Solutuions to Final Exam

ECON 772001 - Math for Economists
Boston College, Department of Economics

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Fall 2023

Due Tuesday, December 19

1. Endogenous Growth: Continuous Time

Given the initial capital stock $K(0)$, the social planner chooses $C(t)$ for all $t \in [0, \infty)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left(\frac{1}{1-\sigma} \right) C(t)^{1-\sigma} \text{ d}t, \tag{1}$$

subject to the constraint

$$(A - \delta)K(t) - C(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0, \infty)$.

a. With the maximized present-value Hamiltonian for the social planner's problem defined as

$$H(K(t), \pi(t); t) = \max_{C(t)} \; e^{-\rho t} \left(\frac{1}{1-\sigma} \right) C(t)^{1-\sigma} + \pi(t)[(A - \delta)K(t) - C(t)],$$

the maximum principle implies that the solution to the problem must satisfy the first-order condition

$$e^{-\rho t} C(t)^{-\sigma} - \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_K(K(t), \pi_t; t) = -\pi(t)(A - \delta)$$

and

$$\dot{K}(t) = (A - \delta)K(t) - C(t)$$

for all $t \in [0, \infty)$.

b. Rewrite the first-order condition as

$$\pi(t) = e^{-\rho t} C(t)^{-\sigma}$$

and differentiate both sides with respect to t to obtain

$$\dot{\pi}(t) = -\rho e^{-\rho t} C(t)^{-\sigma} - \sigma e^{-\rho t} C(t)^{-\sigma-1} \dot{C}(t).$$

Now substitute these two expressions into the differential equation for $\pi(t)$ to obtain

$$-\rho e^{-\rho t} C(t)^{-\sigma} - \sigma e^{-\rho t} C(t)^{-\sigma-1} \dot{C}(t) = -(A - \delta) e^{-\rho t} C(t)^{-\sigma}$$

or, more simply

$$\frac{\dot{C}(t)}{C(t)} = g_c = \frac{A - \delta - \rho}{\sigma}.$$

2. Endogenous Growth with Comparison Utility: Continuous Time

Taking the initial capital stock $K(0)$ and the trajectory for $Z(t)$ as given, the consumer chooses $C(t)$ for all $t \in [0, \infty)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left(\frac{1}{1-\sigma}\right) \left[\frac{C(t)}{Z(t)^\gamma}\right]^{1-\sigma} \, dt, \tag{3}$$

subject to the constraint

$$(A-\delta)K(t) - C(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0, \infty)$.

a. With the maximized present-value Hamiltonian for the social planner's problem defined as

$$H(K(t), \pi(t); t) = \max_{C(t)} \; e^{-\rho t} \left(\frac{1}{1-\sigma}\right) C(t)^{1-\sigma} Z(t)^{-\gamma(1-\sigma)} + \pi(t)[(A-\delta)K(t) - C(t)],$$

the maximum principle implies that the solution to the problem must satisfy the first-order condition

$$e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)} - \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_K(K(t), \pi_t; t) = -\pi(t)(A-\delta)$$

and

$$\dot{K}(t) = (A-\delta)K(t) - C(t)$$

for all $t \in [0, \infty)$.

b. Rewrite the first-order condition as

$$\pi(t) = e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)}$$

and differentiate both sides with respect to t to obtain

$$\begin{aligned} \dot{\pi}(t) = &- \rho e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)} \\ &- \sigma e^{-\rho t} C(t)^{-\sigma-1} Z(t)^{-\gamma(1-\sigma)} \dot{C}(t) - \gamma(1-\sigma) e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)-1} \dot{Z}(t). \end{aligned}$$

Now substitute these two expressions into the differential equation for $\pi(t)$ to obtain

$$\begin{aligned} (A-\delta) e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)} = &\; \rho e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)} \\ &+ \sigma e^{-\rho t} C(t)^{-\sigma-1} Z(t)^{-\gamma(1-\sigma)} \dot{C}(t) \\ &+ \gamma(1-\sigma) e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)-1} \dot{Z}(t). \end{aligned}$$

or, more simply,

$$A - \delta - \rho = \sigma \left[\frac{\dot{C}(t)}{C(t)}\right] + \gamma(1-\sigma) \left[\frac{\dot{Z}(t)}{Z(t)}\right].$$

Hence, along a steady-state growth path with

$$\dot{C}(t)/C(t) = \dot{Z}(t)/Z(t) = g_c$$

it must be that

$$A - \delta - \rho = \sigma g_c + \gamma(1-\sigma)g_c$$

or

$$g_c = \frac{A - \delta - \rho}{\sigma + \gamma(1-\sigma)}.$$

3. Endogenous Growth: Discrete Time

Given the initial capital stock K_0, the social planner chooses C_t for $t = 0, 1, 2, \ldots$ and K_t for $t = 1, 2, 3, \ldots$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right) C_t^{1-\sigma}, \tag{5}$$

subject to the constraint

$$AK_t + (1-\delta)K_t - C_t \geq K_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$.

a. With the Bellman equation written as

$$v(K_t) = \max_{K_{t+1}} \left(\frac{1}{1-\sigma}\right) [(A + 1 - \delta)K_t - K_{t+1}]^{1-\sigma} + \beta v(K_{t+1}),$$

dynamic programming arguments imply that the solution to the planner's problem must satisfy the first-order condition

$$-[(A + 1 - \delta)K_t - K_{t+1}]^{-\sigma} + \beta v'(K_{t+1}) = 0$$

for all $t = 0, 1, 2, \ldots$, the envelope condition

$$v'(K_t) = (A + 1 - \delta)[(A + 1 - \delta)K_t - K_{t+1}]^{-\sigma},$$

for all $t = 1, 2, 3, \ldots$, and the binding constraint

$$K_{t+1} = (A + 1 - \delta)K_t - C_t$$

for all $t = 0, 1, 2, \ldots$.

b. Use the binding constraint to rewrite the first-order and envelope conditions more simply as

$$C_t^{-\sigma} = \beta v'(K_{t+1})$$

and

$$v'(K_t) = (A + 1 - \delta)C_t^{-\sigma}.$$

Since this last equation must hold for all $t = 1, 2, 3, \ldots$, it also implies that

$$v'(K_{t+1}) = (A + 1 - \delta)C_{t+1}^{-\sigma}$$

for all $t = 0, 1, 2, \ldots$. Substituting this expression into the first-order condition yields

$$C_t^{-\sigma} = \beta(A + 1 - \delta)C_{t+1}^{-\sigma}$$

or

$$\frac{C_{t+1}}{C_t} = G_c = [\beta(A + 1 - \delta)]^{1/\sigma}.$$

4. Endogenous Growth with Comparison Utility: Discrete Time

Taking K_0 and the trajectory for Z_t as given, the consumer chooses C_t for $t = 0, 1, 2, \ldots$ and K_t for $t = 1, 2, 3, \ldots$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right) \left(\frac{C_t}{Z_t^{\gamma}}\right)^{1-\sigma}, \qquad (7)$$

subject to the constraint

$$AK_t + (1 - \delta)K_t - C_t \geq K_{t+1} \qquad (6)$$

for all $t = 0, 1, 2, \ldots$.

a. With the Bellman equation written as

$$v(K_t) = \max_{K_{t+1}} \left(\frac{1}{1-\sigma}\right) [(A + 1 - \delta)K_t - K_{t+1}]^{1-\sigma} Z_t^{-\gamma(1-\sigma)} + \beta v(K_{t+1}),$$

dynamic programming arguments imply that the solution to the planner's problem must satisfy the first-order condition

$$-[(A + 1 - \delta)K_t - K_{t+1}]^{-\sigma} Z_t^{-\gamma(1-\sigma)} + \beta v'(K_{t+1}) = 0$$

for all $t = 0, 1, 2, \ldots$, the envelope condition

$$v'(K_t) = (A + 1 - \delta)[(A + 1 - \delta)K_t - K_{t+1}]^{-\sigma} Z_t^{-\gamma(1-\sigma)},$$

for all $t = 1, 2, 3, \ldots$, and the binding constraint

$$K_{t+1} = (A + 1 - \delta)K_t - C_t$$

for all $t = 0, 1, 2, \ldots$.

b. Use the binding constraint to rewrite the first-order and envelope conditions more simply as

$$C_t^{-\sigma} Z_t^{-\gamma(1-\sigma)} = \beta v'(K_{t+1})$$

and

$$v'(K_t) = (A + 1 - \delta) C_t^{-\sigma} Z_t^{-\gamma(1-\sigma)}.$$

Since this last equation must hold for all $t = 1, 2, 3, \ldots$, it also implies that

$$v'(K_{t+1}) = (A + 1 - \delta) C_{t+1}^{-\sigma} Z_{t+1}^{-\gamma(1-\sigma)}$$

for all $t = 0, 1, 2, \ldots$. Substituting this expression into the first-order condition yields

$$C_t^{-\sigma} Z_t^{-\gamma(1-\sigma)} = \beta (A + 1 - \delta) C_{t+1}^{-\sigma} Z_{t+1}^{-\gamma(1-\sigma)}$$

or, more simply,

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} \left(\frac{Z_{t+1}}{Z_t}\right)^{\gamma(1-\sigma)} = \beta(A + 1 - \delta)$$

Hence, along a steady-state growth path with

$$C_{t+1}/C_t = Z_{t+1}/Z_t = G_c$$

it must be that

$$G_c^{\sigma} G_c^{\gamma(1-\sigma)} = \beta(A + 1 - \delta)$$

or

$$G_c = [\beta(A + 1 - \delta)]^{1/[\sigma + \gamma(1-\sigma)]}.$$