Final Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Tuesday, December 21 at 12noon

This exam has three questions on seven pages; before you begin, please check to make sure that your copy has all three questions and all seven pages. Each part of each question is worth ten points. Therefore, questions 1 with one part is worth 10 points, question 2 with four parts is worth 40 points, and question 3 with five parts is worth 50 points, for a total of 100 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, the papers referenced, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. The Tobin Effect

How does inflation affect the process of economic growth? James Tobin proposed one answer to this question in his article, "Money and Economic Growth" (*Econometrica* 33 (1965): 671-684). In Tobin's framework, consumers hold wealth in the form of physical capital and money. When inflation rises, they substitute out of money and into capital. Hence, via this "Tobin effect," output rises as well.

One would not want to take this argument too far. No economist or central banker would argue that hyperinflation is optimal monetary policy because it causes consumers to hold almost all of their wealth in the form of physical capital and almost none as money. But, maybe the Tobin effect operates at modest rates of inflation, explaining why periods when inflation runs low – such as the Great Depression of the 1930s or the more recent period following the financial crisis and Great Recession of 2008-9 – are also periods when output is disappointingly low as well.

Tobin's argument can be formalized using a monetary version of the Solow ("A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics* 70 (1956): 65-94) growth model. Consider an economy in which, at each time $t \in [0, \infty)$, output $Y(t)$ is produced with $K(t)$ units of capital and $N(t)$ workers according to the Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} N(t)^{1-\alpha},$$

where $0 < \alpha < 1$. Suppose, also, that the supply of money – "dollars" – is $D(t)$ at time $t \in [0, \infty)$. Suppose that the government increases the money supply over time by making transfers $X(t) = \mu D(t)$ to the public at time t. Then the nominal money supply grows at the constant rate μ over time.

Let $P(t)$ denote the aggregate nominal price level at time t, so that

$$M(t) = D(t)/P(t)$$

denotes real money balances held by the public at the same date. If δ denotes the rate of physical capital depreciation and $\pi(t) = \dot{P}(t)/P(t)$ denotes the inflation rate (here, as always, $\dot{P}(t)$ denotes the derivative of $P(t)$ with respect to time t), then the public's disposable income at time t is

$$Y^d(t) = K(t)^\alpha N(t)^{1-\alpha} + \frac{X(t)}{P(t)} - \delta K(t) - \pi(t)M(t).$$

The first term on the right-hand side of this expression is real output and the second is the real value of the government transfer. The third term subtracts off the value of depreciated capital, and the fourth corrects for the erosion due to inflation of the real value of money held by the public. Remembering that $X(t) = \mu D(t)$ and $M(t) = D(t)/P(t)$, this last expression can be written more simply as

$$Y^d(t) = K(t)^\alpha N(t)^{1-\alpha} - \delta K(t) + [\mu - \pi(t)]M(t).$$

Now, following Solow (1956), assume that the savings rate out of disposable income is constant and equal to s between zero and one: $0 < s < 1$. Extending Solow's model as suggested by Tobin (1965), however, total saving is used here to increase both the stock of physical capital and real money balances. Therefore,

$$\dot{K}(t) + \dot{M}(t) = sY^d(t) = s\{K(t)^\alpha N(t)^{1-\alpha} - \delta K(t) + [\mu - \pi(t)]M(t)\}, \tag{1}$$

where, again as always, $\dot{K}(t)$ and $\dot{M}(t)$ denote the derivatives of $K(t)$ and $M(t)$ with respect to time t.

It is helpful to rewrite (1) in per-capita terms. Let $k(t) = K(t)/N(t)$ and $m(t) = M(t)/N(t)$ denote capital and real money balances per worker. And assume, again following Solow (1956), that the population/labor force expands at the constant rate $n > 0$. Then

$$\dot{k}(t) + \dot{m}(t) = s\{k(t)^\alpha - \delta k(t) + [\mu - \pi(t)]m(t)\} - nk(t) - nm(t). \tag{2}$$

In a steady state where the stocks of physical capital and real balances per capita are constant, with $\dot{k}(t) = \dot{m}(t) = 0$, $k(t) = k$, and $m(t) = m$. The steady-state condition $\dot{m}(t) = 0$ also determines the constant inflation rate $\pi(t) = \pi$ as

$$\pi = \mu - n.$$

Hence, in steady state, (2) requires

$$0 = s(k^\alpha - \delta k) - nk - (1 - s)nm. \tag{3}$$

To complete our analysis, we need a second steady-state condition, describing how the public divides its financial wealth between real money and physical capital. A simple specification that captures the essence of Tobin's (1965) argument assumes that the steady-state ratio m/k is inversely related to the inflation rate π, so that

$$m = \phi(\pi)k, \tag{4}$$

where the function ϕ satisfies $\phi(\pi) > 0$ and $\phi'(\pi) < 0$ for all steady-state inflation rates π.

Combining (3) and (4) yields

$$0 = s(k^\alpha - \delta k) - nk - (1-s)n\phi(\pi)k. \tag{5}$$

a. Use the total differential of (5) to compute $dk/d\pi$. Assuming that the stability condition

$$s(\alpha k^{\alpha-1} - \delta) - n < 0$$

from the original Solow (1956) model holds here as well, you should be able to use your result to show that

$$\frac{dk}{d\pi} > 0$$

as conjectured by Tobin (1965).

2. An Optimizing Model of Saving and Asset Allocation

The model from question 1 might be criticized for simply assuming that the savings rate is constant and that the steady-state real money-to-capital ratio is a simple function of the inflation rate without explicitly deriving those behavioral patterns from a utility-maximization problem. The constant savings rate assumption, in particular, might be playing an important role in shaping the results: if the savings rate falls when inflation rises, for example, an increase in inflation could decrease physical capital as well as real money balances, offsetting or even reversing the Tobin effect.

What happens when we extend our analysis from question 1 by formulating and solving the dynamic optimization problem faced by a household that saves to accumulate both physical capital and real money balances over time? Miguel Sidrauski answered this question in his paper "Rational Choice and Patterns of Growth in a Monetary Economy" (*American Economic Review* 57 (1967): 534-544). Here, we'll use his framework to see for ourselves.

Consider an infinitely-lived representative household consisting of $N(t)$ members, with preferences described by

$$\int_0^\infty e^{-\rho t} N(t) U\left(\frac{C(t)}{N(t)}, \frac{D(t)}{P(t)N(t)}\right) \mathrm{d}t,$$

where $\rho > 0$ is the discount rate, $C(t)$ is total consumption, and as before, $D(t)$ are the total number of dollars held and $P(t)$ is the aggregate nominal price level. Hence, according to this specification, the household enjoys consumption of goods and also benefits from holding real money balances, perhaps because doing so saves time in transacting with other households.

Assume, again, that the population grows at the constant rate $n > 0$, and for simplicity, normalize the population at $t = 0$ to equal one: $N(0) = 1$. Let $c(t) = C(t)/N(t)$ denote per-capita consumption, and let $M(t) = D(t)/P(t)$ and $m(t) = M(t)/N(t)$ denote total and per-capital real money balances. Then the utility function can be written more simply as

$$\int_0^\infty e^{-(\rho-n)t} U(c(t), m(t)) \mathrm{d}t. \tag{6}$$

During each period t, the household divides its real wealth $A(t)$ up into an amount $K(t)$ to be held as physical capital and an amount $M(t)$ to be held as real money balances. With $a(t) = A(t)/N(t)$ and $k(t) = K(t)/N(t)$, this asset-allocation constraint can be written in per-capita terms as

$$a(t) \geq k(t) + m(t) \tag{7}$$

for all $t \in [0, \infty)$. In (7), the constraint appears as an inequality to allow for the possibility of free disposal. If the single-period utility function U is increasing in both its arguments, however, (7) will always bind when the household is optimizing.

As before, as sources of funds in each period t, the household has output, produced according to a Cobb-Douglas production function with capital's share $0 < \alpha < 1$, and a government transfer of newly-printed dollars $X(t)$ having real value $X(t)/P(t)$. Also as before, the household's capital stock depreciates at the constant rate δ and the real value of its money holdings is eroded by $\pi(t)M(t)$ because of inflation $\pi(t)$. And as an additional use of funds, not explicitly considered before, the household has its consumption $C(t)$. The household, therefore, accumulates real assets subject to the constraint

$$K(t)^\alpha N(t)^{1-\alpha} + X(t)/P(t) - \delta K(t) - \pi(t)M(t) - C(t) \geq \dot{A}(t),$$

or, in per-capita terms,

$$k(t)^\alpha + x(t) - \delta k(t) - \pi(t)m(t) - c(t) - na(t) \geq \dot{a}(t), \tag{8}$$

for all $t \in [0, \infty)$, where $x(t) = X(t)/P(t)$ denotes per-capita real money transfers received during period t.

The household's optimization problem can now be stated as: taking initial wealth $a(0)$ as given, choose $c(t)$, $k(t)$, and $m(t)$ for all $t \in [0, \infty)$ and $a(t)$ for all $t \in (0, \infty)$ to maximize utility in (6) subject to the constraints (7) and (8) for all $t \in [0, \infty)$. Note that in this problem, $a(t)$ appears as a stock variable and $c(t)$, $k(t)$, and $m(t)$ appear as flows. If treating capital and money as flow variables seems counterintuitive, think instead of $k(t)$ and $m(t)$ as measuring service flows from physical capital and real money balances received by the household using its stock of financial assets $a(t)$ according to the constraint in (7).

a. As a first step in characterizing the household's optimizing behavior using the maximum principle, write down an expression for the maximized Hamiltonian for its problem. *Note:* For this problem, it may be easiest to work with the Hamiltonian in current value form. But of course you'll reach the same conclusions starting from the Hamiltonian in present value form. So use whichever formulation you prefer; just make sure the optimality conditions you list in part (b), below, are consistent with your chosen definition of the Hamiltonian.

b. Next, write down the first-order conditions for $c(t)$, $k(t)$, and $m(t)$ and the pair of differential equations for $a(t)$ and the associated multiplier from the Hamiltonian that, according to the maximum principle, help characterize the solution to the household's problem.

c. In any steady-state, all per-capita variables will be constant, with $c(t) = c$, $k(t) = k$, $m(t) = m$, and $a(t) = a$ for all t. And if, as in question 1, the government sets the transfer $x(t) = \mu m(t)$ to provide for constant money growth at rate μ, the steady-state condition $\dot{m}(t) = 0$ once again determines the constant inflation rate $\pi(t) = \pi$ as

$$\pi = \mu - n.$$

Use these steady-state conditions, together with your optimality conditions from part (b), to obtain a solution for the steady-state capital stock k in terms of the model's parameters: ρ, n, α, δ, and π. *Note:* You should be able to solve for k using the steady-state and optimality conditions without necessarily having to find the steady-state values of other variables: this was one of the surprising results derived by Sidrauski (1965). In addition, the value of k will not depend on all of the parameters, just a subset of them.

d. Finally, use the solution you just derived for k to answer the question posed at outset: does the Tobin effect of inflation on the capital stock still appear when savings and asset allocation decisions are derived from optimizing behavior as modeled here and by Sidrauski (1967)?

3. Precautionary Saving

This question asks you to work through a version of a model developed by Ricardo Caballero in his article "Consumption Puzzles and Precautionary Savings" (*Journal of Monetary Economics* 25 (1990): 113-136), in which an optimizing consumer saves to insure against adverse shocks to income. With a set of restrictions imposed on Caballero's more general framework, it is possible to solve explicitly for the value function that enters into the dynamic programming formulation of the consumer's problem via the guess-and-verify method.

The consumer in this problem seeks to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t [-\frac{1}{\theta} \exp(-\theta c_t)]. \tag{9}$$

where the discount factor lies between zero and one, $0 < \beta < 1$, and the positive parameter $\theta = -u''(c_t)/u'(c_t) > 0$ measures the consumer's constant coefficient of absolute risk aversion.

The consumer's bank account balance at the beginning of each period t is A_t. During the period, the consumer decides how much to withdraw from the bank in order to finance his or her consumption c_t. Unspent balances earn interest at the constant, gross rate $R > 1$, which satisfies $\beta R = 1$.

At the very start of the following period $t + 1$, the consumer learns how much income Y_{t+1} he or she will earn during the following period $t + 1$. Thus, the consumer must decide on spending c_t and saving $A_t - c_t$ before knowing Y_{t+1}. Cabellero allows Y_{t+1} to follow a more general stochastic process; here, instead, we will assume that Y_{t+1} is independently and identically distributed and, in particular, normally distributed with mean $\bar{Y}$ and variance σ^2:

$$Y_{t+1} \sim N(\bar{Y}, \sigma^2)$$

for all $t = 0, 1, 2, \ldots$. Our goal is to find out how the consumer's spending and saving decisions depend on the coefficient of absolute risk aversion θ and the variance σ^2 of income.

The consumer's problem can be stated as: given A_0, choose contingency plans for c_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$ in order to maximize expected utility in (9) subject to the constraints

$$R(A_t - c_t) + Y_{t+1} \geq A_{t+1} \tag{10}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of Y_{t+1}.

a. As a first step in characterizing the consumer's optimal spending and savings decisions via dynamic programming, write down the Bellman equation for this problem. *Note:* In doing this, it is helpful to observe that since income Y_{t+1} is iid and since c_t must be chosen before the value of Y_{t+1} is known, the stock variable A_t completely summarizes the state of the world as it appears to the consumer at the beginning of period t. Thus, in the dynamic programming formulation, c_t can serve as the flow or control variable, A_t as the stock or state variable, and the value function $v(A_t)$ for period t will depend only on A_t.

b. Now guess that the value function takes the specific form

$$v(A_t) = -\frac{1}{\theta F} \exp[-\theta(FA_t + G)],$$

where F and G are constant coefficients to be determined. Rewrite the Bellman equation using this guess, and use the Bellman equation to derive the the first-order condition for c_t and the envelope condition for A_t that help characterize the solution to the consumer's problem. In doing this, it is helpful to note that since the constraint (10) will always bind at the optimum, the guess for the value function also implies that

$$\begin{aligned} v(A_{t+1}) &= -\frac{1}{\theta F} \exp[-\theta(FA_{t+1} + G)] \\ &= -\frac{1}{\theta F} \exp[-\theta FR(A_t - c_t) - \theta FY_{t+1} - \theta G] \\ &= -\frac{1}{\theta F} \exp(-\theta G) \exp[-\theta FR(A_t - c_t)] \exp(-\theta FY_{t+1}) \end{aligned}$$

Moreover, because A_t and c_t are both known at time t,

$$E_t v(A_{t+1}) = -\frac{1}{\theta F} \exp(-\theta G) \exp[-\theta F R(A_t - c_t)] E_t[\exp(-\theta F Y_{t+1})].$$

Finally, since $Y_{t+1} \sim N(\bar{Y}, \sigma^2)$,

$$-\theta F Y_{t+1} \sim N(-\theta F \bar{Y}, \theta^2 F^2 \sigma^2)$$

and therefore $\exp(-\theta F Y_{t+1})$ is log-normally distributed with

$$E_t[\exp(-\theta F Y_{t+1})] = \exp\left(-\theta F \bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right).$$

It follows that the second term on the right-hand side of the Bellman equation can be rewritten as

$$\beta E_t v(A_{t+1}) = -\frac{\beta}{\theta F} \exp(-\theta G) \exp\left(-\theta F \bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right) \exp[-\theta F R(A_t - c_t)]$$

before deriving the first-order and envelope conditions.

c. Next, combine the first-order and envelope conditions to show that the optimal choice for c_t must be given by

$$c_t = F A_t + G.$$

d. It still remains to determine how the unknown coefficients F and G depend on the parameters of the consumer's optimization problem. To do this, substitute $c_t = FA_t + G$ back into the envelope condition. Then, find a solution for F so that the terms involving A_t on the left-hand side are equal to the terms involving A_t on the right-hand side. This solution should indicate that F depends only on R, the gross interest rate on savings. After finding the solution for F, find the solution for G so that the remaining terms on the left-hand side of the envelope condition match the remaining terms on the right-hand side of the envelope condition. This solution should indicate that G depends on the parameters $\bar{Y}$, R, θ, and σ^2.

e. Finally, to complete our analysis, substitute your solutions for F and G back into the expression $c_t = FA_t + G$ to see how consumption c_t and saving $A_t - c_t$ depend on the coefficient of absolute risk aversion θ and the variance of income σ^2. These expressions should indicate that the consumer saves more to guard against adverse income shocks when he or she is more risk averse (that is, when θ is larger) or when the volatility of income goes up (that is, when σ^2 is larger).