{1-\alpha} - c_t} = 0,$$

and the envelope condition for k_t is

$$\frac{F}{k_t} = \frac{\alpha \beta F z_t k_t^{\alpha-1} n_t^{1-\alpha}}{z_t k_t^{\alpha} n_t^{1-\alpha} - c_t}.$$

b. The first-order condition for c_t can be rearranged to yield

$$c_t = \left(\frac{1}{1+\beta F}\right) z_t k_t^{\alpha} n_t^{1-\alpha}.$$

Substituting this expression for c_t into the envelope condition for k_t and simplifying yields

$$\frac{1}{1+\beta F} = 1 - \alpha\beta$$

Hence,

$$F = \frac{\alpha}{1-\alpha\beta},$$

$$c_t = (1-\alpha\beta) z_t k_t^{\alpha} n_t^{1-\alpha},$$

and, using the binding constraint,

$$k_{t+1} = \alpha\beta z_t k_t^{\alpha} n_t^{1-\alpha}.$$

These last two results confirm that the consumer optimally chooses to consume and save the fixed fractions $1 - s$ and s of output, where $s = \alpha\beta$.

c. Using the solutions for F and c_t, the first-order condition for n_t implies

$$(1-\alpha\beta)\gamma\alpha\beta z_t k_t^{\alpha} n_t^{1-\alpha} = (1-\alpha)\beta\alpha z_t k_t^{\alpha} n_t^{-\alpha}.$$

Simplifying yields the solution for the constant labor supply:

$$n_t = n = \frac{1}{\gamma}\left(\frac{1-\alpha}{1-\alpha\beta}\right).$$

d. Finally, substituting the solutions for F, c_t, n_t, and k_t back into the Bellman equation (17) yields

$$\begin{aligned} E + G\ln(z_t) \quad = \quad & \ln(1-\alpha\beta) + \ln(z_t) + (1-\alpha)\ln(n) - \gamma n + \beta E \\ & + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(z_t) \\ & + (1-\alpha)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(n) + \beta G\rho\ln(z_t) \end{aligned}$$

after cancelling out the terms in $\ln(k_t)$ that appear on both sides. Since this last equation must hold for all values of $\ln(z_t)$, it requires

$$G = 1 + \frac{\alpha\beta}{1-\alpha\beta} + \beta G\rho,$$

which provides the solution for

$$G = \frac{1}{(1-\alpha\beta)(1-\beta\rho)}.$$

For the sake of completeness, one can also use the Bellman equation to solve for

$$E = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta) - \gamma n + \left(\frac{1-\alpha}{1-\alpha\beta}\right)\ln(n)\right],$$

where n depends on γ, α, and β as shown above.