Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, December 20

1. Optimal Pollution Abatement

The consumer's problem is: given $k(0)$ and $E(0)$, choose $c(t)$ and $a(t)$ for all $t \in [0, \infty)$ and $k(t)$ and $E(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t}[\ln(c(t)) + \beta \ln(E(t))]dt$$

subject to the constraints

$$k(t)^\alpha - \delta k(t) - c(t) - a(t) \geq \dot{k}(t)$$

and

$$\gamma a(t) - k(t)^\alpha \geq \dot{E}(t)$$

for all $t \in [0, \infty)$.

a. The maximized current-value Hamiltonian for the consumer's problem is

$$\begin{aligned} H(k(t), E(t), \theta(t), \varphi(t)) = \max_{c(t), a(t)} \; & \ln(c(t)) + \beta \ln(E(t)) \\ & + \theta(t)[k(t)^\alpha - \delta k(t) - c(t) - a(t)] \\ & + \varphi(t)[\gamma a(t) - k(t)^\alpha] \end{aligned}$$

b. With the maximized Hamiltonian as defined above, the maximum principle implies that the solution to the consumer's problem is characterized by the first order conditions

$$\frac{1}{c(t)} - \theta(t) = 0$$

and

$$-\theta(t) + \gamma\varphi(t) = 0$$

for all $t \in [0, \infty)$, the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta] + \varphi(t)\alpha k(t)^{\alpha-1}$$

and

$$\dot{\varphi}(t) = \rho\varphi(t) - \frac{\beta}{E(t)}$$

for all $t \in [0, \infty)$ and the binding constraints

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t) - a(t)$$

and

$$\dot{E}(t) = \gamma a(t) - k(t)^\alpha$$

for all $t \in [0, \infty)$.

c. In a steady-state, $c(t) = c$, $a(t) = a$, $k(t) = k$, $E(t) = E$, $\theta(t) = \theta$, and $\varphi(t) = \varphi$ are all constant. Since the first-order condition for $a(t)$ implies

$$\varphi = (1/\gamma)\theta,$$

the differential equation for $\dot{\theta}(t)$ requires

$$0 = \rho\theta - \theta(\alpha k^{\alpha-1} - \delta) + (1/\gamma)\theta\alpha k^{\alpha-1},$$

which simplifies to

$$0 = \rho - \alpha k^{\alpha-1} + \delta + (1/\gamma)\alpha k^{\alpha-1}$$

and leads to the desired expression for the steady-state capital stock:

$$k = \left[\frac{\rho + \delta}{\alpha\left(1 - \frac{1}{\gamma}\right)}\right]^{1/(\alpha-1)}.$$

Although this expression for k is the only one needed to answer the questions in part (d), we can for the sake of completeness we can use the remaining optimality conditions to find steady-state values for the other variables. In particular, the constraint for $\dot{E}(t)$ determines

$$a = (1/\gamma)k^\alpha,$$

which shows why the restriction $\gamma > 1$ is needed: to make spending on pollution abatement a fraction instead of a multiple of output. The constraint for $\dot{k}(t)$ then implies

$$c = \left(1 - \frac{1}{\gamma}\right)k^\alpha - \delta k.$$

The first-order condition for $c(t)$ determines

$$\theta = 1/c,$$

and the differential equation for $\dot{\varphi}(t)$ determines

$$E = \frac{\beta\gamma}{\rho\theta}.$$

Thus, once the value of k is found, the values for c, a, E, θ, and φ can all be computed as well.

d. Returning to the expression for steady-state capital,

$$k = \left[\frac{\rho + \delta}{\alpha \left(1 - \frac{1}{\gamma}\right)} \right]^{1/(\alpha - 1)},$$

the restriction $\gamma > 1$ implies that this value is smaller than corresponding value

$$k = \left(\frac{\rho + \delta}{\alpha} \right)^{1/(\alpha - 1)}$$

from the neoclassical model without pollution and abatement. Intuitively, the optimal steady-state capital stock falls when production is accompanied by pollution. The steady-state capital stock here increases when the productivity parameter γ increases and converges to the steady-state capital stock from model without pollution as γ becomes arbitrarily large. Again, this is intuitive: we can think of the original neoclassical model as one in which, even if pollution occurs, abatement is costless. For extensions of and elaborations on the relatively simple model of pollution abatement considered here, see Emmett Keeler, Michael Spence, and Richard Zeckhauser ("The Optimal Control of Pollution," *Jorurnal of Economic Theory* 1971).

2. Investment Under Uncertainty

The Bellman equation for the firm's problem is

$$v(K_t, A_t) = \max_{I_t, K_{t+1}} A_t K_t - I_t - \left(\frac{\phi}{2} \right) I_t^2 + \left(\frac{1}{1+r} \right) E_t v(K_{t+1}, A_{t+1})$$
$$\text{subject to } (1 - \delta) K_t + I_t \geq K_{t+1}.$$

a. Use λ_t to denote the Lagrange multiplier on the constraint from the static problem on the right-hand side of the Bellman equation. The first-order conditions for I_t and K_{t+1} are obtained by differentiating the Lagrangian for the the static problem on the right-hand side of the Bellman equation partially with respect to I_t

$$-1 - \phi I_t + \lambda_t = 0$$

and with respect to K_{t+1}.

$$\left(\frac{1}{1+r} \right) E_t v_1(K_{t+1}, A_{t+1}) - \lambda_t = 0.$$

The envelope condition for K_t is found by differentiating both sides of the Bellman equation with respect to K_t, once again after forming the Lagrangian for the the static problem on the right-hand side:

$$v_1(K_t, A_t) = A_t + \lambda_t (1 - \delta).$$

b. As the envelope condition must hold for all periods $t = 1, 2, 3, \ldots$, it implies that

$$v_1(K_{t+1}, A_{t+1}) = A_{t+1} + \lambda_{t+1}(1-\delta)$$

for all $t = 0, 1, 2, \ldots$. Substituting this expression into the first-order condition for K_{t+1} yields

$$\lambda_t = \left(\frac{1}{1+r}\right) E_t A_{t+1} + \left(\frac{1-\delta}{1+r}\right) E_t \lambda_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

c. Note that this last result also implies

$$\lambda_{t+1} = \left(\frac{1}{1+r}\right) E_{t+1} A_{t+2} + \left(\frac{1-\delta}{1+r}\right) E_{t+1} \lambda_{t+2}$$

which can be substituted into the right-hand side of the original expression for λ_t to get

$$\lambda_t = \left(\frac{1}{1+r}\right) E_t A_{t+1} + \left(\frac{1}{1+r}\right)\left(\frac{1-\delta}{1+r}\right) E_t E_{t+1} A_{t+2} + \left(\frac{1-\delta}{1+r}\right)^2 E_t E_{t+1} \lambda_{t+2}$$

or, using the law of iterated expectations to simplify,

$$\lambda_t = \left(\frac{1}{1+r}\right) E_t A_{t+1} + \left(\frac{1}{1+r}\right)\left(\frac{1-\delta}{1+r}\right) E_t A_{t+2} + \left(\frac{1-\delta}{1+r}\right)^2 E_t \lambda_{t+2}$$

Since, likewise,

$$\lambda_{t+2} = \left(\frac{1}{1+r}\right) E_{t+2} A_{t+3} + \left(\frac{1-\delta}{1+r}\right) E_{t+2} \lambda_{t+3},$$

we can continue the forward recursive substitution to get

$$\begin{aligned} \lambda_t = {} & \left(\frac{1}{1+r}\right) E_t A_{t+1} + \left(\frac{1}{1+r}\right)\left(\frac{1-\delta}{1+r}\right) E_t A_{t+2} \\ & + \left(\frac{1}{1+r}\right)\left(\frac{1-\delta}{1+r}\right)^2 E_t A_{t+2} + \left(\frac{1-\delta}{1+r}\right)^3 E_t \lambda_{t+3} \end{aligned}$$

Repeating this process and using the transversality condition

$$\lim_{j\to\infty} \left(\frac{1-\delta}{1+r}\right)^j E_t \lambda_{t+j} = 0$$

then implies

$$\lambda_t = \left(\frac{1}{1+r}\right) \sum_{j=1}^{\infty} \left(\frac{1-\delta}{1+r}\right)^{j-1} E_t A_{t+j}.$$

d. Since the autoregressive law of motion for productivity implies that

$$E_t A_{t+j} = A + \rho^j (A_t - A)$$

for all $j = 1, 2, 3, \ldots$, the solution for λ_t implies

$$\begin{aligned}\lambda_t &= \left(\frac{1}{1+r}\right) \sum_{j=1}^{\infty} \left(\frac{1-\delta}{1+r}\right)^{j-1} A + \left(\frac{\rho}{1+r}\right) \sum_{j=1}^{\infty} \left[\frac{\rho(1-\delta)}{1+r}\right]^{j-1} (A_t - A) \\ &= \left(\frac{1}{r+\delta}\right) A + \left[\frac{\rho}{1+r-\rho(1-\delta)}\right] (A_t - A).\end{aligned}$$

The first-order condition for I_t then implies that

$$I_t = \frac{1}{\phi} \left\{ \left(\frac{1}{r+\delta}\right) A + \left[\frac{\rho}{1+r-\rho(1-\delta)}\right] (A_t - A) - 1 \right\}$$

for all $t = 0, 1, 2, \ldots$. Assuming that $r > 0$ and $1 > \rho \geq 0$, this expression implies that optimal investment I_t depends positively on average productivity A. Assuming also that $\rho > 0$, the expression implies that investment I_t is larger during periods when realized productivity A_t is larger. When $\rho = 0$, however, I_t is constant and, in particular, independent of A_t. This is because investment is forward-looking: it depends on expected future productivity instead of today's productivity. And with $\rho = 0$, future productivity is unforecastable. In general, the value of A_t today matters only to the extent that is helps forecast future values of productivity A_{t+j}, $j = 1, 2, 3, \ldots$. That the solution for optimal investment depends importantly on the persistence parameter ρ from the law of motion for productivity provides an example of the "cross-equation restrictions" that Lars Hansen and Thomas Sargent ("Formulating and Estimating Dynamic Linear Rational Expectations Models," *Journal of Economic Dynamics and Control* 1980) called the "hallmark of rational expectations models."