Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2023

Due Tuesday, December 19

This exam has four questions on five pages; before you begin, please check to make sure that your copy has all four questions and all five pages. The four questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, my notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class. The answers you submit must be yours and yours alone.

1. Endogenous Growth: Continuous Time

As we've seen, the Ramsey (neoclassical growth) model implies that, in the absence of exogenous technological change, the economy converges to a steady state in which consumption and the capital stock remain constant. A straightforward modification of that model, suggested by Sergio Rebelo ("Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy*, Vol.99, June 1991) dispenses with the traditional assumption of diminishing returns to capital accumulation to obtain a variant that is consistent with continuing long-run growth. In his paper (footnote 7, p.507), Rebelo explains that his assumption of constant returns is consistent with a broader interpretation of the capital stock that includes human and well as physical capital and therefore implies that in effect there are no fixed factors of production.

Following Rebelo, assume that the representative consumer's preferences are described by the utility function

$$\int_0^\infty e^{-\rho t}\left(\frac{1}{1-\sigma}\right)C(t)^{1-\sigma}\ \mathrm{d}t, \tag{1}$$

where $C(t)$ is consumption at time $t \in [0,\infty)$ and the discount rate and curvature parameters $\rho > 0$ and $\sigma > 0$ are both strictly positive. *Note:* In the special case where $\sigma = 1$, the instantaneous utility function in (1) should be replaced by the natural log $\ln(C(t))$. For the purposes of solving this problem, however, you can simply assume that $\sigma \neq 1$ to avoid having to the consider the limiting case separately.

Output at time t is a linear function $AK(t)$ of the capital stock, where $A > 0$. The capital stock depreciates at the constant rate $\delta > 0$. Capital accumulation is therefore governed by

$$(A-\delta)K(t) - C(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0,\infty)$.

Like the Ramsey model, Rebelo's describes an economic environment in which the two welfare theorems apply. Equilibrium allocations can therefore be characterized by solving a social planner's problem: Given the initial capital stock $K(0)$, choose $C(t)$ for all $t \in [0, \infty)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function (1) subject to the constraint (2) for $t \in [0, \infty)$.

a. Define (write down) the maximized Hamiltonian for the social planner's problem, and use the maximum principle to derive the set of optimality conditions (the first-order conditions and pair of differential equations) that describe the solution to this problem. *Note:* For this problem, you can use whichever form of the Hamiltonian – present or current value – you find most convenient. Either way, however, just make sure the optimality conditions you write down are consistent with your choice.

b. Interestingly, this endogenous growth model has no transition dynamics. It implies, in particular, that consumption always grows at a constant rate, with

$$\dot{C}(t)/C(t) = g_c$$

for all $t \in [0, \infty)$. Use the optimality conditions you derived in part (a) to solve for the constant growth rate g_c in terms of the model's parameters: ρ, σ, A, and δ.

2. Endogenous Growth with Comparison Utility: Continuous Time

Christoper Carroll, Jody Overland, and David Weil ("Comparison Utility in a Growth Model," *Journal of Economic Growth*, Vol.2, December 1997) develop an interesting variant of Rebelo's endogenous growth model in which the representative consumer's utility depends on his or her own consumption relative to an economy-wide average. They trace this idea back to James Duesenberry's PhD dissertation (published as *Income, Saving, and the Theory of Consumer Behavior*, Harvard University Press, 1949).

In Carroll, Overland, and Weil's model, capital accumulation is still governed by the constraint in (2), but the representative consumer's preferences are described by

$$\int_0^\infty e^{-\rho t} \left(\frac{1}{1-\sigma} \right) \left[\frac{C(t)}{Z(t)^\gamma} \right]^{1-\sigma} \mathrm{d}t, \tag{3}$$

where $Z(t)$ is an index of past aggregate consumption that the representative consumer takes as given. The preference parameters in (3) are now assumed to satisfy $\rho > 0$, $\sigma > 1$, and $\gamma \in [0, 1)$. Capital accumulation continues to be governed by the constraint in (2).

Because this model features a kind of negative externality, through which a rise in aggregate consumption impacts adversely on the representative consumer's utility, the welfare theorems of economics no longer hold. Here, we'll focus on equilibrium allocations by solving the individual consumer's problem: Taking the initial capital stock $K(0)$ and the trajectory for $Z(t)$ as given, choose $C(t)$ for all $t \in [0, \infty)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function (3) subject to the constraint (2) for $t \in [0, \infty)$.

a. Define (write down) the maximized Hamiltonian for the consumer's problem, and use the maximum principle to derive the set of optimality conditions (the first-order conditions and pair of differential equations) that describe the solution to this problem. *Note:* Again, you can use whichever form of the Hamiltonian – present or current value – you find most convenient. Either way, however, just make sure the optimality conditions you write down are consistent with your choice.

b. In deriving the implications of their model for long-run growth rates, Carroll, Overland, and Weil go on to assume that the aggregate index of consumption $Z(t)$ evolves according to

$$\dot{Z}(t) = \alpha[\bar{C}(t) - Z(t)],$$

where $\bar{C}(t)$ is average consumption per capita and the speed-of-adjustment parameter $\alpha > 0$ is positive. If all consumers in the economy are identical, then $\bar{C}(t) = C(t)$ in equilibrium, and this adjustment equation becomes

$$\dot{Z}(t) = \alpha[C(t) - Z(t)]. \tag{4}$$

Equation (4) reveals that this model is similar to one in which there is habit formation in preferences although, here, each consumer neglects the effects that his or her consumption choice today has on the future habit stock $Z(t)$. Carroll, Overland, and Weil show that when the dynamics of $Z(t)$ are governed by (4), the economy converges to a steady-state growth path in which consumption and the habit stock $Z(t)$ grow at the same constant rate, with

$$\dot{C}(t)/C(t) = \dot{Z}(t)/Z(t) = g_c$$

for all t. Use the optimality conditions you derived in part (a) to solve for the constant growth rate g_c in terms of the model's parameters: ρ, σ, γ, A, and δ. Your answer should show that under the assumption that $\sigma > 1$, the steady-state growth rate of consumption g_c increases as the parameter γ measuring the importance of the habit stock $Z(t)$ in utility becomes larger.

3. Endogenous Growth: Discrete Time

In a discrete-time version of Rebelo's model, the representative consumer's preferences are described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right) C_t^{1-\sigma}, \tag{5}$$

where $\beta \in (0,1)$ and $\sigma > 0$. Again, when $\sigma = 1$ the single-period utility function in (5) can be replaced by the natural log $\ln(C_t)$ but here, for simplicity, you can just assume that $\sigma \neq 1$ to avoid having to consider the limiting case separately.

Capital accumulation takes place according to the constraint

$$AK_t + (1-\delta)K_t - C_t \geq K_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$, where $A > 0$ and $\delta > 0$. The social planner takes K_0 as given and chooses C_t for $t = 0, 1, 2, \ldots$ and K_t for $t = 1, 2, 3, \ldots$ to maximize the utility function (5) subject to the constraint (6) for all $t = 0, 1, 2, \ldots$.

a. Define (write down) the Bellman equation for the social planner's problem, and use dynamic programming to derive the set of optimality conditions (the first-order and envelope conditions, plus the binding constraint) that describe the solution to this problem. *Note:* There are several ways to set up the Bellman equation; feel free to use whichever you find most convenient. Perhaps the easiest approach, though, is to start by substituting the binding constraint from (6) into the utility function in (5), and then to use K_t as the state variable and K_{t+1} as the control variable.

b. Once again, this model implies that consumption growth will be constant, with

$$C_{t+1}/C_t = G_c$$

for all $t = 0, 1, 2, \ldots$. Use your optimality conditions from part (a) to solve for the constant growth rate G_c in terms of the model's parameters β, σ, A, and δ.

4. Endogenous Growth with Comparison Utility: Discrete Time

In a discrete-time version of Carroll, Overland, and Weil's model, the representative consumer's preferences are described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right) \left(\frac{C_t}{Z_t^{\gamma}}\right)^{1-\sigma}, \tag{7}$$

where $\beta \in (0, 1)$, $\sigma > 1$, and $\gamma \in [0, 1)$. Capital accumulation continues to be governed by the constraint in (6).

The consumer takes the initial capital stock K_0 and the trajectory for Z_t as given, and chooses C_t for $t = 0, 1, 2, \ldots$ and K_t for $t = 1, 2, 3, \ldots$ to maximize the utility function (7) subject to the constraint (6) for all $t = 0, 1, 2, \ldots$.

a. Define (write down) the Bellman equation for the consumer's problem, and use dynamic programming to derive the set of optimality conditions (the first-order and envelope conditions, plus the binding constraint) that describe the solution to this problem. *Note:* Once again, there are several ways to set up the Bellman equation; feel free to use whichever you find most convenient. Perhaps the easiest approach, though, is to start by substituting the binding constraint from (6) into the utility function in (7), and then to use K_t as the state variable and K_{t+1} as the control variable.

b. When the dynamics of the aggregate consumption index are governed by

$$Z_{t+1} = \alpha \bar{C}_t + (1-\alpha) Z_t$$

with $\alpha \in (0,1)$ and when average per capita consumption $\bar{C}_t$ equals the representative consumer's consumption C_t for all $t = 0, 1, 2, \ldots$, the discrete-time version of (4)

$$Z_{t+1} = \alpha C_t + (1-\alpha) Z_t$$

will hold in equilibrium and the economy will once again converge to a steady-state growth path in which consumption C_t and the habit stock Z_t grow at the same constant rate, with

$$C_{t+1}/C_t = Z_{t+1}/Z_t = G_c$$

for all $t = 0, 1, 2, \ldots$. Use your optimality conditions from part (a) to solve for the constant growth rate G_c in terms of the model's parameters: β, σ, γ, A, and δ.