Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
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Thursday, December 6, 1:30 - 2:45pm

1. Learning By Doing

The monopolist takes the initial condition $x(0) = 0$ as given, and chooses continuously differentiable functions $y(t)$ for $t \in [0, \infty)$ and $x(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-rt} \left\{ y(t)^{1-\alpha} - \frac{y(t)}{[1 + x(t)]^\gamma} \right\} \mathrm{d}t$$

subject to

$$y(t) \geq \dot{x}(t) \tag{1.2}$$

for all $t \in [0, \infty)$.

a. In present value form, the maximized Hamiltonian for the monopolist's problem is

$$H(x(t), \pi(t); t) = \max_{y(t)} \; e^{-rt} \left\{ y(t)^{1-\alpha} - \frac{y(t)}{[1 + x(t)]^\gamma} \right\} + \pi(t) y(t).$$

b. According to the maximum principle, the solution to the monopolist's problem is characterized by the first-order condition

$$e^{-rt} \left\{ (1-\alpha) y(t)^{-\alpha} - \frac{1}{[1 + x(t)]^\gamma} \right\} + \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_x(x(t), \pi(t); t) = -\gamma e^{-rt} \left\{ \frac{y(t)}{[1 + x(t)]^{1+\gamma}} \right\}$$

and

$$\dot{x}(t) = H_\pi(x(t), \pi(t); t) = y(t)$$

for all $t \in [0, \infty)$.

c. Under perfect competition, output during each period $t \in [0, \infty)$ is described by

$$y(t) = \dot{x}(t) = [1 + x(t)]^{\gamma/\alpha}. \tag{1.5}$$

In the first-order condition for the dynamic problem, the term inside brackets.

$$(1-\alpha) y(t)^{-\alpha} - \frac{1}{[1 + x(t)]^\gamma}$$

shows that, without learning-by-doing, the monopolist would set marginal revenue equal to marginal cost by choosing

$$y(t) = (1-\alpha)^{1/\alpha}[1+x(t)]^{\gamma/\alpha}.$$

Since $0 < \alpha < 1$, for any given value of $x(t)$, this choice for $y(t)$ is smaller than under perfect competition. Since the monopolist also takes into account the effects that learning-by-doing today has on future marginal costs, however, the first-order condition dictates that term inside brackets

$$(1-\alpha)y(t)^{-\alpha} - \frac{1}{[1+x(t)]^{\gamma}}$$

be set equal to $-e^{rt}\pi(t) < 0$ instead of zero. This second effect pushes the monopolist in the opposite direction, towards choosing a larger value of $y(t)$.

2. Optimal Growth via Human Capital Accumulation

The representative consumer takes the initial stock of human capital $k_0 > 0$ as given and chooses the sequence $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{ \frac{[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} \right\}.$$

To solve this problem using dynamic programming, it is most convenient to treat k_t as the period-t state variable and k_{t+1} as the period-t control variable, and then to write the Bellman equation as

$$v(k_t;t) = \max_{k_{t+1}} \frac{[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} + \beta v(k_{t+1};t+1). \tag{2.2}$$

a. For this problem, the value function takes the specific, time-invariant form

$$v(k_t;t) = v(k_t) = \frac{Ak_t^{\alpha(1-\sigma)}}{1-\sigma},$$

where A is an undetermined coefficient. Substituting this guess into the Bellman equation yields

$$\frac{Ak_t^{\alpha(1-\sigma)}}{1-\sigma} = \max_{k_{t+1}} \frac{[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} + \frac{\beta Ak_{t+1}^{\alpha(1-\sigma)}}{1-\sigma}.$$

The first-order condition for k_{t+1} is then

$$\alpha[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)-1}\varphi'(k_{t+1}/k_t) + \alpha\beta Ak_{t+1}^{\alpha(1-\sigma)-1} = 0$$

and the envelope condition for k_t is

$$\alpha Ak_t^{\alpha(1-\sigma)-1} = \alpha[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)-1}[\varphi(k_{t+1}/k_t) - (k_{t+1}/k_t)\varphi'(k_{t+1}/k_t)].$$

b. Note that the first-order and envelope conditions from part (b), above, can be rewritten as

$$\varphi(k_{t+1}/k_t)^{\alpha(1-\sigma)-1}\varphi'(k_{t+1}/k_t) + \beta A(k_{t+1}/k_t)^{\alpha(1-\sigma)-1} = 0$$

and

$$A = \varphi(k_{t+1}/k_t)^{\alpha(1-\sigma)-1}[\varphi(k_{t+1}/k_t) - (k_{t+1}/k_t)\varphi'(k_{t+1}/k_t)].$$

Besides the constant coefficient A and the constant parameters α, β, and σ, these expressions depend only on the growth rate k_{t+1}/k_t of human capital, suggesting that it is optimal for the consumer to choose $k_{t+1} = \theta k_t$ for all $t = 0, 1, 2, \ldots$, so that this growth rate is constant and equal to θ. The two equations

$$\varphi(\theta)^{\alpha(1-\sigma)-1}\varphi'(\theta) + \beta A\theta^{\alpha(1-\sigma)-1} = 0$$

and

$$A = [\varphi(\theta)]^{\alpha(1-\sigma)-1}[\varphi(\theta) - \theta\varphi'(\theta)]$$

then determine solutions for the undetermined coefficients A and θ.

c. Although it is not possible to solve the pair of equations from part (b), above, to obtain explicit, closed-form expressions for A and θ in terms of the parameters α, β, and σ, the production function

$$y_t = (k_t h_t)^\alpha = [k_t\varphi(k_{t+1}/k_t)]^\alpha \tag{2.1}$$

implies that with human capital growing at the constant rate θ, output will also grow at a constant rate, equal to

$$\frac{y_{t+1}}{y_t} = \frac{[k_{t+1}\varphi(k_{t+2}/k_{t+1})]^\alpha}{[k_t\varphi(k_{t+1}/k_t)]^\alpha} = \frac{[k_{t+1}\varphi(\theta)]^\alpha}{[k_t\varphi(\theta)]^\alpha} = \left(\frac{k_{t+1}}{k_t}\right)^\alpha = \theta^\alpha.$$

3. Saving with a Random Return

The consumer takes his or her initial assets A_0 as given and chooses contingency plans for s_t for all $t = 0, 1, 2, \ldots$ and A_t for all $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t),$$

with $0 < \beta < 1$, subject to the constraints

$$R_{t+1}s_t \geq A_{t+1},$$

which must hold for all $t = 0, 1, 2, \ldots$ and all possible realizations of R_{t+1}.

a. In the dynamic programming formulation of this problem, the value function takes the specific form

$$v(A_t) = E + F\ln(A_t),$$

where E and F are undetermined coefficients. Using this guess, the Bellman equation for the consumer's problem is

$$E + F\ln(A_t) = \max_{s_t} \ \ln(A_t - s_t) + \beta E + \beta F E_t \ln(R_{t+1}s_t).$$

But because s_t must be chosen based on time-t information, and because the random asset return R_{t+1} is assumed to be distributed such that

$$E_t[\ln(R_{t+1})] = 0$$

for all $t = 0, 1, 2, \ldots$ the Bellman equation can be simplified to

$$E + F\ln(A_t) = \max_{s_t} \ \ln(A_t - s_t) + \beta E + \beta F E_t \ln(s_t).$$

The first-order condition for s_t is therefore

$$-\frac{1}{A_t - s_t} + \frac{\beta F}{s_t} = 0,$$

and the envelope condition for A_t is

$$\frac{F}{A_t} = \frac{1}{A_t - s_t}.$$

b. Although there are a number of ways to use the first-order and envelope conditions from part (a) to obtain this result, perhaps the easiest is to note that, together, they imply

$$\frac{\beta F}{s_t} = \frac{1}{A_t - s_t} = \frac{F}{A_t}$$

and hence

$$s_t = \beta A_t$$

and

$$c_t = A_t - s_t = (1 - \beta)A_t.$$

Evidently, the consumer finds it optimal to always consume an amount equal to the fraction $1 - \beta$ of his or her assets.