Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2024

Due Tuesday, December 17

This exam has two questions on five pages; before you begin, please check to make sure that your copy has both questions and all five pages. The two questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, my notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class. The answers you submit must be yours and yours alone.

1. Technical Knowledge Spillovers and Long-Run Economic Growth

As we've seen, the Ramsey (neoclassical growth) model implies that, in the absence of exogenous technological change, the economy converges to a steady state in which consumption and the capital stock remain constant. An interesting and influential modification to that model, suggested by Paul Romer ("Increasing Returns and Long-Run Growth," *Journal of Political Economy*, Vol.94, October 1986), endogenizes the process of technological change to obtain a variant that is consistent with continuing long-run economic growth. A special case of Romer's model, considered here, is one in which the differential equations implied by the maximum principle can be solved, subject to boundary conditions supplied by the initial capital stock and the transversality condition, so as to characterize explicitly the equilibrium growth path.

Suppose that, as in the version of the Ramsey model we studied in class, the representative consumer's preferences are described by the utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t)) \, \mathrm{d}t, \tag{1}$$

where $c(t)$ is consumption at time $t \in [0, \infty)$ and the discount rate satisfies $\rho > 0$. Suppose also that output produced by the representative consumer at each time t is a Cobb-Douglas function of the stock of capital $k(t)$ owned by the consumer and the economy-wide stock of technical knowledge $x(t)$, which each consumer takes as given. The representative consumer's capital stock is therefore governed by

$$k(t)^\alpha x(t)^{1-\alpha} - \delta k(t) - c(t) \geq \dot{k}(t) \tag{2}$$

for all $t \in [0, \infty)$, where the share parameter and the depreciation rate both lie between zero and one: $0 < \alpha < 1$ and $0 < \delta < 1$. Given his or her initial capital stock $k(0)$, the consumer chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize the utility function (1) subject to the constraint (2) for $t \in [0, \infty)$.

a. Define (write down) the maximized Hamiltonian for the consumer's problem, and use the maximum principle to derive the set of optimality conditions (the first-order condition and pair of differential equations) that describe the solution to this problem. *Note:* For this problem, you can use whichever form of the Hamiltonian – present or current value – you find most convenient. Either way, however, just make sure the optimality conditions you write down are consistent with your choice.

b. Now suppose, following Romer but departing from the original Ramsey model, that instead of growing exogenously, the stock of technical knowledge expands through a process of learning-by-doing, modeled here through the simple assumption that

$$x(t) = k(t) \tag{3}$$

for all $t \in [0, \infty)$. Note that in solving the representative consumer's problem, above, we assumed that the individual consumer takes $x(t)$ as given, ignoring its dependence on $k(t)$. Thus, with (2) and (3), we're assuming that private capital accumulation provides a positive externality, by increasing the stock of technical knowledge that all other consumers and producers benefit from. The presence of this positive externality means that equilibrium and optimal resource allocations no longer coincide. Here, we're focusing on equilibrium allocations. Substitute the equilibrium condition (3) into the differential equations you derived from the maximum principle in part (a), above. Then, as we did for the Ramsey model in class, use the first-order condition to eliminate reference to the multiplier in the differential equations. The result will be a pair of differential equations: the Euler equation describing the growth rate of consumption $\dot{c}(t)/c(t)$ and the capital accumulation constraint describing the behavior of $\dot{k}(t)$.

c. The Euler equation you just derived in part (b) should reveal that, in this version of Romer's model, equilibrium consumption growth is constant, with

$$\frac{\dot{c}(t)}{c(t)} = g \tag{4}$$

for all $t \in [0, \infty)$. Use the Euler equation to express the constant consumption growth rate g in terms of the model's parameters: ρ, α, and δ.

d. The constant growth rate for consumption summarized by equation (4) implies that

$$c(t) = c(0)e^{gt} \tag{5}$$

for $t \in [0, \infty)$. If you substitute (5) into the differential equation for capital that you derived in part (b), you should get

$$\dot{k}(t) = (1 - \delta)k(t) - c(0)e^{gt}. \tag{6}$$

It turns out that the differential equation in (6) has a general solution of the form

$$k(t) = Ae^{(1-\delta)t} + \left[\frac{c(0)}{1-\delta-g}\right] e^{gt}, \tag{7}$$

for any value of A. To see this, differentiate both sides of (7) with respect to time:

$$\begin{aligned}
\dot{k}(t) &= (1-\delta)Ae^{(1-\delta)t} + g\left[\frac{c(0)}{1-\delta-g}\right] e^{gt} \\
&= (1-\delta)\left\{k(t) - \left[\frac{c(0)}{1-\delta-g}\right] e^{gt}\right\} + g\left[\frac{c(0)}{1-\delta-g}\right] e^{gt} \\
&= (1-\delta)k(t) - (1-\delta-g)\left[\frac{c(0)}{1-\delta-g}\right] e^{gt} \\
&= (1-\delta)k(t) - c(0)e^{gt}
\end{aligned} \tag{8}$$

as required by (6). The general solutions (5) and (7) involve two as-yet unknown values, $c(0)$ and A, which must be pinned down using the model's two boundary conditions: the initial condition for the capital stock $k(0)$ and the transversality condition from the representative consumer's dynamic optimization problem. Regardless of whether you set up the Hamiltonian for the consumer's problem in present or current value form, the transversality condition, when combined with the first-order condition for $c(t)$, requires that

$$\lim_{t\to\infty} e^{-\rho t}\left[\frac{1}{c(t)}\right] k(t) = 0. \tag{9}$$

Substituting the general solutions (5) and (7) into (9) yields

$$\begin{aligned}
0 &= \lim_{t\to\infty}\left[\frac{1}{c(0)}\right] e^{-(g+\rho)t}\left\{Ae^{(1-\delta)t} + \left[\frac{c(0)}{1-\delta-g}\right] e^{gt}\right\} \\
&= \left[\frac{1}{c(0)}\right] \lim_{t\to\infty}\left\{Ae^{(1-\delta-g-\rho)t} + \left[\frac{c(0)}{1-\delta-g}\right] e^{-\rho t}\right\}
\end{aligned} \tag{10}$$

The second term inside brackets in (10), involving $e^{-\rho t}$, goes to zero as $t \to \infty$ since $\rho > 0$. Thus, (10) requires

$$0 = \frac{A}{c(0)} \lim_{t\to\infty} e^{(1-\delta-g-\rho)t} \tag{11}$$

Use the formula for the constant growth rate of consumption g that you derived in part (c), above, to verify that the exponential growth rate in (11) satisfies

$$1 - \delta - g - \rho > 0. \tag{12}$$

Note: We can conclude from (11) and (12) that the transversality condition requires $A = 0$. Now, it only remains to pin down $c(0)$. But this can be done by returning to (7), after setting $A = 0$ as required by the TVC. For $t = 0$, this condition requires

$$c(0) = (1 - \delta - g)k(0). \tag{13}$$

To summarize: with $k(0)$ given and $c(0)$ determined by (13), consumption and the capital stock both grow at the constant rate g is equilibrium, according to (5) and (7) with $A = 0$.

2. Stochastic Growth with Labor Supply and Serially Correlated Shocks

This problem asks you to use dynamic programming to solve a version of the stochastic growth model from problem set 14, extended to include an optimal labor supply decision and serially correlated productivity shocks. Maintaining the assumption that physical capital depreciates fully between periods continues to allow an explicit solution for the value function to be found using the guess-and-verify method.

In this version of the model, the representative consumer chooses contingency plans for consumption c_t and labor supply n_t (for "number of hours worked") for all $t = 0, 1, 2, \ldots$ and physical capital k_t for all $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln(c_t) - \gamma n_t], \tag{14}$$

where the discount factor and the constant disutility of labor satisfy $0 < \beta < 1$ and $\gamma > 0$, subject to the constraints k_0 given and

$$z_t k_t^{\alpha} n_t^{1-\alpha} \geq c_t + k_{t+1}, \tag{15}$$

where the share parameter in the Cobb-Douglas production function satisfies $0 < \alpha < 1$. The natural log of productivity shock z_t follows the autoregressive process

$$\ln(z_{t+1}) = \rho \ln(z_t) + \varepsilon_{t+1}, \tag{16}$$

with persistence parameter satisfying $0 \leq \rho < 1$, where the serially uncorrelated innovation ε_{t+1} satisfies $E_t(\varepsilon_{t+1}) = 0$ for all $t = 0, 1, 2, \ldots$. Hence, the value of z_t is known when c_t, n_t, and k_{t+1} are chosen during period t, but the value of z_{t+1}, though partly forecastable via (16), still contains the element of randomness due to ε_{t+1}. The constraint in (15) must hold for all $t = 0, 1, 2, \ldots$ and all possible realizations of z_t.

For this problem, the Bellman equation takes the general form

$$v(k_t, z_t) = \max_{c_t, n_t} \ln(c_t) - \gamma n_t + \beta E_t v(z_t k_t^{\alpha} n_t^{1-\alpha} - c_t, z_{t+1}).$$

Using the same conjectured form of the value function as in problem set 14, namely,

$$v(k_t, z_t) = E + F \ln(k_t) + G \ln(z_t)$$

the Bellman equation becomes

$$E + F \ln(k_t) + G \ln(z_t) = \max_{c_t, n_t} \ln(c_t) - \gamma n_t + \beta E + \beta F \ln(z_t k_t^{\alpha} n_t^{1-\alpha} - c_t) + \beta G \rho \ln(z_t), \tag{17}$$

where, in the very last term on the right-hand side, use has been made of (16), which implies

$$E_t[\ln(z_{t+1})] = \rho \ln(z_t).$$

a. Starting from (17), derive (write down) the first-order conditions for c_t and n_t and the envelope condition for k_t that help characterize the solution to the consumer's problem.

b. Next, use the first-order condition for c_t, the envelope condition for k_t, and the binding constraint

$$k_{t+1} = z_t k_t^{\alpha} n_t^{1-\alpha} - c_t$$

to derive expressions that show how the constant F from the value function depends on the parameters α and β and how optimal consumption and savings are determined as constant fractions of output, with

$$c_t = (1-s) z_t k_t^{\alpha} n_t^{1-\alpha}$$

and

$$k_{t+1} = s z_t k_t^{\alpha} n_t^{1-\alpha},$$

where s also depends on the parameters α and β.

c. Now use the first-order condition for n_t to show that optimal labor supply is a constant, which depends on the parameters γ, α, and β.

d. Finally, substitute your solutions for F, c_t, n_t, and k_t back into the Bellman equation (17) and, by observing that the equation must hold for all values of $\ln(z_t)$, use the result to find a solution for G in terms the parameters α, β, and ρ.

Note: It is also possible – but not necessary for the purposes of this exam – to use (17) to find the solution for E in terms of γ, α, and β.