Final Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Thursday, December 10, at 12 noon

This exam has two questions on five pages; please check to make sure that your copy has all five pages. Each question has four parts and each part of each question is worth five points, for a total of $2 \times 4 \times 5 = 40$ points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Consumption and Labor Supply in Continuous Time

This problem asks you to use the maximum principle to solve a consumer's problem with an infinite horizon in continuous time.

The consumer enters each period $t \in [0, \infty)$ with bonds worth $B(t)$ dollars that pay interest at the nominal rate $r(t)$ (in net terms so that $r(t)$ is a number like 0.05). The consumer supplies $h(t)$ units of labor at t, earning $W(t)h(t)$ dollars in labor income, where $W(t)$ in the nominal wage rate. The consumer uses his or her interest and labor income to purchase $c(t)$ units of consumption at total cost of $P(t)c(t)$ dollars, where $P(t)$ is the dollar price of goods (the aggregate nominal price level) at t, according to the budget constraint

$$r(t)B(t) + W(t)h(t) - P(t)c(t) \geq \dot{B}(t). \tag{1}$$

This constraint indicates that if the consumer spends less then he or she earns in interest and labor income during period t, he or she is saving, so that $\dot{B}(t) > 0$. Conversely, if the consumer spends more than he or she earns during period t, he or she is dissaving (either borrowing or running down his or her stock of bonds) so that $\dot{B}(t) < 0$.

Taking his or her initial stock of bonds $B(0)$ as given, the consumer chooses consumption $c(t)$ and labor supply $h(t)$ for all $t \in [0, \infty)$ and bond holdings $B(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1 - \sigma} - \frac{\alpha h(t)^{1+\varphi}}{1 + \varphi} \right] dt$$

subject to the constraint in (1) for all $t \in [0, \infty)$. In the utility function, $\rho > 0$ is the consumer's rate of time preference, $\sigma > 0$ governs the consumer's willingness to substitute between consumption at different points in time, $\alpha > 0$ is a parameter that governs the weight placed on leisure versus consumption in preferences, and $\varphi > 0$ is a parameter that governs the consumer's elasticity of labor supply. Specifically, higher values of ρ correspond to decreased patience, higher values of σ reduce the intertemporal elasticity of substitution in consumption, higher values of α imply a stronger preference for leisure versus consumption, and higher values of φ correspond to a smaller elasticity of labor supply.

In current-value form, the maximized Hamiltonian for the consumer's problem is

$$H(B(t), \theta(t); t) = \max_{c(t), h(t)} \frac{c(t)^{1-\sigma} - 1}{1 - \sigma} - \frac{\alpha h(t)^{1+\varphi}}{1 + \varphi} + \theta(t)[r(t)B(t) + W(t)h(t) - P(t)c(t)].$$

Note that, strictly speaking, the maximized current value Hamiltonian depends on t in addition to $B(t)$ and $\theta(t)$ because of time-variation in the interest and wage rates $r(t)$ and $W(t)$ and in the price level $P(t)$.

a. Using the expression for the maximized current-value Hamiltonian given above, write down the first-order conditions for $c(t)$ and $h(t)$ and the pair of differential equations for $\theta(t)$ and $B(t)$ that, according to the maximum principle, help characterize the solution to the consumer's problem.

b. Next, combine two first-order conditions you derived in part (a) to show that the optimizing consumer chooses $c(t)$ and $h(t)$ at each t to equate his or her marginal rate of substitution between consumption and leisure (that is, the ratio of the marginal disutilty of labor supply to the marginal utility of consumption) to the real wage rate $W(t)/P(t)$.

c. Now let

$$\pi(t) = \frac{\dot{P}(t)}{P(t)}$$

denote the rate of price inflation during each period $t \in [0, \infty)$. Use this definition, together with your optimality conditions from part (a), to derive the Euler equation that links the consumer's optimal rate of consumption growth, $\dot{c}(t)/c(t)$, to the parameter σ governing the consumer's intertemporal elasticity of substitution in consumption, the parameter ρ governing the consumer's degree of patience or impatience, and the real (inflation-adjusted) interest rate $r(t) - \pi(t)$.

d. Finally, use your result from part (c) to answer the following question: under what circumstances will the consumer find it optimal to choose a constant level of consumption, with $\dot{c}(t) = 0$ at t?

2. Uncovered Interest Rate Parity?

Consider an investor, residing in the United States, who can trade in bonds issued by the US government and bond issued by the Japanese government. A US government bond costs one dollar today (at time t) and returns R^{usd} next period (at time $t+1$), where R^{usd} is the US dollar interest rate (in gross terms, so R^{usd} is a number like 1.05). A Japanese government bond costs one yen today (at time t) and returns R^{yen} next period (at time $t+1$), where R^{yen} is the Japanese yen interest rate (also in gross terms). The notation reflects an assumption, made here for simplicity, that the US dollar and Japanese yen interest rates remain constant over time.

Since the investor resides in the US, he or she cares about asset prices and payoffs denominated in dollars. Let e_t denote the dollar-per-yen exchange rate today (at time t); that is, e_t is the number of dollars that must be exchanged for one yen today. Likewise, let e_{t+1} denote the dollar-per-yen exchange rate next period (at time $t+1$). With this notation, we can see that purchasing a Japanese government bond requires e_t dollars today; the R^{yen} returned by the bond can then be converted back to $e_{t+1}R^{yen}$ dollars next period.

If we assume that equilibrium in global financial markets requires the expected returns on these two strategies – investing in US government bonds and investing in Japanese government bonds – to be equal, then

$$R^{usd} = R^{yen} E_t \left(\frac{e_{t+1}}{e_t} \right),$$

where E_t denotes investors' expectation at time t. Rearranging this equation yields what international economists call the "uncovered interest parity" or "UIP" condition

$$\frac{R^{usd}}{R^{yen}} = E_t \left(\frac{e_{t+1}}{e_t} \right). \tag{2}$$

This result says that if interest rates on US government bonds are observed to be higher than interest rates on Japanese government bonds, it must be that investors expect the US dollar to depreciate, that is, fall in value so that $E_t(e_{t+1})$ is larger than e_t. Otherwise, investors could earn higher expected returns by selling Japanese bonds and buying US bonds. Conversely, if interest rates on US government bonds are observed to be lower than interest rates on Japanese government bonds, it must be that investors expect the US dollar to appreciate, that is, rise in value so that $E_t(e_{t+1})$ is smaller than e_t.

While intuitively appealing, uncovered interest parity rarely holds in the data. To the contrary, returns on bonds denominated in "high interest rate currencies" tend to have higher average returns than bonds denominated in "low interest rate currencies." With reference to the specific example considered here, the historical experience suggests that, since interest rates are higher in the US than in Japan today, investors in US government bonds will earn higher expected returns than investors in Japanese government bonds, even after taking exchange rate movements into account.

Should we be surprised that the UIP condition is rejected by the data? To find out, this problem asks you to use dynamic programming to solve an asset allocation problem faced by the investor located in the US, who can invest in both US government bonds and Japanese government bonds. In this problem, there are three sources of uncertainty. The investor's labor income W_t, the US price level P_t, and the dollar-per-yen exchange rate e_t all fluctuate randomly, governed by stochastic processes that have the Markov property: from the perspective of time t, the distributions of W_{t+1}, P_{t+1}, and e_{t+1} can depend on the realized values of W_t, P_t, and e_t but not on additional lagged realizations W_{t-j}, P_{t-j}, and e_{t-j} with $j = 1, 2, 3, \ldots$.

The investor enters each period $t = 0, 1, 2, \ldots$ with financial wealth A_t, measured in US dollars. At the beginning of period t, the values of W_t, P_t, and e_t are observed by the investor; however, the values of W_{t+1}, P_{t+1}, and e_{t+1} are still viewed as random. During period t, the investor divides his or her financial wealth A_t, augmented by his or her labor income W_t, into amounts used to purchase consumption goods, US government bonds, and Japanese government bonds.

More specifically, since each consumption good sells for P_t dollars during period t, the investor purchases c_t units of consumption at the total cost of $P_t c_t$ dollars. As above, the interest rates R^{usd} and R^{yen} on US government and Japanese government bonds are assumed to be constant over time. Thus, if B_t^{usd} and B_t^{yen} denote the number of US and Japanese bonds purchased by the investor, the investor spends B_t^{usd} dollars on US bonds and $e_t B_t^{yen}$ dollars on Japanese bonds during period t. These bonds then return $B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}$ dollars to the investor at the beginning of period $t+1$.

Thus, the investor takes initial wealth A_0 as given, and chooses contingency plans for c_t, B_t^{usd}, and B_t^{yen} for $t = 0, 1, 2, \ldots$ and A_t for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

with discount factor β between 0 and 1, subject to the constraints

$$A_t + W_t \geq P_t c_t + B_t^{usd} + e_t B_t^{yen}$$

and

$$B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen} \geq A_{t+1},$$

where each of these two constraints must hold for all $t = 0.1, 2, \ldots$. and all possible realizations of the random shocks.

As a first step in solving the investor's problem, note that if the utility function u is strictly increasing, both constraints will always hold with equality. Thus, to simplify the Bellman equation, the first constraint can be solved for c_t and substituted into the utility function and the second constraint can be solved for A_{t+1} and substituted into the value function for period $t+1$, yielding

$$\begin{aligned} v(A_t, W_t, P_t, e_t) &= \max_{B_t^{usd}, B_t^{yen}} u\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right) \\ &\quad + \beta E_t v(B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}, W_{t+1}, P_{t+1}, e_{t+1}). \end{aligned}$$

a. Using the Bellman equation from above, derive (write down) the first-order and envelope conditions that characterize the investor's optimal choices of B_t^{usd}, B_t^{yen}, and A_t.

b. Next, use the envelope condition together with the binding constraints from the original formulation of the investor's problem to eliminate reference to the unknown value function in the two first-order conditions for B_t^{usd} and B_t^{yen}.

c. Now let

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

denote the investor's intertemporal marginal rate of substitution (IMRS) between t and $t+1$ and let

$$\pi_{t+1} = \frac{P_{t+1}}{P_t}$$

denote the US inflation rate between t and $t+1$. Rearrange your first-order conditions for B_t^{usd} and B_t^{yen} so that the first links the US dollar interest rate R^{usd} to the conditional expectation

$$E_t\left(\frac{m_{t+1}}{\pi_{t+1}}\right)$$

and the second links the Japanese yen interest rate R^{yen} to the conditional expectation

$$E_t\left[\left(\frac{m_{t+1}}{\pi_{t+1}}\right)\left(\frac{e_{t+1}}{e_t}\right)\right].$$

d. Finally, recall that the definition of covariance implies that

$$E_t\left[\left(\frac{m_{t+1}}{\pi_{t+1}}\right)\left(\frac{e_{t+1}}{e_t}\right)\right] = E_t\left(\frac{m_{t+1}}{\pi_{t+1}}\right)E_t\left(\frac{e_{t+1}}{e_t}\right) + cov_t\left(\frac{m_{t+1}}{\pi_{t+1}}, \frac{e_{t+1}}{e_t}\right).$$

Use this definition, together with your results from part (c), to obtain a more general expression that shows that the UIP condition (2) will only hold if the rate of dollar depreciation e_{t+1}/e_t is uncorrelated with the ratio m_{t+1}/π_{t+1} of the US investor's IMRS to the US inflation rate. This special case will hold if, for example, either the Federal Reserve or Bank of Japan conducts monetary policy in a way that fixes the dollar-per-yen exchange rate over time, so that $e_{t+1}/e_t = 1$ for sure for all $t = 0, 1, 2, \ldots$. But under a regime of floating exchange rates, it seems highly unlikely that the dollar-per-yen exchange rate will be uncorrelated with US inflation and the broader US economic conditions that will affect the investor's intertemporal marginal rate of substitution. Again, the message of this fully dynamic and stochastic model is that we should not be surprised that econometric tests routinely reject the UIP condition (2) – in light of this model, in fact, it would be surprising if, to the contrary, UIP *did* appear to hold in the data from countries with floating exchange rates!