Solutions to Final Exam

ECON 772001 - Math for Economists | Peter Ireland
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Due Tuesday, December 21 at 12noon

1. The Tobin Effect

The condition linking the steady-state value k of physical capital to the inflation rate π is

$$0 = s(k^\alpha - \delta k) - nk - (1-s)n\phi(\pi)k, \tag{5}$$

where $\phi(\pi) > 0$ and $\phi'(\pi) < 0$ for all π.

a. The total differential of (5) is

$$0 = [s(\alpha k^{\alpha-1} - \delta) - n - (1-s)n\phi(\pi)]dk - (1-s)n\phi'(\pi)kd\pi,$$

which can be rearranged to obtain the expression

$$\frac{dk}{d\pi} = \frac{(1-s)n\phi'(\pi)k}{s(\alpha k^{\alpha-1} - \delta) - n - (1-s)n\phi(\pi)}.$$

Since $\phi'(\pi) < 0$, the numerator is negative. And since $\phi(\pi) > 0$, the denominator is also negative if the stability condition from the original Solow (1956) model is assumed to hold. Therefore, the derivative is positive, confirming Tobin's conjecture.

2. An Optimizing Model of Saving and Asset Allocation

The household takes initial wealth $a(0)$ as given and chooses $c(t)$, $k(t)$, and $m(t)$ for all $t \in [0, \infty)$ and $a(t)$ for all $t \in (0, \infty)$ to maximize utility

$$\int_0^\infty e^{-(\rho-n)t} U(c(t), m(t)) \mathrm{d}t. \tag{6}$$

subject to the constraints

$$a(t) \geq k(t) + m(t) \tag{7}$$

and

$$k(t)^\alpha + x(t) - \delta k(t) - \pi(t)m(t) - c(t) - na(t) \geq \dot{a}(t) \tag{8}$$

for all $t \in [0, \infty)$.

a. In current value form, the maximized Hamiltonian for the household's problem is

$$\begin{aligned} H(a(t),\theta(t);t) = \max_{c(t),k(t),m(t)} \; & U(c(t),m(t)) \\ & + \theta(t)[k(t)^\alpha + x(t) - \delta k(t) - \pi(t)m(t) - c(t) - na(t)] \\ & \text{subject to } a(t) \geq k(t) + m(t). \end{aligned}$$

Note that the maximized Hamiltonian depends on t as well as $a(t)$ and $\theta(t)$ because inflation $\pi(t)$ may in general depend on t.

b. According to the maximum principle, the solution to the household's problem is characterized by the first-order condition for $c(t)$,

$$U_1(c(t),m(t)) - \theta(t) = 0,$$

the first-order condition for $k(t)$,

$$\theta(t)[\alpha k(t)^{\alpha-1} - \delta] - \lambda(t) = 0,$$

and the first-order condition for $m(t)$,

$$U_2(c(t),m(t)) - \theta(t)\pi(t) - \lambda(t) = 0,$$

along with the pair of differential equations

$$\dot{\theta}(t) = (\rho - n)\theta(t) + \theta(t)n - \lambda(t) = \rho\theta(t) - \lambda(t).$$

and

$$\dot{a}(t) = k(t)^\alpha + x(t) - \delta k(t) - \pi(t)m(t) - c(t) - na(t).$$

In these optimality conditions, U_1 and U_2 denote the partial derivatives of the utility function U with respect to consumption and real money balances per capita and $\lambda(t)$ is the Lagrange multiplier on the constraint from the static problem defining the maximized Hamiltonian.

c. In any steady-state, all per-capita variables will be constant, including $c(t) = c$, $k(t) = k$, and $m(t) = m$. Moreover, the inflation rate $\pi(t) = \pi$ is constant as well. Thus, the first-order condition for $c(t)$ implies that $\theta(t) = \theta$ is constant, and then the first-order condition for $m(t)$ implies that $\lambda(t) = \lambda$ is also constant. Since $\dot{\theta}(t) = 0$, the first differential equation requires that

$$\lambda = \rho\theta.$$

Substituting this result into the first-order condition for $k(t)$ shows that

$$\alpha k^{\alpha-1} - \delta - \rho = 0.$$

Interestingly, this expression for the steady-state capital is the same as the one from the non-monetary Ramsey model. It implies that the steady-state capital stock is

$$k = \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}.$$

d. Since the expression of k just derived does not depend on the inflation rate π, in this model the Tobin effect disappears. Instead, money is "superneutral" in the sense that changes in the inflation have no effect on steady-state capital or output.

It should be noted that while inflation has no effect on the capital stock in the steady state of Sidrauski's (1967) model, it will affect capital accumulation along the transition path from an arbitrary value of $a(0)$ to the steady state. This was shown by Stanley Fischer in his article "Capital Accumulation on the Transition Path in a Monetary Optimizing Model" (*Econometrica* 47 (1979): 1433-1439). It should be noted, as well, that further modifications of or extensions to Sidrauski's (1967) model can re-introduce effects of steady-state inflation on the capital stock, with some model variants implying that inflation increases capital and others implying that inflation decreases capital. For a partial survey, see Athanasios Orphanides and Robert M. Solow's "Money, Inflation and Growth" (Chapter 6 in Volume 1 of the *Handbook of Monetary Economics*, edited by B.M. Friedman and F.H. Hahn, Elsevier, 1990).

3. Precautionary Saving

The consumer takes A_0 as given, and chooses contingency plans for c_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$ in order to maximize expected utility

$$E_0 \sum_{t=0}^{\infty} \beta^t [-\frac{1}{\theta} \exp(-\theta c_t)] \tag{9}$$

subject to the constraints

$$R(A_t - c_t) + Y_{t+1} \geq A_{t+1} \tag{10}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of $Y_{t+1} \sim N(\bar{Y}, \sigma^2)$.

a. The Bellman equation for the consumer's problem is

$$v(A_t) = \max_{c_t} \; -\frac{1}{\theta} \exp(-\theta c_t) + \beta E_t v[R(A_t - c_t) + Y_{t+1}].$$

b. Using the conjecture that the value function takes the specific form

$$v(A_t) = -\frac{1}{\theta F} \exp[-\theta(FA_t + G)],$$

along with the algebraic manipulations shown previously, the Bellman equation specializes to

$$\begin{aligned} &-\frac{1}{\theta F} \exp[-\theta(FA_t + G)] \\ &= \max_{c_t} \; -\frac{1}{\theta} \exp(-\theta c_t) - \frac{\beta}{\theta F} \exp(-\theta G) \exp\left(-\theta F \bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right) \exp[-\theta F R(A_t - c_t)]. \end{aligned}$$

The first-order condition for c_t is then

$$\exp(-\theta c_t) - \exp(-\theta G)\exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp[-\theta FR(A_t - c_t)] = 0$$

and the envelope condition is

$$\exp[-\theta(FA_t + G)] = \exp(-\theta G)\exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp[-\theta FR(A_t - c_t)],$$

where, in both cases, use has been made of the assumption that $\beta R = 1$.

c. Use the envelope condition to replace the second term in the first-order condition and thereby obtain the much simpler expression

$$\exp(-\theta c_t) - \exp[-\theta(FA_t + G)] = 0,$$

which implies that

$$c_t = FA_t + G$$

and hence

$$A_t - c_t = (I - F)A_t - G.$$

d. Substitute the expression for saving $A_t - c_t$ just derived into the right-hand side of the envelope condition to obtain

$$\exp[-\theta(FA_t + G)] = \exp(-\theta G)\exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp\{-\theta FR[(I - F)A_t - G]\},$$

or

$$\exp(-\theta FA_t) = \exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp[-\theta FR(I - F)A_t]\exp(\theta FRG).$$

If the term on the left involving A_t is to equal the term on the right involving A_t, it must be that

$$1 = R(1 - F).$$

This expression leads to the solution

$$F = \frac{R - 1}{R}.$$

With this solution substituted in, the envelope condition simplifies to

$$1 = \exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp(\theta FRG),$$

which requires that

$$-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2} + \theta FRG = 0.$$

This expression leads to the solution

$$G = \frac{\bar{Y}}{R} - \frac{\theta F \sigma^2}{2R} = \frac{\bar{Y}}{R} - \frac{\theta(R - 1)\sigma^2}{2R^2}.$$

e. Substituting the solutions for F and G into the expressions for consumption and saving derived previously yields

$$c_t = \left(\frac{R-1}{R}\right) A_t + \frac{\bar{Y}}{R} - \frac{\theta(R-1)\sigma^2}{2R^2}$$

and

$$A_t - c_t = \left(\frac{1}{R}\right) A_t - \frac{\bar{Y}}{R} + \frac{\theta(R-1)\sigma^2}{2R^2}.$$

From the last term in this last expression, we can confirm that the consumer saves more when he or she is more risk averse and when the volatility of income goes up.