Final Exam

ECON 772001 - Math for Economists — Peter Ireland
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Due Thursday, December 19, at 12 noon

This exam has two questions on six pages; please check to make sure that your copy has all six pages. Each question has four parts and each part of each question is worth five points, for a total of $2 \times 4 \times 5 = 40$ points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Consumption, Investment, and Capital Accumulation in a Small Open Economy

This question asks you to characterize optimal resource allocations in a "small open economy" version of the Ramsey (neoclassical growth) model. The economy under consideration is "open" because it can borrow from abroad to finance consumption and investment; it is "small" because it takes the worldwide real interest rate as given.

Suppose, in particular, that a representative household in the small open economy can save or borrow by buying or issuing internationally-traded bonds. Each bond sells for one unit of output or consumption at time t and earns or pays interest at the constant global real interest rate r. Let $B(t)$ and $K(t)$ denote the number of bonds and the units of capital owned by the household at each time $t \in [0, \infty)$. Bond holdings $B(t)$ can be positive or negative: if $B(t)$ is positive, the household is saving, or lending to the rest of the world, and if $B(t)$ is negative, the household is borrowing from the rest of the world. As in the closed-economy Ramsey model, the household uses its $K(t)$ units of capital to produce $K(t)^\alpha$ units of output during each period $t \in [0, \infty)$, where $0 < \alpha < 1$. Also as in the closed-economy Ramsey model, this production function can be interpreted as a special case of the more general Cobb-Douglas specification, $K(t)^\alpha L(t)^{1-\alpha}$, where labor is supplied inelastically by the household, so that $L(t) = 1$ holds for all $t \in [0, \infty)$.

Let $C(t)$ and $I(t)$ denote the household's consumption and investment in each period $t \in [0, \infty)$, and assume that the household faces a quadratic cost $(\phi/2)I(t)^2$, with $\phi > 0$, measured in units of output, of adjusting its capital stock during each period $t \in [0, \infty)$. In this model, $I(t)$ can be positive, in which case the household is installing new capital, or negative, in which case it is consuming or selling off existing capital; but, either way, the formulation implies that it will incur adjustment costs in making these changes. Again as in the closed-economy Ramsey model, physical capital depreciates at the constant rate δ, with

$0 < \delta < 1$, at each date $t \in [0, \infty)$.

Given the initial conditions $B(0)$ and $K(0)$, the household's bond holdings and capital then evolve over time according to

$$rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2 \geq \dot{B}(t) \tag{1}$$

and

$$I(t) - \delta K(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0, \infty)$. Equation (1) shows that, during each period $t \in [0, \infty)$, the household receives interest income $rB(t)$ if $B(t) > 0$ or pays the interest expense $rB(t)$ if $B(t) < 0$, which it then combines with income $K(t)^\alpha$ from production in order to finance spending $C(t)$ on consumption and $I(t)$ on investment and to cover the capital adjustment cost $(\phi/2)I(t)^2$. The constraint in (1), which will always bind at the optimum, determines the change $\dot{B}(t) = dB(t)/dt$ in the household's bond holdings over time. Equation (2), meanwhile, shows how new investment $I(t)$ replaces depreciated capital $\delta K(t)$ before adding to the capital stock at each $t \in [0, \infty)$. This constraint, which will also bind at the optimum, determines the change $\dot{K}(t) = dK(t)/dt$ in the household's capital stock over time.

The household's preferences are described by the additively time-separable utility function

$$\int_0^\infty e^{-\rho t} \ln(C(t))\, \mathrm{d}t, \tag{3}$$

where $\rho > 0$ is the constant discount rate. Over the infinite horizon, the household chooses paths for the flow variables $C(t)$ and $I(t)$ for all $t \in [0, \infty)$ and the stock variables $B(t)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function in (3) subject to the constraints in (1) and (2).

Because the household chooses two flow variables and two stock variables, this problem is slightly more complicated than those we studied in class. Nevertheless, our analysis from class can be extended by defining the maximized current value Hamiltonian

$$\begin{aligned} H(B(t), K(t), \theta(t), q(t)) = \max_{C(t), I(t)} \{&\ln(C(t)) \\ &+ \theta(t)[rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2] \\ &+ q(t)[I(t) - \delta K(t)]\} \end{aligned} \tag{4}$$

and observing that, according to the maximum principle, the values of $C(t)$, $I(t)$, $B(t)$, $K(t)$ that solve the dynamic optimization problem, together with the associated values of the multipliers $\theta(t)$ and $q(t)$, must satisfy the first-order conditions for the values of $C(t)$ and $I(t)$ that solve the static, unconstrained optimization problem on the right-hand side of (4) and the differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_B(B(t), K(t), \theta(t), q(t)),$$

$$\dot{q}(t) = \rho q(t) - H_K(B(t), K(t), \theta(t), q(t)),$$

$$\dot{B}(t) = H_\theta(B(t), K(t), \theta(t), q(t)),$$

and

$$\dot{K}(t) = H_q(B(t), K(t), \theta(t), q(t))$$

where the partial derivatives of the maximized current value Hamiltonian can be computed by applying the envelope theorem to the optimization problem in (4).

a. To characterize more sharply the solution to the household's problem, write down the first-order conditions for $C(t)$ and $I(t)$ that solve the static, unconstrained optimization problem on the right-hand side of (4). Then use the envelope theorem to rewrite the differential equations for $\theta(t)$, $q(t)$, $B(t)$, and $K(t)$ so that they, too, describe the solution to the household's dynamic optimization problem.

b. Suppose that $\rho = r$, so that the household's discount rate exactly equals the worldwide real interest rate. Although this might seem like a knife-edged restriction, recall from some of your homework assignments that restrictions like this one can often be re-interpreted as equilibrium conditions if, in this case, the model is extended to describe how a large number of small open economies borrow and lend in perfectly competitive markets for the internationally-traded bond. In any case, when $\rho = r$, what do the optimality conditions you derived in part (a) imply about the behavior of consumption $C(t)$ and the multiplier $\theta(t)$?

c. Still assuming that $\rho = r$, combine the first-order condition for $I(t)$ and the differential equation for $\dot{q}(t)$ that you derived in part (a), so as to eliminate reference to the multiplier $q(t)$ and obtain a differential equation involving $I(t)$ and $K(t)$ alone.

d. Combined with the differential equation for $I(t)$ that you derived in part (c), the differential equation for $\dot{K}(t)$ that you derived in part (a), which simply restates the binding capital accumulation constraint, forms a system of two differential equations describing the optimal paths for investment $I(t)$ and the capital stock $K(t)$ in this small open economy. Use this pair of differential equations to draw a phase diagram that illustrates the following property of the solution to the household's problem: starting from any value $K(0) > 0$ for the initial capital stock, there is a unique value of investment $I(0)$ such that, starting from $I(0)$ and $K(0)$, the optimally-chosen paths for $I(t)$ and $K(t)$ converge to steady state values I^* and K^*. In drawing this phase diagram, it may be helpful to note that while investment $I(t)$ can take on positive or negative values, depending on whether the household is increasing or decreasing the domestic capital stock, the capital stock $K(t)$ itself must always remain positive.

2. Random Walk Consumption and the Marginal Propensity to Consume

This question asks you to use dynamic programming to deduce what, in a famous 1978 article from the *Journal of Political Economy*, Robert Hall called the "stochastic implications of the life-cycle-permanent income hypothesis." The example is similar to the one featuring "saving with multiple random returns" that we studied in class, except that risk is introduced here through random fluctuations in the consumer's labor income instead through random variation in asset returns.

To begin, let A_t denote a consumer's bank account balance at the beginning of each period $t = 0, 1, 2, \ldots$. The consumer takes A_0 as given, but can choose negative values of A_t for any $t = 1, 2, 3, \ldots$, in which case he or she is borrowing from the bank instead of saving.

At the beginning of period t, the consumer receives labor income y_t. Assume that labor income varies randomly over time, according to a Markov process by which the expected value $E_t y_{t+1}$ of y_{t+1} at time t depends only on y_t and not on additional lags $y_{t-1}, y_{t-2}, y_{t-3}, \ldots$. During period t, knowing current income y_t but still taking future income y_{t+j} for $j = 1, 2, 3, \ldots$ as random, the consumer chooses consumption c_t.

As in the example from class, it is convenient in setting up the consumer's dynamic programming problem to define his or her gross savings during period t as

$$s_t = A_t + y_t - c_t \tag{5}$$

Assuming that the interest rate r on savings and borrowing is constant over time, the consumer's bank account balance then evolves according to

$$(1 + r)s_t \geq A_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$.

Now, if the consumer's preferences are described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where the constant discount factor β satisfies $0 < \beta < 1$, the consumer's problem can be described as one of choosing contingency plans for s_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

subject to A_0 given, the constraint in (6) for all $t = 0, 1, 2, \ldots$, and the Markov process generating the random income stream y_t for all $t = 0, 1, 2, \ldots$. The Bellman equation for this problem is

$$v(A_t, y_t) = \max_{s_t} \; u(A_t + y_t - s_t) + \beta E_t \{v[(1 + r)s_t, y_{t+1}]\}.$$

a. Using the Bellman equation from above, write down the first-order condition for s_t and the envelope condition for A_t that help characterize the solution to the consumer's dynamic, stochastic optimization problem.

b. Next, combine your first-order and envelope conditions from part (a) with the help of the constraints shown in (5) and (6) to obtain a single optimality condition (sometimes called the"Euler equation") that links the consumer's intertemporal marginal rate of substitution to the real interest rate.

c. Now assume that the household's discount factor and the constant real interest rate satisfy $\beta(1+r) = 1$. Again, this may seem like a knife-edged restriction, but the relationship emerges as an equilibrium condition in a more complicated model where a large number of individual households borrow and lend in a competitive market for bonds. Assume, as well, that the consumer's single-period utility function is quadratic, with

$$u(c_t) = -(1/2)(c_t - b)^2$$

for some satiation, or "bliss," point b that is large enough so that $c_t - b < 0$ will always hold. Use these assumptions, together with your optimality condition from part (b), to re-derive Hall's most famous result: that consumption should follow a random walk (more precisely, a "martingale"), with

$$c_t = E_t c_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

d. By combining (5) and (6) to obtain

$$A_t + y_t - c_t \geq \frac{A_{t+1}}{1+r},$$

iterating by forward substitution, imposing some finite limit on borrowing to rule out Ponzi schemes, and invoking the transversality condition ruling out an overaccumulation of savings, one can – as we did for a simpler model in class – derive the consumer's present value budget constraint

$$A_t + \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} = \sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j}.$$

Here in this stochastic model, this present value budget constraint must hold for all possible realizations $\{y_{t+j}\}_{j=0}^{\infty}$ of the path for future income. Therefore, the same equality must hold in expected value at time t, so that

$$A_t + \sum_{j=0}^{\infty} \frac{E_t y_{t+j}}{(1+r)^j} = \sum_{j=0}^{\infty} \frac{E_t c_{t+j}}{(1+r)^j}.$$

Using the result from part (c) that optimal consumption follows a martingale, the restriction that $\beta = 1/(1+r)$, and Euclid's formula

$$\sum_{j=0}^{\infty} \beta^j = \frac{1}{1-\beta}$$

for the infinite sum, this version of the present-value budget constraint pins down the optimal level of consumption at each date $t = 0, 1, 2, \ldots$ as

$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty} \beta^j E_t y_{t+j}\right].$$

Suppose now that income follows a first-order autoregressive process, with

$$y_{t+1} = \bar{y} + \rho(y_t - \bar{y}) + \varepsilon_{t+1},$$

where $\bar{y}$ is the long-run average level of income, the parameter ρ, satisfying $0 \le \rho < 1$, governs the persistence of fluctuations of income above or below its long-run average, and ε_{t+1} is a serially uncorrelated shock with mean zero. This law of motion for income implies that

$$E_t y_{t+j} = \bar{y} + \rho^j (y_t - \bar{y})$$

and hence that

$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty} \beta^j \bar{y} + \sum_{j=0}^{\infty} (\beta\rho)^j (y_t - \bar{y})\right].$$

By applying Euclid's formula to the two infinite sums that remain in this equation, show that Hall's model has two additional implications. Show, first, that the change in c_t brought about by a change in $\bar{y}$ holding $y_t - \bar{y}$ constant (that is, the marginal propensity to consume out of permanent income) equals one. Then show, also, that the change in c_t brought about by a change in $y_t - \bar{y}$ holding $\bar{y}$ constant (that is, the marginal propensity to consume out of deviations of income from its long-run average) is less than one and gets smaller as ρ, measuring the persistence of those deviations, declines.