Solutions to Final Exam

ECON 772001 - Math for Economists | Peter Ireland
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Due Thursday, December 19, at 12 noon

1. Consumption, Investment, and Capital Accumulation in a Small Open Economy

The representative household in the small open economy chooses paths for the flow variables $C(t)$ and $I(t)$ for all $t \in [0, \infty)$ and the stock variables $B(t)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \ln(C(t))\, dt, \tag{3}$$

subject to the constraints

$$rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2 \geq \dot{B}(t) \tag{1}$$

and

$$I(t) - \delta K(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0, \infty)$, taking the initial conditions $B(0)$ and $K(0)$ as given.

The maximized current value Hamiltonian for this problem is

$$\begin{aligned} H(B(t), K(t), \theta(t), q(t)) = \max_{C(t), I(t)} \{ &\ln(C(t)) \\ &+ \theta(t)[rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2] \\ &+ q(t)[I(t) - \delta K(t)]\}. \end{aligned} \tag{4}$$

a. From the current-value Hamiltonian, the solution to the dynamic problem must satisfy the first-order conditions

$$\frac{1}{C(t)} - \theta(t) = 0$$

and

$$-\theta(t)[1 + \phi I(t)] + q(t) = 0$$

and the differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_B(B(t), K(t), \theta(t), q(t)) = \rho\theta(t) - \theta(t)r,$$

$$\dot{q}(t) = \rho q(t) - H_K(B(t), K(t), \theta(t), q(t)) = \rho q(t) - \theta(t)\alpha K(t)^{\alpha-1} + q(t)\delta,$$

$$\dot{B}(t) = H_\theta(B(t), K(t), \theta(t), q(t)) = rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2,$$

and

$$\dot{K}(t) = H_q(B(t), K(t), \theta(t), q(t)) = I(t) - \delta K(t).$$

b. If $\rho = r$, the first differential equation from part (a) implies that $\dot{\theta}(t) = 0$, so that $\theta(t)$ is equal to some constant θ for all $t \in [0, \infty)$. The first-order condition for $C(t)$ then implies that consumption is also constant for all $t \in [0, \infty)$. These conditions hold even as investment and capital are still in transition towards their steady-state values. The household borrower or lends internationally to smooth out its consumption over time, consistent with the permanent income hypothesis.

c. Still assuming that $\rho = r$, so that $\theta(t) = \theta$ is constant, the first-order condition for $I(t)$ implies that

$$q(t) = \theta[1 + \phi I(t)]$$

and

$$\dot{q}(t) = \theta\phi\dot{I}(t)$$

for all $t \in [0, \infty)$. Substituting these conditions into the second differential equation from part (a) yields

$$\theta\phi\dot{I}(t) = r\theta[1 + \phi I(t)] - \theta\alpha K(t)^{\alpha-1} + \delta\theta[1 + \phi I(t)].$$

Dividing through by $\theta\phi$ yields the differential equation

$$\dot{I}(t) = (1/\phi)\{(\delta + r)[1 + \phi I(t)] - \alpha K(t)^{\alpha-1}\}$$

involving $I(t)$ and $K(t)$ alone.

d. Combined with the fourth differential equation from part (a),

$$\dot{K}(t) = I(t) - \delta K(t),$$

the differential equation just derived in part (c),

$$\dot{I}(t) = (1/\phi)\{(\delta + r)[1 + \phi I(t)] - \alpha K(t)^{\alpha-1}\}$$

form a system of two differential equations that coincide, exactly, with those that characterize the solution to the "investment with adjustment costs" example from problem set 11. Intuitively, the domestic household's ability to borrow from abroad to smooth consumption also allows it to choose time paths for investment and capital that maximize the present value of output net of investment and adjustment costs, discounted at the world real interest rate. Exactly as in problem set 11, the differential equation for $K(t)$ implies that

$$\dot{K}(t) = 0 \text{ when } I(t) = \delta K(t),$$

$$\dot{K}(t) > 0 \text{ when } I(t) > \delta K(t),$$

and

$$\dot{K}(t) < 0 \text{ when } I(t) < \delta K(t),$$

and differential equation for $I(t)$ implies that

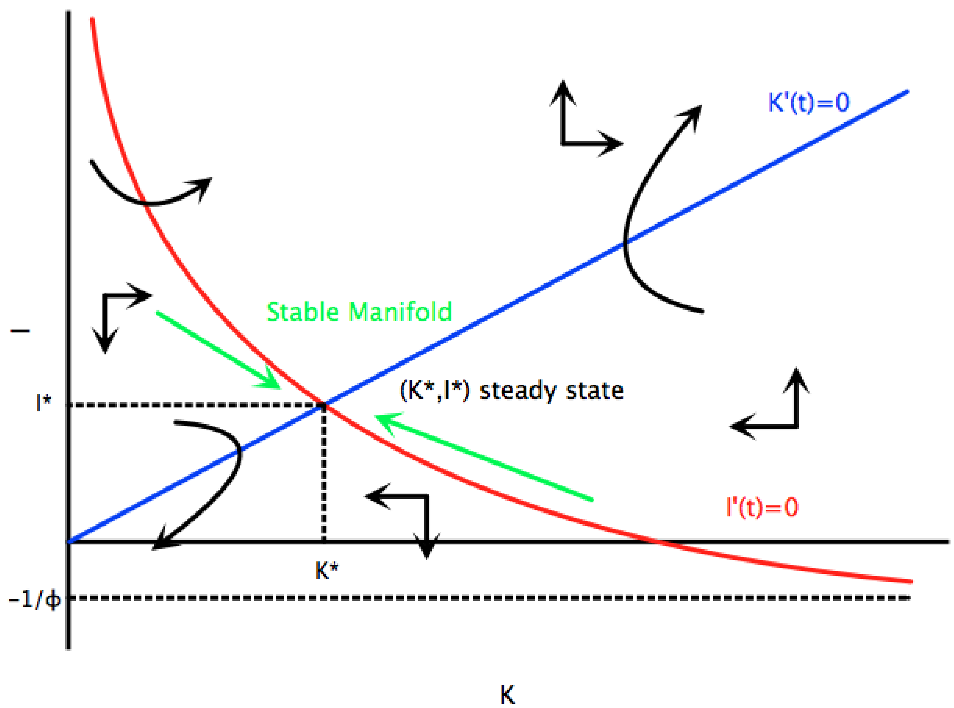
$$\dot{I}(t) = 0 \text{ when } I(t) = \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right],$$

$$\dot{I}(t) > 0 \text{ when } I(t) > \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right],$$

and

$$\dot{I}(t) < 0 \text{ when } I(t) < \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right].$$

The phase diagram shown below illustrates these conditions and also reveals that starting from any value $K(0) > 0$ for the initial capital stock, there is a unique value of investment $I(0)$ such that, starting from $I(0)$ and $K(0)$, the optimally-chosen paths for $I(t)$ and $K(t)$ converge to the steady-state values I^* and K^*.



2. Random Walk Consumption and the Marginal Propensity to Consume

The consumer chooses contingency plans for s_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

subject to A_0 given, the constraint

$$(1 + r)s_t \geq A_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$, and the Markov process generating the random income stream y_t for all $t = 0, 1, 2, \ldots$. The Bellman equation for this problem is

$$v(A_t, y_t) = \max_{s_t} \; u(A_t + y_t - s_t) + \beta E_t\{v[(1 + r)s_t, y_{t+1}]\}.$$

a. Using the Bellman equation from above, the first-order condition for s_t,

$$-u'(A_t + y_t - s_t) + \beta(1 + r)E_t\{v_1[(1 + r)s_t, y_{t+1}]\} = 0,$$

and the envelope condition for A_t,

$$v_1(A_t, y_t) = u'(A_t + y_t - s_t),$$

b. Using the definition

$$s_t = A_t + y_t - c_t \tag{5}$$

of gross savings s_t and the binding constraint from (6), the first-order and envelope conditions from part (a) can be written more simply as

$$u'(c_t) = \beta(1 + r)E_t[v_1(A_{t+1}, y_{t+1})]$$

and

$$v_1(A_t, y_t) = u'(c_t).$$

Since the envelope condition must hold for all $t = 1, 2, 3, \ldots$, it also implies that

$$v_1(A_{t+1}, y_{t+1}) = u'(c_{t+1}),$$

for all $t = 0, 1, 2, \ldots$. Hence, it can be substituted into the first-order condition to obtain the optimality condition

$$u'(c_t) = \beta(1 + r)E_t[u'(c_{t+1})],$$

or

$$\frac{1}{1+r} = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right],$$

linking the consumer's intertemporal marginal rate of substitution to the real interest rate.

c. When $\beta(1 + r) = 1$ and

$$u(c_t) = -(1/2)(c_t - b)^2$$

the optimality condition from part (b) specializes to

$$-c_t - b = -E_t(c_{t+1} - b),$$

implying that consumption follows a martingale:

$$c_t = E_t c_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

d. By combining (5) and (6) to obtain

$$A_t + y_t - c_t \geq \frac{A_{t+1}}{1+r},$$

iterating by forward substitution, imposing some finite limit on borrowing to rule out Ponzi schemes, and invoking the transversality condition ruling out an overaccumulation of savings, one can derive the consumer's present value budget constraint

$$A_t + \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} = \sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j}.$$

Here in this stochastic model, this present value budget constraint must hold for all possible realizations $\{y_{t+j}\}_{j=0}^{\infty}$ of the path for future income. Therefore, the same equality must hold in expected value at time t, so that

$$A_t + \sum_{j=0}^{\infty} \frac{E_t y_{t+j}}{(1+r)^j} = \sum_{j=0}^{\infty} \frac{E_t c_{t+j}}{(1+r)^j}.$$

Using the result from part (c) that optimal consumption follows a martingale, the restriction that $\beta = 1/(1+r)$, and Euclid's formula

$$\sum_{j=0}^{\infty} \beta^j = \frac{1}{1-\beta}$$

for the infinite sum, this version of the present-value budget constraint pins down the optimal level of consumption at each date $t = 0, 1, 2, \ldots$ as

$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty} \beta^j E_t y_{t+j}\right].$$

Suppose now that income follows a first-order autoregressive process, with

$$y_{t+1} = \bar{y} + \rho(y_t - \bar{y}) + \varepsilon_{t+1},$$

where $\bar{y}$ is the long-run average level of income, the parameter ρ, satisfying $0 \leq \rho < 1$, governs the persistence of fluctuations of income above or below its long-run average, and ε_{t+1} is a serially uncorrelated shock with mean zero. This law of motion for income implies that

$$E_t y_{t+j} = \bar{y} + \rho^j (y_t - \bar{y})$$

and hence that

$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty} \beta^j \bar{y} + \sum_{j=0}^{\infty} (\beta\rho)^j (y_t - \bar{y})\right].$$

Applying Euclid's formula to the two infinite sums that remain in this equation, yields

$$c_t = (1-\beta)A_t + \bar{y} + \left(\frac{1-\beta}{1-\beta\rho}\right)(y_t - \bar{y}).$$

This last expression reveals that the marginal propensity to consume out of permanent income $\bar{y}$ holding $y_t - \bar{y}$ constant equals one, whereas the marginal propensity to consume out of deviations of income from its long-run average $y_t - \bar{y}$ holding $\bar{y}$ constant is

$$\frac{1-\beta}{1-\beta\rho} < 1$$

and gets smaller as ρ, measuring the persistence of those deviations, declines.