Final Exam

ECON 772001 - Math for Economists
Boston College, Department of Economics

Peter Ireland

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This exam has two questions on four pages; before you begin, please check to make sure that your copy has both questions and all four pages. The two questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, my notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class. The answers you submit must be yours and yours alone.

1. Optimal Pollution Abatement

Consider a variant of the neoclassical growth model in which more production inevitably creates more pollution, but also where the representative consumer can allocate resources to pollution abatement as well as consumption and investment. The infinitely-lived consumer has preferences described by the utility function

$$\int_0^\infty e^{-\rho t}[\ln(c(t)) + \beta \ln(E(t))]dt,$$

where $c(t)$ and $E(t)$ are consumption and environmental quality at each time $t \in [0, \infty)$, $\rho > 0$ is the discount rate, and $\beta > 0$ is the weight that the consumer places on environmental quality relative to consumption.

At each time $t \in [0, \infty)$, the consumer uses the capital stock $k(t)$ to produce $k(t)^\alpha$ units of output, where $0 < \alpha < 1$. The capital stock depreciates in use at rate $\delta > 0$. In addition to consumption $c(t)$, the consumer allocates $a(t)$ units of output to pollution reduction (abatement), and invests in new capital. The capital stock evolves according to

$$k(t)^\alpha - \delta k(t) - c(t) - a(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$.

As noted above, production in this economy inevitably leads to pollution, which degrades environmental quality. The consumer's spending on abatement, however, helps offset this by improving environmental quality. Specifically, environmental quality evolves according to

$$\gamma a(t) - k(t)^\alpha \geq \dot{E}(t)$$

for all $t \in [0, \infty)$, where the parameter $\gamma > 1$ measures the productivity of resources allocated to pollution abatement (the restriction that γ is greater than one is needed to ensure that the economy has a well-defined steady state).

The consumer's problem is therefore: given $k(0)$ and $E(0)$, choose $c(t)$ and $a(t)$ for all $t \in [0, \infty)$ and $k(t)$ and $E(t)$ for all $t \in (0, \infty)$ to maximize the utility function subject to the constraints determining $\dot{k}(t)$ and $\dot{E}(t)$.

a. Define (write down) the maximized Hamiltonian for the consumer's problem, using $k(t)$ and $E(t)$ as stock variables, $c(t)$ and $a(t)$ as flow variables, and remembering to introduce a separate multiplier for each of the two constraints, as illustrated more generally in section 6 of the notes on the maximum principle and more specifically in examples from question 1 on the 2017 final exam and question 1 on the 2019 final exam. For this problem, you can use whichever form of the Hamiltonian – present or current value – you find most convenient, although it might be easiest to use the current-value formulation. Either way, however, just make sure the optimality conditions you'll write down next, for part (b), are consistent with your choice.

b. Now write down the first-order conditions for $c(t)$ and $a(t)$ and the two pairs of differential equations for the multipliers and stock variables that, according to the maximum principle, characterize the solution to the consumer's problem.

c. Now consider a steady state, in which $c(t) = c$, $a(t) = a$, $k(t) = k$, and $E(t) = E$ are all constant. Use your optimality conditions from part (b) to find an expression that shows how the steady-state capital stock k depends on the parameters ρ, δ, α, γ, and β (*Note:* k may depend on some, but not all of these parameters).

d. How does the steady-state capital stock in this model of pollution and abatement compare to the steady-state capital stock in the original (without pollution and abatement) neoclassical (Ramsey) model with the same values for ρ, δ, and α? How does the steady-state capital stock in this model change as the parameter γ increases? What happens to the steady-state capital stock in this model as γ becomes arbitrarily large?

2. Investment Under Uncertainty

Consider a firm that produces output Y_t with capital K_t according to the linear technology

$$Y_t = A_t K_t,$$

where productivity A_t fluctuates randomly around a long-run mean A according to

$$A_{t+1} - A = \rho(A_t - A) + \varepsilon_{t+1},$$

with persistence governed by the parameter ρ, satisfying $0 < \rho < 1$, and where ε_{t+1} is serially uncorrelated with mean zero and constant variance σ^2.

During each period $t = 0, 1, 2, \ldots$, the firm chooses its optimal level of investment I_t after observing the realized value of ε_t and hence knowing the value of A_t. It earns profits

$$A_t K_t - I_t - \left(\frac{\phi}{2}\right) I_t^2,$$

during period t, where $\phi > 0$ governs the magnitude of adjustment costs for capital. It then carries

$$K_{t+1} = (1 - \delta)K_t + I_t$$

units of capital unto period $t+1$, where the depreciation rate δ satisfies $0 < \delta < 1$. The firm discounts future profits at the constant rate r.

The firm's problem is therefore: given K_0 and A_0, choose contingency plans for I_t, $t = 0, 1, 2, \ldots$, and K_{t+1}, $t = 1, 2, 3, \ldots$, to maximize

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right) \left[A_t K_t - I_t - \left(\frac{\phi}{2}\right) I_t^2\right]$$

subject to the constraints

$$(1 - \delta)K_t + I_t \geq K_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of A_{t+1}.

Although there are a number of ways to solve this problem via dynamic programming, perhaps the easiest starts by writing the Bellman equation as

$$v(K_t, A_t) = \max_{I_t, K_{t+1}} A_t K_t - I_t - \left(\frac{\phi}{2}\right) I_t^2 + \left(\frac{1}{1+r}\right) E_t v(K_{t+1}, A_{t+1})$$
$$\text{subject to } (1 - \delta)K_t + I_t \geq K_{t+1}.$$

Note that in the static, constrained optimization problem on the right-hand side of this Bellman equation, I_t and K_{t+1} appear as choice variables while K_t appears as a parameter.

a. Using λ_t to denote the Lagrange multiplier on the constraint from the problem on the right-hand side of the Bellman equation, derive (write down) the first-order conditions for I_t and K_{t+1} and the envelope condition for K_t that characterize the solution to the firm's problem.

b. As the next step in characterizing the firm's optimal investment, combine the first-order condition for K_{t+1} with the envelope condition for K_t so as to eliminate reference in the optimality conditions to the unknown value functions $v(K_t, A_t)$ and $v(K_{t+1}, A_{t+1})$.

c. Your result from part (b) should take the form of a difference equation that links λ_t to the expected future values $E_t A_{t+1}$ and $E_t \lambda_{t+1}$. Use recursive forward substitution to obtain an expression that shows how λ_t depends on the expected future values of productivity $E_t A_{t+j}$ for all $j = 1, 2, 3, \ldots$ stretching out into the infinite future. To derive this result, you'll need to use the law of iterated expectations, which implies that

$$E_t(E_{t+j-1} A_{t+j}) = E_t A_{t+j}$$

and

$$E_t(E_{t+j-1} \lambda_{t+j}) = E_t \lambda_{t+j}$$

for all $j = 1, 2, 3, \ldots$. You can also assume that the transversality condition for the firm's infinite-horizon problem implies that

$$\lim_{j \to \infty} \left(\frac{1-\delta}{1+r} \right)^j E_t \lambda_{t+j} = 0$$

so that the infinite sum in your expression for λ_t converges.

d. Finally, notice that the autoregressive law of motion for productivity implies that

$$E_t A_{t+j} = A + \rho^j (A_t - A)$$

for all $j = 1, 2, 3, \ldots$. Use this forecasting equation, together with your solution for λ_t from part (c) and your first-order condition for I_t from part (a), to show how the firm's optimal investment I_t depends on the current level of productivity A_t, the average level of productivity A, and the parameters ρ, δ, r and ϕ.