Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Thursday, December 10, at 12 noon

1. Consumption and Labor Supply in Continuous Time

Taking his or her initial stock of bonds $B(0)$ as given, the consumer chooses consumption $c(t)$ and labor supply $h(t)$ for all $t \in [0,\infty)$ and bond holding $B(t)$ for all $t \in (0,\infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t}\left[\frac{c(t)^{1-\sigma}-1}{1-\sigma} - \frac{\alpha h(t)^{1+\varphi}}{1+\varphi}\right] dt$$

subject to the constraint constraint

$$r(t)B(t) + W(t)h(t) - P(t)c(t) \geq \dot{B}(t). \tag{1}$$

for all $t \in [0,\infty)$. In current-value form, the maximized Hamiltonian for this problem is

$$H(B(t),\theta(t)) = \max_{c(t),h(t)} \frac{c(t)^{1-\sigma}-1}{1-\sigma} - \frac{\alpha h(t)^{1+\varphi}}{1+\varphi} + \theta(t)[r(t)B(t) + W(t)h(t) - P(t)c(t)].$$

a. According to the maximum principle, the solution to the consumer's problem is characterized by the first-order conditions

$$c(t)^{-\sigma} - \theta(t)P(t) = 0$$

and

$$-\alpha h(t)^{\varphi} + \theta(t)W(t) = 0$$

for $c(t)$ and $h(t)$ and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_B(B(t),\theta(t)) = \rho\theta(t) - \theta(t)r(t)$$

and

$$\dot{B}(t) = H_\theta(B(t),\theta(t)) = r(t)B(t) + W(t)h(t) - P(t)c(t).$$

for $\theta(t)$ and $B(t)$.

b. Rearrange the first-order condition for $c(t)$ to read

$$\theta(t) = \frac{c(t)^{-\sigma}}{P(t)}$$

and substitute this expression into the first-order condition for $h(t)$ to obtain

$$\frac{\alpha h(t)^{\varphi}}{c(t)^{-\sigma}} = \frac{W(t)}{P(t)}.$$

The numerator of the fraction on the left-hand side is the marginal disutility of labor or, equivalently, the marginal utility of leisure. The denominator is the marginal utility of consumption. Hence, the ratio itself is the marginal rate of substitution between consumption and leisure, which the optimizing consumer sets equal to the real wage $W(t)/P(t)$.

c. Rewrite the first-order condition for $c(t)$ as

$$c(t)^{-\sigma} = \theta(t)P(t)$$

and differentiate both sides with respect to t to obtain

$$-\sigma c(t)^{-\sigma-1}\dot{c}(t) = \dot{\theta}(t)P(t) + \theta(t)\dot{P}(t).$$

Next, use the first-order condition for $c(t)$ to replace $c(t)^{-\sigma}$ with $\theta(t)P(t)$ on the left-hand side and the differential equation for $\theta(t)$ to replace $\dot{\theta}(t)$ with $[\rho - r(t)]\theta(t)$ on the right-hand side. Now

$$-\sigma\theta(t)P(t)\left[\frac{\dot{c}(t)}{c(t)}\right] = [\rho - r(t)]\theta(t)P(t) + \theta(t)\dot{P}(t).$$

Divide both sides of this last expression by $-\sigma\theta(t)P(t)$ and use the notation $\pi(t) = \dot{P}(t)/P(t)$ for inflation to obtain

$$\frac{\dot{c}(t)}{c(t)} = (1/\sigma)[r(t) - \pi(t) - \rho],$$

which is the Euler equation for consumption.

d. The Euler equation implies that it will be optimal for the consumer to hold consumption constant, with $\dot{c}(t) = 0$, whenever the real interest rate $r(t) - \pi(t)$ equals the rate of time preference ρ.

2. Uncovered Interest Rate Parity?

The investor takes initial wealth A_0 as given, and chooses contingency plans for c_t, B_t^{usd}, and B_t^{yen} for $t = 0, 1, 2, \ldots$ and A_t for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the constraints

$$A_t + W_t \geq P_t c_t + B_t^{usd} + e_t B_t^{yen}$$

and

$$B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen} \geq A_{t+1},$$

where each of these two constraints must hold for all $t = 0.1, 2, \ldots$. and all possible realizations of the random shocks.

The Bellman equation for this problem is

$$\begin{aligned} v(A_t, W_t, P_t, e_t) &= \max_{B_t^{usd}, B_t^{yen}} u\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right) \\ &\quad + \beta E_t v(B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}, W_{t+1}, P_{t+1}, e_{t+1}). \end{aligned}$$

a. Using the Bellman equation from above, the first-order conditions for B_t^{usd} and B_t^{yen} are

$$\begin{aligned} &\left(\frac{1}{P_t}\right) u'\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right) \\ &= \beta R^{usd} E_t[v_1(B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}, W_{t+1}, P_{t+1}, e_{t+1})] \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{e_t}{P_t}\right) u'\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right) \\ &= \beta R^{yen} E_t[e_{t+1} v_1(B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}, W_{t+1}, P_{t+1}, e_{t+1})], \end{aligned}$$

while the envelope condition for A_t is

$$v_1(A_t, W_t, P_t, e_t) = \left(\frac{1}{P_t}\right) u'\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right).$$

Note that, in the first-order conditions, the constant interest rates R^{usd} and R^{yen} can be pulled outside of the conditional expectations, but the unknown future exchange rate e_{t+1} must remain inside the expectation.

b. The binding constraints can be used to simplify the first-order and envelope conditions so that they read, more compactly,

$$\frac{u'(c_t)}{P_t} = \beta R^{usd} E_t[v_1(A_{t+1}, W_{t+1}, P_{t+1}, e_{t+1})],$$

$$\frac{e_t u'(c_t)}{P_t} = \beta R^{yen} E_t[e_{t+1} v_1(A_{t+1}, W_{t+1}, P_{t+1}, e_{t+1})],$$

and

$$v_1(A_t, W_t, P_t, e_t) = \frac{u'(c_t)}{P_t}.$$

Since the envelope condition must hold for all $t = 1, 2, 3, \ldots$, it also implies that

$$v_1(A_{t+1}, W_{t+1}, P_{t+1}, e_{t+1}) = \frac{u'(c_{t+1})}{P_{t+1}}.$$

Substituting these expressions into the first-order conditions yields

$$\frac{u'(c_t)}{P_t} = \beta R^{usd} E_t \left[\frac{u'(c_{t+1})}{P_{t+1}} . \right]$$

and

$$\frac{e_t u'(c_t)}{P_t} = \beta R^{yen} E_t \left[\frac{e_{t+1} u'(c_{t+1})}{P_{t+1}} . \right] .$$

c. Using $m_{t+1} = \beta u'(c_{t+1})/u'(c_t)$ to denote the consumer IMRS and $\pi_{t+1} = P_{t+1}/P_t$ to denote the US inflation rate, the first-order conditions can be written even more compactly as

$$\frac{1}{R^{usd}} = E_t \left(\frac{m_{t+1}}{\pi_{t+1}} \right)$$

and

$$\frac{1}{R^{yen}} = E_t \left[\left(\frac{m_{t+1}}{\pi_{t+1}} \right) \left(\frac{e_{t+1}}{e_t} \right) \right] .$$

d. We can now see that the key to understanding observed deviations from the UIP condition (2) is the fact that the expected value of the product of m_{t+1}/π_{t+1} and e_{t+1}/e_t is not the same as the product of the expected values of m_{t+1}/π_{t+1} and e_{t+1}/e_t. In fact, the first-order conditions imply that

$$\begin{aligned} \frac{1}{R^{yen}} &= E_t \left[\left(\frac{m_{t+1}}{\pi_{t+1}} \right) \left(\frac{e_{t+1}}{e_t} \right) \right] \\ &= E_t \left(\frac{m_{t+1}}{\pi_{t+1}} \right) E_t \left(\frac{e_{t+1}}{e_t} \right) + cov_t \left(\frac{m_{t+1}}{\pi_{t+1}}, \frac{e_{t+1}}{e_t} \right) \\ &= \left(\frac{1}{R^{usd}} \right) E_t \left(\frac{e_{t+1}}{e_t} \right) + cov_t \left(\frac{m_{t+1}}{\pi_{t+1}}, \frac{e_{t+1}}{e_t} \right) \end{aligned}$$

so that instead of (2), we have the more general relation

$$\frac{R^{usd}}{R^{yen}} = E_t \left(\frac{e_{t+1}}{e_t} \right) + R^{usd} cov_t \left(\frac{m_{t+1}}{\pi_{t+1}}, \frac{e_{t+1}}{e_t} \right) .$$

This relation makes clear that UIP will only hold in the very special case in which m_{t+1}/π_{t+1} and e_{t+1}/e_t are uncorrelated. This special case would arise, most plausibly, under a regime of fixed exchange rates in which $e_{t+1}/e_t = 1$ for sure for all $t = 0, 1, 2, \ldots$. But suppose the US dollar depreciates, so that e_{t+1}/e_t is large, whenever the US economy experiences a period of deflationary recession that is, an episode where m_{t+1} is large because consumption growth is slow and π_{t+1} is small because inflation is low. Now Japanese bonds provide "insurance" against a deflationary recession in the US: the dollar depreciation under those circumstances means that Japanese bonds pay off particularly well, in dollar terms, exactly when US investors need the payoffs most. Under those circumstances, US investors will still be willing to hold Japanese bonds, despite their low yields and even without the expectation that the dollar will depreciate on average.