Final Exam

ECON 772001 - Math for Economists — Peter Ireland
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Thursday, December 6, 1:30 - 2:45pm

This exam has three questions on five pages; please check to make sure that your copy has all three questions and all five pages. Questions 1 and 2 each have three parts; question 3 has two parts. Each part of each question is worth 5 points, for a total of $15 + 15 + 10 = 40$ points overall.

1. Learning By Doing

Consider an industry in which marginal costs of production depend inversely on the total volume of past production, reflecting the effects of learning by doing. Suppose in particular, that these costs are given by

$$c(x(t)) = [1 + x(t)]^{-\gamma}, \tag{1.1}$$

with $\gamma > 0$, where $x(t)$, the total stock of past production, depends on $y(t)$, the flow of output at each date $t \in [0, \infty)$, according to

$$y(t) \geq \dot{x}(t), \tag{1.2}$$

$\dot{x}(t) = dx(t)/dt$, as always, denoting the derivative of the function $x(t)$ with respect to t. Consumer demand for the product is described by the inverse demand curve

$$p(y(t)) = y(t)^{-\alpha}, \tag{1.3}$$

where $0 < \alpha < 1$, so that $-1/\alpha < -1$ measures the constant price elasticity of demand, assumed to be greater than one in absolute value.

If the good is produced and sold in a perfectly competitive market by a large number of firms, each of which ignores the effect that today's production has on future marginal costs, current price will equal market cost:

$$p(y(t)) = c(x(t)) = [1 + x(t)]^{-\gamma} \tag{1.4}$$

Combining (1.4) with (1.1)-(1.3), the dynamics of output under perfect competition are described by

$$y(t) = \dot{x}(t) = [1 + x(t)]^{\gamma/\alpha}. \tag{1.5}$$

Even under perfect competition, learning by doing results in rising output and falling prices over time.

Suppose now that, instead of a large number of perfectly competitive firms, the good is produced by a single monopolist, who takes into account the effects of today's learning by doing on future marginal costs and therefore solves a dynamic optimization problem, choosing continuously differentiable functions $y(t)$ for $t \in [0, \infty)$ and $x(t)$ for $t \in (0, \infty)$ to maximize the present discounted value of profits over the infinite horizon

$$\int_0^\infty e^{-rt}[y(t)p(t) - y(t)c(c(t))]\, \mathrm{d}t$$

or, using the specific functional forms introduced in (1.1) and (1.3)

$$\int_0^\infty e^{-rt} \left\{ y(t)^{1-\alpha} - \frac{y(t)}{[1 + x(t)]^\gamma} \right\} \mathrm{d}t,$$

subject to the constraint in (1.2) for all $t \in [0, \infty)$ and taking the initial condition $x(0) = 0$ as given, where $r > 0$ denotes a constant interest rate.

a. As a first step in solving the monopolist's problem using the maximum principle, write down the maximized Hamiltonian for this problem.

b. Next, write down the first-order condition and the pair of differential equations that, according to the maximum principle, help characterize the solution to the monopolist's problem.

c. If you compare the first-order condition for the monopolist's optimal choice of $y(t)$ that you derived in part (b), above, to the expression for output under perfect competition shown earlier in (1.5), you will notice two differences. First, like any monopolist operating with or without learning-by-doing, the one in this problem accounts for the fact that higher production today results in a lower output price today; that is, the monopolist compares marginal revenue, instead of price, to marginal cost when setting production. Second, unlike the competitive firms, the monopolist in this dynamic problem internalizes the effects that learning-by-doing today has on future marginal costs. Explain briefly (two or three sentences is all is should take) how these two effects on the monopolist's optimal choice of $y(t)$ are reflected in the first-order condition that you derived in part (b).

2. Optimal Growth via Human Capital Accumulation

Consider an economy in which output y_t depends on "quality adjusted" hours worked $k_t h_t$ during each period $t = 0, 1, 2, \ldots$, where h_t denotes the actual number of hours worked and k_t denotes the level of a representative consumer's human capital. Suppose that the consumer can accumulate human capital in any period only by reducing the total number of hours worked, so that

$$h_t = \varphi(k_{t+1}/k_t)$$

for all $t = 0, 1, 2, \ldots$, where φ is strictly positive, strictly decreasing, strictly concave, and continuously differentiable.

Assume, in particular, that

$$y_t = (k_t h_t)^\alpha = [k_t \varphi(k_{t+1}/k_t)]^\alpha, \tag{2.1}$$

where $0 < \alpha < 1$. Since this simple model of growth via human capital accumulation abstracts from physical capital accumulation, consumption equals output in each period $t = 0, 1, 2, \ldots$. Thus, if the representative consumer's preference are described by the additively-time separable utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} \right)$$

where the discount factor lies between zero and one, with $0 < \beta < 1$, and the constant coefficient of relative risk aversion is positive and different from one, with $\sigma > 0$ and $\sigma \neq 1$, then (2.1) implies that the representative consumer's problem can be stated as: given the initial stock of human capital $k_0 > 0$, choose the sequence $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{ \frac{[k_t \varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} \right\}.$$

To solve this problem using dynamic programming, it is most convenient to treat k_t as the period-t state variable and k_{t+1} as the period-t control variable, and then to write the Bellman equation as

$$v(k_t; t) = \max_{k_{t+1}} \frac{[k_t \varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} + \beta v(k_{t+1}; t+1). \tag{2.2}$$

a. For this problem, the value function takes the specific, time-invariant form

$$v(k_t; t) = v(k_t) = \frac{A k_t^{\alpha(1-\sigma)}}{1-\sigma}$$

for all $t = 0, 1, 2, \ldots$, where A is an undetermined coefficient. Substitute this guess for the value function into the Bellman equation (2.2). Then, using the guess, write down the first-order and envelope conditions that characterize the solution to the problem.

b. The optimality conditions you just derived imply that it is optimal for the consumer to choose $k_{t+1} = \theta k_t$ for all $t = 0, 1, 2, \ldots$, so that the growth rate of human capital is constant and equal to a second undetermined coefficient θ. Substitute this conjectured solution into the first-order and envelope conditions that are derived in part (a), above, to obtain a pair of equations that relate the undetermined coefficients A and θ to the model's parameters α, β, and σ.

c. In general, it is not possible to use the pair of equations you derived in part (b), above, to obtain explicit, closed-form expressions for A and θ in terms of α, β, and σ. Since the growth rate of human capital is constant and equal to θ, however, we can conclude in any case that the growth rate of output, y_{t+1}/y_t, will be constant as well. Use the aggregate production function (2.1) to obtain an expression for the constant output growth rate in terms of θ and one or more of the parameters α, β, and σ.

3. Saving with a Random Return

Let A_t denote an infinitely-lived consumer's assets at the beginning of each period $t = 0, 1, 2, \ldots$. Suppose that, during each period t, the consumer divides these assets up into an amount c_t to be consumed and an amount s_t to be saved. Then, between t and $t+1$, the consumer earns a return on his or her savings at the random, gross rate R_{t+1}, which does not become known until after the consumer chooses c_t and s_t during period t. For simplicity, assume that R_{t+1} is independently and identically distributed with

$$E_t[\ln(R_{t+1})] = 0$$

for all $t = 0, 1, 2, \ldots$.

Thus, the consumer takes his or her initial assets A_0 as given and chooses contingency plans for s_t for all $t = 0, 1, 2, \ldots$ and A_t for all $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t),$$

with $0 < \beta < 1$, subject to the constraints

$$R_{t+1} s_t \geq A_{t+1},$$

which must hold for all $t = 0, 1, 2, \ldots$ and all possible realizations of R_{t+1}.

a. In the dynamic programming formulation of this problem, the value function takes the specific form

$$v(A_t) = E + F \ln(A_t),$$

where E and F are undetermined coefficients. Using this guess, write down the Bellman equation for the consumer's problem. Then, write down the first-order and envelope conditions that characterize the solution to the problem.

b. Use the optimality conditions you derived in part (a), above, together with the binding constraint

$$c_t = A_t - s_t$$

to obtain an expression that shows how the optimal choice of c_t will depend, at each date $t = 0, 1, 2, \ldots$, on the consumer's stock of assets A_t and the discount factor β.