Final Exam

Economics 720: Mathematics for Economists

Fall 1999

This exam has three questions on three pages; before you begin, please check to make sure that your copy has all three questions and all three pages. Each question has five parts. Each part of each question is worth 6 points, for a total of $6 \times 5 \times 3 = 90$ points.

1 The Maximum Principle: Optimal Growth

This question asks you to apply the maximum principle to solve a social planner's problem in continuous time and with an infinite horizon. Consider an economy in which output is produced with capital during each period $t \in [0, \infty)$ according to the production function

$$F(k(t)) = k(t)^{\alpha},$$

with $1 > \alpha > 0$. Let $c(t)$ denote the consumption of a representative consumer, and let δ denote the depreciation rate of capital, where $1 > \delta > 0$. Then the capital stock evolves according to

$$k(t)^{\alpha} - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. The social planner takes the initial capital stock, $k(0)$, as given and chooses functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize the representative consumer's utility, given by

$$\int_0^{\infty} e^{-\rho t} \left[\frac{c(t)^{1-\sigma}}{1-\sigma} \right] dt$$

with $\rho > 0$, $\sigma > 0$, and $\sigma \neq 1$, subject to the constraint governing the evolution of $k(t)$ for all $t \in [0, \infty)$.

a) To solve the social planner's problem using the maximum principle, begin by setting up the Hamiltonian.

b) Write down the first order condition for the static optimization problem that appears in the definition of the Hamiltonian.

c) Next, write down the two differential equations that, according to the maximum principle, must be satisfied by the problem's solution.

d) Combine your results from parts (b) and (c) above to obtain a system of two differential equations involving $c(t)$ and $k(t)$ that describe the problem's solution.

e) Your results from part (d) above should imply that starting from any initial $k(0)$, the economy will converge to a steady state in which $k(t)$ remains constant at some value k^* and $c(t)$ remains constant at some value c^*. Use your results from part (d) to derive solutions for the steady-state values k^* and c^* in terms of the model's parameters: α, δ, ρ, and σ.

2 Stochastic Dynamic Programming: Saving with a Random Return

This problem asks you to use dynamic programming to solve a representative consumer's problem when savings earn a random rate of return. Let time be discrete and the horizon be infinite, so that periods are indexed by $t = 0, 1, 2, \ldots$. Let A_t denote the consumer's assets at the beginning of period t. During each period, the consumer divides these assets up into an amount c_t to be consumed and an amount s_t to be saved. The consumer's savings earn interest at the gross rate R_{t+1}, where R_{t+1} is random, possibly serially correlated, and does not become known until the beginning of period $t+1$. Thus, the consumer must choose s_t before knowing the realized value of R_{t+1}. The consumer takes the initial stock of assets A_0 as given and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{A_t\}_{t=1}^{\infty}$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t u(A_t - s_t)$$

subject to the constraint

$$A_{t+1} = R_{t+1} s_t,$$

which must hold for all $t = 0, 1, 2, \ldots$ as well as for all possible realizations of R_{t+1}.

a) Set up the Bellman equation for this problem, using A_t as the state variable, s_t as the control variable, and recognizing that R_{t+1} is random.

b) Now suppose that the consumer's single-period utility function takes the form

$$u(c_t) = u(A_t - s_t) = \frac{(A_t - s_t)^{1-\sigma}}{1-\sigma},$$

where $\sigma > 0$ and $\sigma \neq 1$. Suppose also that the random interest rate R_{t+1} is independently and identically distributed over time with

$$E_t R_{t+1}^{1-\sigma} = 1$$

for all $t = 0, 1, 2, \ldots$. Guess that under these conditions, the value function depends only on A_t, where

$$v(A_t) = \frac{K A_t^{1-\sigma}}{1-\sigma}$$

for some unknown constant K. Use this guess, together with the assumptions on $u(c_t)$ and R_{t+1} stated above, to rewrite your Bellman equation from part (a). Then derive the first order and envelope conditions for the consumer's problem.

c) Use your results from part (b) above to derive expressions that show how the optimal values of c_t and s_t depend on the stock of assets A_t, the parameters β and σ, and the unknown constant K. In doing this, it may be helpful to note that since s_t is known at time t, $E_t s_t = s_t$.

d) Use your results from parts (b) and (c) above to solve for the unknown K in terms of the parameters β and σ.

e) Finally, combine your results from parts (c) and (d) above to derive a solution that shows how the optimal c_t depends on A_t and the parameters β and σ.

3 A System of Linear Differential Equations

Consider the system of linear differential equations consisting of

$$\dot{x}_1(t) = -x_1(t) + 3x_2(t)$$

and

$$\dot{x}_2(t) = 2x_1(t).$$

This system can be written more simply as

$$\dot{x}(t) = Ax(t),$$

where

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix},$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

a) As a first step in solving this system, find the two eigenvalues of the matrix A.

b) Next, find eigenvectors corresponding to the two eigenvalues that you identified in part (a) above.

c) Use your results from parts (a) and (b) above to construct a nonsingular matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

d) Use your results from part (c) above to find the general solutions for $x_1(t)$ and $x_2(t)$.

e) Use your results from above to identify the stationary solution to the system. Is the stationary solution asymptotically stable or unstable?

Solutions to Final Exam

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1 The Maximum Principle: Optimal Growth

a) The Hamiltonian for the social planner's problem is

$$H(k(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \left[\frac{c(t)^{1-\sigma}}{1-\sigma} \right] + \pi(t)[k(t)^\alpha - \delta k(t) - c(t)]$$

b) The first order condition for the static optimization problem that appears in the definition of $H(k(t), \pi(t))$ stated above is

$$e^{-\rho t} c(t)^{-\sigma} - \pi(t) = 0.$$

c) The differential equations that, according to the maximum principle, must be satisfied by the problem's solution are

$$\dot{\pi}(t) = -H_k(k(t), \pi(t); t) = -\pi(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = k(t)^\alpha - \delta k(t) - c(t).$$

d) Take the first order condition from part (b) and rewrite it as

$$e^{-\rho t} = c(t)^\sigma \pi(t).$$

Differentiate both sides of this equation with respect to t to obtain

$$-\rho e^{-\rho t} = \sigma c(t)^{\sigma-1} \dot{c}(t) \pi(t) + c(t)^\sigma \dot{\pi}(t).$$

Use the first order condition again, together with the differential equation for $\dot{\pi}(t)$ from part (c), to rewrite this last result as

$$-\rho c(t)^\sigma \pi(t) = \sigma c(t)^{\sigma-1} \dot{c}(t) \pi(t) - c(t)^\sigma \pi(t)[\alpha k(t)^{\alpha-1} - \delta].$$

Now divide through by $\sigma \pi(t) c(t)^{\sigma-1}$ and isolate $\dot{c}(t)$ on one side to obtain

$$\dot{c}(t) = (1/\sigma) c(t)[\alpha k(t)^{\alpha-1} - \delta - \rho],$$

which together with the differential equation for $\dot{k}(t)$ in part (c),

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t),$$

forms a system of two differential equations that determine the optimal $c(t)$ and $k(t)$.

e) The differential equation

$$\dot{c}(t) = (1/\sigma)c(t)[\alpha k(t)^{\alpha-1} - \delta - \rho]$$

from part (d) implies that in a steady-state where $\dot{c}(t) = 0$, $k(t)$ must satisfy

$$k(t) = k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}.$$

The differential equation

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t)$$

from part (d) then implies that when $k(t) = k^*$ and $\dot{k}(t) = 0$,

$$c(t) = c^* = \left(\frac{\delta + \rho}{\alpha}\right)^{\alpha/(\alpha-1)} - \delta \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}.$$

2 Stochastic Dynamic Programming: Saving with a Random Return

a) The Bellman equation for the consumer's problem is

$$v(A_t, R_t) = \max_{s_t} u(A_t - s_t) + \beta E_t v(R_{t+1}s_t, R_{t+1}).$$

b) Using the assumptions on $u(c_t)$ and R_{t+1} and the suggested guess for $v(A_t)$, the Bellman equation becomes

$$\frac{KA_t^{1-\sigma}}{1-\sigma} = \max_{s_t} \frac{(A_t - s_t)^{1-\sigma}}{1-\sigma} + \frac{\beta K E_t[(R_{t+1}s_t)^{1-\sigma}]}{1-\sigma}.$$

The first order condition is

$$-(A_t - s_t)^{-\sigma} + \beta K E_t(R_{t+1}^{1-\sigma} s_t^{-\sigma}) = 0.$$

The envelope condition is

$$KA_t^{-\sigma} = (A_t - s_t)^{-\sigma}.$$

c) Since s_t is known at time t and since $E_t R_{t+1}^{1-\sigma} = 1$, the first order condition from part (b) can be rewritten as

$$(A_t - s_t)^{-\sigma} = \beta K s_t^{-\sigma}.$$

Use this expression to solve for

$$s_t = \left[\frac{(\beta K)^{1/\sigma}}{1 + (\beta K)^{1/\sigma}}\right] A_t$$

and

$$c_t = A_t - s_t = \left[\frac{1}{1 + (\beta K)^{1/\sigma}}\right] A_t.$$

d) Substituting the results from part (c) back into the Bellman equation yields

$$\begin{aligned}\frac{KA_t^{1-\sigma}}{1-\sigma} &= \left(\frac{1}{1-\sigma}\right)\left[\frac{1}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} A_t^{1-\sigma} \\ &\quad + \left(\frac{\beta K}{1-\sigma}\right)\left[\frac{(\beta K)^{1/\sigma}}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} A_t^{1-\sigma}\end{aligned}$$

or, more simply

$$\begin{aligned}K &= \left[\frac{1}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} + \beta K\left[\frac{(\beta K)^{1/\sigma}}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} \\ &= \left[\frac{1}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} + \beta K(\beta K)^{(1-\sigma)/\sigma}\left[\frac{1}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} \\ &= \left[\frac{1}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} + (\beta K)^{1/\sigma}\left[\frac{1}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} \\ &= [1+(\beta K)^{1/\sigma}]\left[\frac{1}{1+(\beta K)^{1/\sigma}}\right]^{1-\sigma} \\ &= [1+(\beta K)^{1/\sigma}]^{\sigma}.\end{aligned}$$

Hence,

$$K^{1/\sigma} = 1+(\beta K)^{1/\sigma},$$

$$K^{1/\sigma} = \frac{1}{1-\beta^{1/\sigma}},$$

or

$$K = [1-\beta^{1/\sigma}]^{-\sigma}.$$

e) The solution for K from part (d) implies

$$(\beta K)^{1/\sigma} = \frac{\beta^{1/\sigma}}{1-\beta^{1/\sigma}},$$

$$1+(\beta K)^{1/\sigma} = \frac{1}{1-\beta^{1/\sigma}},$$

and

$$\frac{1}{1+(\beta K)^{1/\sigma}} = 1-\beta^{1/\sigma}.$$

Substitute this last result into the solution for c_t from part (c) to obtain

$$c_t = \left[\frac{1}{1+(\beta K)^{1/\sigma}}\right] A_t = (1-\beta^{1/\sigma})A_t,$$

which shows how c_t depends on A_t and the parameters β and σ.

3 A System of Linear Differential Equations

a) By definition, the eigenvalues of A are values of r that satisfy

$$0 = \det \begin{bmatrix} -1-r & 3 \\ 2 & -r \end{bmatrix} = r(1+r) - 6 = r^2 + r - 6 = (r+3)(r-2).$$

Evidently, the eigenvalues are $r_1 = -3$ and $r_2 = 2$.

b) An eigenvector v_1 corresponding to the eigenvalue $r_1 = -3$ must satisfy

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0.$$

One such vector is

$$v_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

An eigenvector v_2 corresponding to the eigenvalue $r_2 = 2$ must satisfy

$$\begin{bmatrix} -3 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0.$$

One such vector is

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

c) Form the matrix P using v_1 and v_2 as its columns:

$$P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$$

Form the matrix D using r_1 and r_2 as its diagonal elements:

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}.$$

It can be verified that P is nonsingular and that

$$P^{-1}AP = D.$$

d) Use the matrices P and D from above to rewrite the system as

$$\dot{z}(t) = Dz(t),$$

where

$$\dot{z}(t) = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = P^{-1}\dot{x}(t)$$

and

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = P^{-1}x(t).$$

Since D is diagonal, the transformed system consists of two self-contained equations

$$\dot{z}_1(t) = r_1 z_1(t) = -3z_1(t)$$

and

$$\dot{z}_2(t) = r_2 z_2(t) = 2z_2(t).$$

We know that these equations have general solutions

$$z_1(t) = k_1 e^{-3t}$$

and

$$z_2(t) = k_2 e^{2t}.$$

Now undo the transformation to solve for $x(t)$:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = x(t) = Pz(t) = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

or

$$x_1(t) = 3k_1 e^{-3t} + k_2 e^{2t}$$

and

$$x_2(t) = -2k_1 e^{-3t} + k_2 e^{2t}.$$

e) The stationary solution is obtained by setting $k_1 = k_2 = 0$, so that

$$x_1(t) = x_2(t) = 0$$

for all t. For any initial conditions $x_1(0)$ and $x_2(0)$ that imply $k_2 \neq 0$, however,

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} (3k_1 e^{-3t} + k_2 e^{2t}) = \lim_{t\to\infty} k_2 e^{2t} = \infty$$

and

$$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} (-2k_1 e^{-3t} + k_2 e^{2t}) = \lim_{t\to\infty} k_2 e^{2t} = \infty,$$

which show that the stationary solution is unstable.

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This exam has four questions on four pages; before you begin, please check to make sure that your copy has all four questions and all four pages. Each part of each question will be given equal weight in determining your total exam score. Since this is a two-hour exam, I suggest that you spend approximately (two hours)/(four questions) = 30 minutes on each question.

1 The Kuhn-Tucker and Envelope Theorems: Cost Minimization

This problem asks you to apply the Kuhn-Tucker and envelope theorems to solve a static cost-minimization problem faced by a firm. The firm produces output with capital k and labor l according to the Cobb-Douglas production function

$$k^{\alpha}l^{1-\alpha}$$

with $0 < \alpha < 1$. Let r denote the rental rate for capital, and let w denote the wage rate for labor. Then the firm's total costs are

$$rk + wl.$$

The firm chooses k and l to minimize its total costs, subject to the requirement that it produce at least y units of output. This problem can be stated formally as

$$\min_{k,l} rk + wl \text{ subject to } k^{\alpha}l^{1-\alpha} \geq y.$$

When solving this problem, the firm takes the factor prices r and w and the output requirement y as given.

a) Set up the Lagrangian for the firm's problem, and write down the first-order conditions for the values k^* and l^* that, according to the Kuhn-Tucker theorem, must be satisfied by the solution to the firm's problem.

b) Together with the binding constraint

$$y = (k^*)^{\alpha}(l^*)^{1-\alpha},$$

your two first-order conditions from part (a) form a system of three equations in three unknowns: the values k^* and l^* that solve the firm's problem and the corresponding value of the multiplier that you introduced into the problem when setting up the Lagrangian. Use these three equations to solve for the three unknowns in terms of the model's four parameters: α, r, w, and y.

c) Now define the minimum cost function as

$$C(\alpha, r, w, y) = \min_{k,l} rk + wl \text{ subject to } k^\alpha l^{1-\alpha} \geq y.$$

Use your results from part (b) to calculate the firm's marginal cost, defined as the derivative $C_4(\alpha, r, w, y)$ of the minimum cost function with respect to output, in terms of the model's parameters α, r, w, and y.

2 The Maximum Principle: Optimal Pricing with Entry

This problem asks you to apply the maximum principle to solve the pricing problem faced by a large firm in an industry where smaller firms gradually enter if the large firm's price becomes too high. Time is continuous and the horizon is infinite, so that periods are indexed by $t \in [0, \infty)$. The demand curve facing the industry as a whole during period t is

$$q(t) = a - bp(t),$$

where $q(t)$ is quantity demanded, $p(t)$ is the large firm's price, and a and b are positive constants.

New, smaller firms begin to enter when the large firm charges a price above some threshold $\bar{p}$. Hence, if $x(t)$ denotes the output produced and sold by the smaller firms, this variable evolves according to

$$\dot{x}(t) = k[p(t) - \bar{p}]$$

for all $t \in [0, \infty)$, where k is also a positive constant. The large firm's sales are then given by

$$q(t) - x(t) = a - bp(t) - x(t)$$

for all $t = [0, \infty)$. If the large firm produces output at the constant marginal cost of c per unit, then its profits during period t equal price minus cost times quantity sold:

$$[p(t) - c][a - bp(t) - x(t)].$$

The large firm's problem can be stated formally as: choose $p(t)$ for all $t \in [0, \infty)$ and $x(t)$ for all $t \in (0, \infty)$ to maximize the present discounted value of profits over the infinite horizon,

$$\int_0^\infty e^{-\rho t}[p(t) - c][a - bp(t) - x(t)]dt,$$

where $\rho > 0$ is the discount rate, subject to the constraints $x(0) = x_0$ given and

$$\dot{x}(t) = k[p(t) - \bar{p}]$$

for all $t \in [0, \infty)$.

a) Set up the Hamiltonian for the large firm's problem, treating $p(t)$ as the flow variable and $x(t)$ as the stock variable. Then write down the first-order condition for the choice of $p(t)$ that solves the static optimization problem on the right-hand side of the Hamiltonian and the two differential equations that, according to the maximum principle, help characterize the solution to the firm's problem.

b) Your results from part (a) form a system of three equations in three unknowns: the flow variable $p(t)$, the stock variable $x(t)$, and the new variable that you introduced when setting up the Hamiltonian. Reduce this three-equation system to a smaller, two-equation system that involves only the flow variable $p(t)$ and the stock variable $x(t)$. One of these two equations should be an equation for $\dot{p}(t)$, and the other should be an equation for $\dot{x}(t)$.

c) Your two-equation system from part (b) should have a steady state, in which $p(t)$ and $x(t)$ are constant, with $p(t) = p^*$, $x(t) = x^*$, $\dot{p}(t) = 0$, and $\dot{x}(t) = 0$. Use your results from above to solve for the steady-state values of p^* and x^* in terms of the model's parameters: a, b, c, k, $\bar{p}$, and ρ.

3 Dynamic Programming: Optimal Resource Depletion

This question asks you to use dynamic programming to characterize the optimal strategy for consuming an exhaustible, or nonrenewable, resource. Time is discrete and the horizon is infinite, so that periods are indexed by $t = 0, 1, 2, ...$ At the beginning of period $t = 0$, the available stock of the resource is given by k_0. During each period $t = 0, 1, 2, ...$, the representative agent consumes c_t units of the resource; since the resource is nonrenewable, this consumption simply subtracts from the available stock according to

$$k_{t+1} = k_t - c_t$$

for all $t = 0, 1, 2, ...$, where k_t is the stock available at the beginning of period t.

Thus, the representative agent chooses $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

with constant discount factor $0 < \beta < 1$, subject to the constraints k_0 given and

$$k_t - c_t \geq k_{t+1}$$

for all $t = 0, 1, 2, ...$

a) Write down the Bellman equation for this problem, using k_t as the state variable, using c_t as the control variable, and guessing that the value function takes the time-invariant form

$$v(k_t) = E + F \ln(k_t),$$

where E and F are constants to be determined. Then write down the first-order and envelope conditions that characterize the solution to the problem.

b) Your first-order and envelope conditions from part (a) can be combined with the binding constraint,

$$k_{t+1} = k_t - c_t,$$

to form a system of three equations in three unknowns: the unknown variables k_t and c_t that solve the original optimization problem and the unknown constant F. Use the equations from this system to solve for the constant F in terms of the model's single parameter: the discount factor β. (Don't worry about trying to solve for E, since this constant does not enter into the expressions for the optimal k_t and c_t.)

c) Now combine your solution for F with the remaining optimality conditions to obtain a system of two equations that can be used to construct the optimal sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$, given the initial condition k_0 and the discount factor β.

4 Differential Equations: Bank Accounts

Let $y(t)$ keep track of a consumer's bank account balances in a continuous-time, infinite-horizon model where periods are indexed by $t \in [0, \infty)$. At each instant in time, these balances grow for two reasons. First, at every instant t, the balances $y(t)$ earn interest at the constant rate r. Second, at every instant t, the consumer deposits a constant amount b of additional funds into the account.

a) Write down the differential equation that describes the evolution of $y(t)$, the bank account balances, over time.

b) Write down the general solution to this differential equation.

c) Now let $y(0) = y_0$ denote the given amount that the consumer initially deposits in the account. Write down the particular solution that satisfies this initial condition.

Solutions to Final Exam

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1 The Kuhn-Tucker and Envelope Theorems: Cost Minimization

a) With the Lagrangian defined as

$$L(k, l, \lambda) = rk + wl - \lambda(k^\alpha l^{1-\alpha} - y),$$

the first-order conditions for the optimal k^* and l^* are

$$r - \alpha\lambda^*(k^*)^{\alpha-1}(l^*)^{1-\alpha} = 0 \tag{1}$$

and

$$w - (1-\alpha)\lambda^*(k^*)^\alpha(l^*)^{-\alpha} = 0. \tag{2}$$

b) Together with the binding constraint

$$y = (k^*)^\alpha(l^*)^{1-\alpha}, \tag{3}$$

the first-order conditions (1) and (2) form a system of three equations in the three unknowns k^*, l^*, and λ^*. There are many ways to solve this system; here is one. Start by using (3) to rewrite (1) and (2) as

$$k^* = \alpha\lambda^*(y/r) \tag{4}$$

and

$$l^* = (1-\alpha)\lambda^*(y/w). \tag{5}$$

Now substitute these results into (3) to solve for λ^* in terms of the parameters:

$$y = \alpha^\alpha(\lambda^*)^\alpha y^\alpha r^{-\alpha}(1-\alpha)^{1-\alpha}(\lambda^*)^{1-\alpha}y^{1-\alpha}w^{-(1-\alpha)}$$

$$1 = \lambda^*\alpha^\alpha r^{-\alpha}(1-\alpha)^{1-\alpha}w^{-(1-\alpha)}$$

$$\lambda^* = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}r^\alpha w^{1-\alpha}. \tag{6}$$

And substitute (6) back into (4) and (5) to solve for k^* and l^* in terms of the parameters:

$$k^* = \alpha^{1-\alpha}(1-\alpha)^{-(1-\alpha)}r^{-(1-\alpha)}w^{1-\alpha}y \tag{7}$$

and

$$l^* = \alpha^{-\alpha}(1-\alpha)^\alpha r^\alpha w^{-\alpha}y. \tag{8}$$

c) The Kuhn-Tucker theorem implies that

$$C(\alpha, r, w, y) = rk^* + wl^* - \lambda^*[(k^*)^\alpha (l^*)^{1-\alpha} - y],$$

and the envelope theorem implies that

$$C_4(\alpha, r, w, y) = \lambda^*.$$

Hence, marginal cost can be found simply by using (6):

$$C_4(\alpha, r, w, y) = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} r^\alpha w^{1-\alpha}. \tag{9}$$

2 The Maximum Principle: Optimal Pricing with Entry

a) The Hamiltonian is

$$H(x(t), \pi(t); t) = \max_{p(t)} e^{-\rho t}[p(t) - c][a - bp(t) - x(t)] + \pi(t)k[p(t) - \bar{p}].$$

The first-order condition for $p(t)$ is

$$e^{-\rho t}[a - bp(t) - x(t)] - be^{-\rho t}[p(t) - c] + \pi(t)k = 0, \tag{10}$$

and the two differential equations for $\pi(t)$ and $x(t)$ are

$$\dot{\pi}(t) = -H_x(x(t), \pi(t); t) = e^{-\rho t}[p(t) - c] \tag{11}$$

and

$$\dot{x}(t) = H_\pi(x(t), \pi(t); t) = k[p(t) - \bar{p}]. \tag{12}$$

b) Rearrange the terms in (10) to obtain

$$\pi(t) = (e^{-\rho t}/k)[2bp(t) + x(t) - (a + bc)],$$

then differentiate both sides with respect to time to obtain

$$\dot{\pi}(t) = -\rho(e^{-\rho t}/k)[2bp(t) + x(t) - (a + bc)] + (e^{-\rho t}/k)[2b\dot{p}(t) + \dot{x}(t)]$$

or, using (11) and (12),

$$\dot{p}(t) = (1/2b)[2b\rho p(t) + \rho x(t) + k(\bar{p} - c) - \rho(a + bc)]. \tag{13}$$

Equations (12) and (13) now form a system of two equations in the two variables $p(t)$ and $x(t)$.

c) In a steady-state with $p(t) = p^*$, $x(t) = x^*$, $\dot{p}(t) = 0$, and $\dot{x}(t) = 0$, (12) implies that

$$0 = k[p(t) - \bar{p}]$$

or

$$p^* = \bar{p}. \tag{14}$$

Equation (13) then implies that

$$0 = (1/2b)[2b\rho p(t) + \rho x(t) + k(\bar{p} - c) - \rho(a + bc)]$$

or

$$x^* = (a + bc) - (k/\rho)(\bar{p} - c) - 2b\bar{p}. \tag{15}$$

3 Dynamic Programming: Optimal Resource Depletion

a) The Bellman equation is

$$E + F\ln(k_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(k_t - c_t).$$

The first-order condition for c_t is

$$\frac{1}{c_t} - \frac{\beta F}{k_t - c_t} = 0, \tag{16}$$

and the envelope condition is

$$\frac{F}{k_t} = \frac{\beta F}{k_t - c_t}. \tag{17}$$

b) To solve for F, use (16) to obtain

$$c_t = \left(\frac{1}{1+\beta F}\right) k_t. \tag{18}$$

Now substitute (18) into (17) to obtain

$$\left(\frac{1}{1+\beta F}\right) = 1 - \beta \tag{19}$$

or

$$F = \frac{1}{1-\beta}, \tag{20}$$

which provides a solution for F in terms of β and verifies that F is a constant.

c) Combine (18) and (19) to obtain

$$c_t = (1-\beta)k_t. \tag{21}$$

Substitute (21) into the binding constraint

$$k_{t+1} = k_t - c_t$$

to obtain

$$k_{t+1} = \beta k_t. \tag{22}$$

Equations (21) and (22) can be used to construct the optimal sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$, given the initial condition k_0 and the discount factor β.

4 Differential Equations: Bank Accounts

a) The differential equation is

$$\dot{y}(t) = ry(t) + b. \tag{23}$$

b) The general solution to the differential equation (23) is

$$y(t) = -b/r + ke^{rt}, \tag{24}$$

since this solution implies that

$$\dot{y}(t) = rke^{rt}$$

equals

$$ry(t) + b = -b + rke^{rt} + b = rke^{rt}$$

as required.

c) Combining the initial condition $y(0) = y_0$ given with the general solution (24) yields

$$y_0 = y(0) = -b/r + k$$

or

$$k = y_0 + b/r.$$

Hence, the particular solution is

$$y(t) = -b/r + (y_0 + b/r)e^{rt}. \tag{25}$$

Midterm Exam

Economics 720: Mathematics for Economists

Fall 2002

This exam has four questions on five pages. Before you begin, please make sure that your copy has all four questions and all five pages. Each question has four parts, and each question will be given equal weight in determining your overall exam score. Since this is an 80-minute exam, I suggest that you spend approximately (80 minutes)/(4 questions) = 20 minutes on each question.

1 The Kuhn-Tucker Theorem and the Cobb-Douglas Utility Function

A consumer likes two goods: good 1 and good 2. The consumer's preferences are described by the Cobb-Douglas utility function

$$U(c_1, c_2) = c_1^a c_2^{1-a},$$

where c_1 denotes consumption of good 1, c_2 denotes consumption of good 2, and the parameter a lies between zero and one: $1 > a > 0$. Let I denote the consumer's income, let p_1 denote the price of good 1, and let p_2 denote the price of good 2. Then the consumer can be viewed as choosing c_1 and c_2 to maximize utility, subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2.$$

a) Define the Lagrangian for this constrained optimization problem.

b) Write down the full set of conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the consumer's optimal choices c_1^* and c_2^* together with the associated value of the Lagrange multiplier.

c) Use your results from above to solve for the optimal choices c_1^* and c_2^* in terms of the model's four parameters: a, I, p_1, and p_2. In doing this, it may help to notice in advance that since the consumer's utility function is strictly increasing, the budget constraint will always bind at the optimum.

d) At the optimum, the consumer spends the fraction

$$\frac{p_1 c_1^*}{I}$$

of his or her income on good 1 and the fraction

$$\frac{p_2 c_2^*}{I}$$

of his or her income on good 2. What do your results from above tell you about the implications of the Cobb-Douglas utility function for these optimal expenditure shares?

2 The Maximum Principle in Discrete Time

Consider the same fairly general discrete-time dynamic optimization problem that we studied in class. The horizon is finite, so that periods are indexed by $t = 0, 1, ..., T$. There is one stock variable y_t and one flow variable z_t. Thus, sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ must be chosen to maximize the objective function

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t),$$

with $1 \geq \beta > 0$, subject to the constraints

$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, ..., T,$$

$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, ..., T,$$

$$y_0 \text{ given},$$

and

$$y_{T+1} \geq y^*.$$

a) Define the Hamiltonian for this problem.

b) Write down the first-order condition for the static optimization problem that appears in the definition of the Hamiltonian.

c) Write down the pair of difference equations that, according to the Maximum Principle, must be satisfied by the solution to the original, dynamic optimization problem.

d) Write down the pair of boundary conditions that, according to the Maximum Principle, must also be satisfied by the solution to the original, dynamic optimization problem.

3 The Maximum Principle in Continuous Time: Customer Flows

Consider the following model, where time is continuous and the horizon is infinite, so that time periods can be indexed by $t \in [0, \infty)$. At each time t, a representative firm has $n(t)$ customers. If the firm sets price $p(t)$, each of its customers purchases $p(t)^{-\theta}$ units of output, where $\theta > 1$ is a parameter that measures the absolute value of the price elasticity of demand.

Hence, the firm's total sales during period t are $n(t)p(t)^{-\theta}$ and its total revenues equal price times quantity sold:

$$p(t)n(t)p(t)^{-\theta} = n(t)p(t)^{1-\theta}.$$

Assume that the representative firm faces increasing marginal costs. More specifically, suppose that producing y units of output costs y^{α}, where $\alpha > 1$. It follows that the firm's total costs during period t are

$$[n(t)p(t)^{-\theta}]^{\alpha} = n(t)^{\alpha}p(t)^{-\theta\alpha}.$$

The firm's profits during period t equal revenues minus costs:

$$n(t)p(t)^{1-\theta} - n(t)^{\alpha}p(t)^{-\theta\alpha}.$$

Total discounted profits over the infinite horizon are therefore

$$\int_0^{\infty} e^{-\rho t}[n(t)p(t)^{1-\theta} - n(t)^{\alpha}p(t)^{-\theta\alpha}],$$

where $\rho > 0$ is the discount rate.

Suppose, for simplicity, that all of the other firms in the industry charge the same price q for their output, and that this price q remains constant over time. Suppose, therefore, that the representative firm gains customers during periods when it charges a price $p(t)$ that is lower than q, and loses customers during periods when it charges a price $p(t)$ that is higher than q. This idea can be summarized by imposing a constraint governing the evolution of $n(t)$:

$$\dot{n}(t) = -\phi[p(t) - q]n(t),$$

where $\phi > 0$. Consistent with our assumptions, this constraint implies that $\dot{n}(t) > 0$, so that the representative firm gains customers, if $p(t) < q$; conversely, $\dot{n}(t) < 0$, so that the representative firm loses customers, if $p(t) > q$. We can generalize this constraint to allow the representative firm to turn away customers; it would then read

$$-\phi[p(t) - q]n(t) \geq \dot{n}(t).$$

We can also assume, however, this constraint will always bind at the optimum.

The representative firm's dynamic optimization problem may now be stated formally as: choose continuously differentiable functions $p(t)$ and $n(t)$ for $t \in [0, \infty)$ to maximize the objective function

$$\int_0^{\infty} e^{-\rho t}[n(t)p(t)^{1-\theta} - n(t)^{\alpha}p(t)^{-\theta\alpha}],$$

subject to the constraints

$$n(0) \text{ given}$$

and

$$-\phi[p(t) - q]n(t) \geq \dot{n}(t)$$

for all $t \in [0, \infty)$.

a) Define the Hamiltonian for the representative firm's problem. In doing this, it may help to view $n(t)$ as the problem's stock variable and $p(t)$ as the problem's flow variable.

b) Write down the first-order condition for the firm's optimal choice of $p(t)$, together with the pair of differential equations that, according to the Maximum Principle, must be satisfied by the solution to the firm's dynamic optimization problem.

c) Now consider a symmetric, steady-state equilibrium for the industry as a whole. In such an equilibrium, all firms, including the representative firm, charge the same constant price q. And if, in addition, there are as many firms as total customers, each firm, including the representative firm, will always serve a single customer. In terms of our notation from above, these steady-state conditions can be written as

$$p(t) = q \text{ and } \dot{p}(t) = 0 \text{ for all } t \in [0, \infty)$$

and

$$n(t) = 1 \text{ and } \dot{n}(t) = 0 \text{ for all } t \in [0, \infty).$$

Substitute these steady-state conditions into the first-order condition and differential equations that you derived above for the representative firm.

d) Finally, use your equations from above to derive a single equation involving only the steady-state equilibrium price q and the model's four parameters: $\theta > 1$, $\alpha > 1$, $\rho > 0$, and $\phi > 0$. IMPORTANT NOTE: It will take too much algebra to solve this equation for q in terms of θ, α, ρ, and ϕ, so don't bother to do this. Just derive the equation itself, and don't worry about its solution.

4 Dynamic Programming Under Certainty: Optimal Monetary Policy

Consider the following macroeconomic model. Time is discrete and the horizon is infinite, so that periods are indexed by $t = 0, 1, 2., ...$ The output gap is denoted by y_t; the inflation rate is denoted by π_t; the nominal interest rate is denoted by r_t; and the real interest rate is calculated as $r_t - \pi_t$.

The central bank sets the nominal interest rate r_t. The output gap is then determined by the Keynesian IS curve,

$$y_t = -\sigma(r_t - \pi_t),$$

with $\sigma > 0$, which just says that output falls when the real interest rate rises; conversely, output rises when the real interest rate falls. This equation implies that the central bank can pick any value it wants for the output gap y_t simply by choosing the appropriate value for r_t. So we can simplify things by just assuming from now on that the central bank chooses y_t; then we can ignore the IS curve throughout the rest of our analysis.

Given the central bank's choice of y_t, inflation is determined by the Phillips curve

$$\pi_{t+1} = \pi_t + \psi y_t,$$

with $\psi > 0$, which just says that inflation rises between t and $t+1$ when the output gap y_t is positive; conversely, inflation falls when the output gap is negative.

Now suppose that the central bank wants to stabilize the inflation rate π_t and the output gap y_t around target values that both equal zero. These policy goals are reflected in the central bank's objective function for period t,

$$-\pi_t^2 - \alpha y_t^2,$$

which penalizes departures of both variables from zero. In this single-period objective function, $\alpha > 0$ measures the weight that the central bank places on its goals for the output gap, relative to its goals for inflation.

The central bank's dynamic optimization problem may now be stated formally as: choose sequences for the output gap $\{y_t\}_{t=0}^{\infty}$ and the inflation rate $\{\pi_t\}_{t=1}^{\infty}$ to maximize the additively time-separable objective function

$$\sum_{t=0}^{\infty} \beta^t [-\pi_t^2 - \alpha y_t^2],$$

with discount factor β, $1 > \beta > 0$, subject to the constraints

$$\pi_0 \text{ given}$$

and

$$\pi_{t+1} = \pi_t + \psi y_t$$

for all $t = 0, 1, 2, ...$

a) Write down the Bellman equation for the dynamic programming formulation of the central bank's problem. In doing so, it may help to view π_t as the state variable and y_t as the control variable.

b) Now guess that the value function takes the time-invariant form

$$v(\pi_t) = -\gamma \pi_t^2,$$

where $\gamma > 0$ is an unknown constant. Using this guess, write down the first-order condition for y_t and the envelope condition for π_t.

c) It will take too much algebra to solve for the unknown constant γ, so DON'T BOTHER TO DO THIS. Instead, just use your first-order condition from above to derive an equation that shows how the central bank's optimal choice for y_t depends on the inflation rate π_t, on the unknown constant γ, and on the model's three parameters: β, α, and ψ.

d) Finally, use your results from above to answer the following question: given that $\gamma > 0$, $1 > \beta > 0$, $\alpha > 0$, and $\psi > 0$, does the central bank find it optimal to "lean against the wind," that is, to choose a negative value for the output gap when inflation is above its target of zero, or does the central bank find it optimal to "lean with the wind," that is, to choose a positive value for the output gap when inflation is above its target of zero?

Solutions to Midterm Exam

Economics 720: Mathematics for Economists

Fall 2002

1 The Kuhn-Tucker Theorem and the Cobb-Douglas Utility Function

a) Define the Lagrangian as

$$L(c_1, c_2, \lambda) = c_1^a c_2^{1-a} + \lambda(I - p_1 c_1 - p_2 c_2). \tag{1.1}$$

b) According to the Kuhn-Tucker theorem, the values c_1^* and c_2^* that solve the consumer's problem and the associated value λ^* of the Lagrange multiplier must satisfy the first-order condition for c_1,

$$L_1(c_1^*, c_2^*, \lambda^*) = a c_1^{*a-1} c_2^{*1-a} - \lambda^* p_1 = 0, \tag{1.2}$$

the first-order condition for c_2^*,

$$L_2(c_1^*, c_2^*, \lambda^*) = (1-a) c_1^{*a} c_2^{*-a} - \lambda^* p_2 = 0, \tag{1.3}$$

the constraint,

$$L_3(c_1^*, c_2^*, \lambda^*) = I - p_1 c_1^* - p_2 c_2^* \geq 0, \tag{1.4}$$

the nonnegativity condition,

$$\lambda^* \geq 0, \tag{1.5}$$

and the complementary slackness condition,

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0. \tag{1.6}$$

c) Begin by dividing (1.2) by (1.3) to eliminate λ^*:

$$\frac{a c_2^*}{(1-a) c_1^*} = \frac{p_1}{P_2}.$$

Now use this result to obtain an expression for c_2^* in terms of c_1^* and the model's parameters:

$$c_2^* = \left(\frac{1-a}{a}\right)\left(\frac{p_1}{p_2}\right) c_1^*.$$

Substitute this expression for c_2^* into the binding constraint to obtain

$$I = p_1 c_1^* + \left(\frac{1-a}{a}\right) p_1 c_1^* = \left(\frac{1}{a}\right) p_1 c_1^*.$$

This last result yields the solution for c_1^*:

$$c_1^* = \frac{aI}{p_1}. \tag{1.7}$$

Substitute (1.7) back into the expression for c_2^* to obtain the solution for c_2^*:

$$c_2^* = \frac{(1-a)I}{p_2}. \tag{1.8}$$

d) Equation (1.7) reveals that the consumer spends the fraction

$$\frac{p_1 c_1^*}{I} = a$$

of his or her income on good 1 and the fraction

$$\frac{p_2 c_2^*}{I} = 1 - a$$

on good 2. A property of the Cobb-Douglas utility function is that it implies that these expenditure shares are constant, and equal to the exponents or weights attached to each good in the utility function itself.

2 The Maximum Principle in Discrete Time

a) Define the Hamiltonian as

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t). \tag{2.1}$$

b) By the Kuhn-Tucker theorem,

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) + \lambda_t [c - G(y_t, z_t; t)],$$

where z_t satisfies the first-order condition

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0. \tag{2.2}$$

c) According to the Maximum Principle, the solution to the original dynamic optimization problem must satisfy the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) = -[\beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t)] \tag{2.3}$$

and

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t) = Q(y_t, z_t; t). \tag{2.4}$$

d) According to the Maximum Principle, the solution to the original dynamic optimization problem must also satisfy two boundary conditions: the initial condition

$$y_0 \text{ given} \tag{2.5}$$

and the terminal, or transversality, condition

$$\pi_{T+1}(y_{T+1} - y^*) = 0. \tag{2.6}$$

3 The Maximum Principle in Continuous Time: Customer Flows

a) Define the Hamiltonian as

$$H(n(t), \pi(t); t) = \max_{p(t)} e^{-\rho t}[n(t)p(t)^{1-\theta} - n(t)^{\alpha}p(t)^{-\theta\alpha}] - \pi(t)\phi[p(t) - q]n(t). \tag{3.1}$$

b) According to the Maximum Principle, the solution to the firm's dynamic optimization problem must satisfy the first-order condition for $p(t)$ on the right-hand side of (3.1),

$$e^{-\rho t}[(1-\theta)n(t)p(t)^{-\theta} + \theta\alpha n(t)^{\alpha}p(t)^{-\theta\alpha-1}] - \pi(t)\phi n(t) = 0 \tag{3.2}$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_n(n(t), \pi(t); t) = -e^{-\rho t}[p(t)^{1-\theta} - \alpha n(t)^{\alpha-1}p(t)^{-\theta\alpha}] + \pi(t)\phi[p(t) - q] \tag{3.3}$$

and

$$\dot{n}(t) = H_{\pi}(n(t), \pi(t); t) = -\phi[p(t) - q]n(t). \tag{3.4}$$

c) Substitute the steady-state conditions into (3.2)-(3.4) to obtain

$$e^{-\rho t}[(1-\theta)q^{-\theta} + \theta\alpha q^{-\theta\alpha-1}] - \pi(t)\phi = 0, \tag{3.5}$$

$$\dot{\pi}(t) = -e^{-\rho t}[q^{1-\theta} - \alpha q^{-\theta\alpha}], \tag{3.6}$$

and

$$\dot{n}(t) = 0. \tag{3.7}$$

d) Equation (3.7) just repeats one of the original steady-state conditions: $\dot{n}(t) = 0$. The problem with (3.5) and (3.6) is that both make reference to the function $\pi(t)$. Moreover, (3.5) refers to $\pi(t)$, while (3.6) refers to $\dot{\pi}(t)$. In order to eliminate all reference to $\pi(t)$, begin by rewriting (3.5) as

$$\pi(t) = (1/\phi)e^{-\rho t}[(1-\theta)q^{-\theta} + \theta\alpha q^{-\theta\alpha-1}].$$

Differentiate both sides of this expression by t to obtain

$$\dot{\pi}(t) = -(\rho/\phi)e^{-\rho t}[(1-\theta)q^{-\theta} + \theta\alpha q^{-\theta\alpha-1}].$$

Now substitute this expression for $\dot{\pi}(t)$ into (3.6) to obtain

$$-(\rho/\phi)e^{-\rho t}[(1-\theta)q^{-\theta}+\theta\alpha q^{-\theta\alpha-1}] = -e^{-\rho t}[q^{1-\theta}-\alpha q^{-\theta\alpha}]$$

or, more simply,

$$(\rho/\phi)[(1-\theta)q^{-\theta}+\theta\alpha q^{-\theta\alpha-1}] = [q^{1-\theta}-\alpha q^{-\theta\alpha}]. \tag{3.8}$$

Equation (3.8) is the equation involving only the steady-state equilibrium price q and the model's parameters. Although it would take some additional algebra, this equation could be used to solve for q in terms of θ, α, ρ, and ϕ.

4 Dynamic Programming Under Certainty: Optimal Monetary Policy

a) The Bellman equation is

$$v(\pi_t;t) = \max_{y_t} -\pi_t^2 - \alpha y_t^2 + \beta v(\pi_t+\psi y_t; t+1). \tag{4.1}$$

b) Using the guess for v, the Bellman equation becomes

$$-\gamma\pi_t^2 = \max_{y_t} -\pi_t^2 - \alpha y_t^2 - \beta\gamma(\pi_t+\psi y_t)^2.$$

The first-order condition for y_t is

$$-2\alpha y_t - 2\beta\gamma\psi(\pi_t+\psi y_t) = 0 \tag{4.2}$$

and the envelope condition for π_t is

$$-2\gamma\pi_t = -2\pi_t - 2\beta\gamma(\pi_t+\psi y_t). \tag{4.3}$$

c) Use the first-order condition to solve for the optimal choice for y_t in terms of the inflation rate π_t, the unknown constant γ, and the model's parameters:

$$\alpha y_t = -\beta\gamma\psi\pi_t - \beta\gamma\psi^2 y_t$$

$$(\alpha+\beta\gamma\psi^2)y_t = -\beta\gamma\psi\pi_t$$

or

$$y_t = -\left(\frac{\beta\gamma\psi}{\alpha+\beta\gamma\psi^2}\right)\pi_t. \tag{4.4}$$

d) Given that γ, β, α, and ψ are all positive, (4.4) reveals that the optimal value for the output gap y_t is negative when inflation is positive; conversely, the optimal y_t is positive when inflation is negative. The central bank finds it optimal to "lean against the wind."

Final Exam

Economics 720: Mathematics for Economists

Fall 2002

This exam has six questions on seven pages. Before you begin, please check to make sure that your copy has all six questions and all seven pages.

Questions 1, 2, 4, and 5 are longer questions, worth 10 points each. Questions 3 and 6 are shorter questions, worth 5 points each.

Since this is a 180-minute exam worth 50 points in total, I suggest that you spend approximately (180 minutes)/(50 points total)×(10 points per long question) = 36 minutes on each long question and (180 minutes)/(50 points total)×(5 points per short question) = 18 minutes on each short question.

1 Dynamic Optimization: Optimal Growth (10 points)

Consider the following version of the optimal growth model. Time is discrete and the horizon is infinite, so that time periods can be indexed by $t = 0, 1, 2, \ldots$. During each period t, output y_t is produced with capital k_t according to the production function

$$y_t = f(k_t),$$

where f is strictly increasing and strictly concave. Let δ, $1 > \delta > 0$, denote the depreciation rate for capital, and let c_t denote the representative household's consumption. Then the capital stock evolves over time according to the constraint

$$k_{t+1} = k_t + f(k_t) - \delta k_t - c_t$$

for all $t = 0, 1, 2, \ldots$, which says that capital available during period $t + 1$ equals capital available during period t, plus output produced during period t, minus the amount of capital that depreciates away during period t, minus the amount of output that is consumed during period t. This constraint can be rewritten as

$$f(k_t) - \delta k_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, 2, \ldots$, to allow for the possibility that capital may be freely disposed of; at the optimum, however, this constraint will always bind.

The representative household has utility over the infinite horizon that is measured by the additively time-separable utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where the discount factor β satisfies $1 > \beta > 0$ and where the single-period utility function u is strictly increasing and strictly concave.

The household's problem can now be stated formally as: choose sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the constraints

$$k_0 \text{ given}$$

and

$$f(k_t) - \delta k_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, 2, ...$

a) Define the Lagrangian for the household's problem. Then write down the first-order conditions and constraints that, according to the Kuhn-Tucker theorem, characterize the solution to the household's problem.

b) Define the Hamiltonian for the household's problem. Then write down the first-order condition and difference equations that, according to the maximum principle, characterize the solution to the household's problem.

c) Write down the Bellman equation for the household's problem. Then write down the first-order condition, envelope condition, and constraint that, according to the dynamic programming approach, characterize the solution to the household's problem.

2 Stochastic Dynamic Programming: Saving with a Serially Correlated, Random Return (10 points)

Consider the following dynamic, stochastic optimization problem in discrete time. An infinitely-lived representative consumer enters each period $t = 0, 1, 2, ...$ with a stock of financial wealth denoted by A_t. During period t, the consumer divides this wealth up into an amount c_t to be consumed and an amount s_t to be saved, subject to the constraint

$$A_t = c_t + s_t,$$

which must hold for all $t = 0, 1, 2, ...$

The gross return R_{t+1} on savings between t and $t + 1$ is random and, more specifically, follows the first-order autoregressive process

$$\ln(R_{t+1}) = \rho \ln(R_t) + \varepsilon_{t+1}$$

where $1 > \rho > 0$ and ε_{t+1} is a shock that satisfies $E_t \varepsilon_{t+1} = 0$ for all $t = 0, 1, 2, ...$ Thus, last period's return R_t is known when the consumer chooses s_t, but R_{t+1} is still viewed as random. Given the consumer's choice of s_t and the realized value of R_{t+1}, the consumer's financial wealth A_{t+1} is determined by the constraint

$$R_{t+1}s_t \geq A_{t+1},$$

which must hold for all $t = 0, 1, 2, \ldots$ and for all possible realizations of R_{t+1}.

The representative consumer seeks to maximize expected utility

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where the discount factor satisfies $1 > \beta > 0$ and where, as indicated, the single-period utility function takes the natural logarithmic form. As a first step in solving the consumer's problem, it is convenient to substitute the constraint $A_t = c_t + s_t$ into the utility function to obtain

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t).$$

The consumer's problem can now be stated formally as: choose contingency plans for s_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$, to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t)$$

subject to the constraints

$$A_0 \text{ given}$$

and

$$R_{t+1}s_t \geq A_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of R_{t+1}.

a) Write down the Bellman equation for the household's problem. In doing so, you might find it helpful to view A_t as the state variable, s_t as the control variable, and R_t as the random variable entering into the problem. You might also find it helpful to recognize that since the consumer's utility function is strictly increasing, the constraint $R_{t+1}s_t \geq A_{t+1}$ will always bind at the optimum.

b) Now guess that the value function for this problem takes the form

$$v(A_t, R_t) = F \ln(A_t) + G \ln(R_t) + H,$$

where F, G, and H are unknown constants. Substitute this guess into your Bellman equation from above, and then write down the first-order condition for s_t and the envelope condition for A_t. In deriving these optimality conditions, you might find it helpful to note that since s_t is known at time t,

$$E_t \ln(s_t) = \ln(s_t).$$

You might also find it helpful to note that since

$$\ln(R_{t+1}) = \rho \ln(R_t) + \varepsilon_{t+1},$$

with $E_t \varepsilon_{t+1} = 0$, it follows that

$$E_t \ln(R_{t+1}) = \rho \ln(R_t).$$

c) Use your first-order condition from above to derive an expression linking the consumer's optimal choice s_t to the state variable A_t, the discount factor β, and the unknown constant F.

d) Next, use your envelope condition from above to solve for the unknown constant F in terms of the discount factor β.

e) Combine your results from above to obtain a set of two equations that can be used to construct the optimal paths for s_t and A_t, given the initial value of A_0 and the realized path for R_{t+1}. NOTE: It should not be necessary to solve for the unknown constants G and H to derive these equations; since solving for G and H requires a lot of extra algebra, DON'T BOTHER TO DO THIS.

3 A Scalar Differential Equation: General Solution and Phase Diagrams (5 points)

Consider the scalar differential equation

$$\dot{y}(t) = ay(t),$$

where $\dot{y}(t)$ is the derivative of the unknown function $y(t)$, and a is a non-zero constant.

a) Write down the general solution to this differential equation.

b) Next, find the unique stationary solution to the differential equation.

c) Suppose now that $a < 0$, and draw the phase diagram for this differential equation. What does the phase diagram tell you about the asymptotic stability or instability of the unique stationary solution?

d) Suppose instead that $a > 0$, and redraw the phase diagram for this alternative case. What does the phase diagram tell you about the asymptotic stability or instability of the unique stationary solution?

4 A System of Linear Differential Equations (10 points)

Consider the system of linear differential equations consisting of

$$\dot{x}_1(t) = -x_1(t) + 3x_2(t)$$

and

$$\dot{x}_2(t) = -2x_1(t) + 4x_2(t).$$

This system can be written more simply as

$$\dot{x}(t) = Ax(t),$$

where

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix},$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and

$$A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}.$$

a) As a first step in solving this system, find the two eigenvalues of the matrix A.

b) Next, find eigenvectors corresponding to the two eigenvalues that you identified above.

c) Use your results from above to construct a nonsingular matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

d) Use your results from above to find the general solutions for $x_1(t)$ and $x_2(t)$.

e) Identify the unique stationary solution to the system. Use your results from above to answer the question: is the stationary solution asymptotically stable or unstable?

f) Finally, use your results from above to find the unique functions $x_1(t)$ and $x_2(t)$ that solve the system of differential equations together with the initial conditions

$$x_1(0) = 2 \text{ and } x_2(0) = 1.$$

5 Linear Difference Equations: Employment and Unemployment Dynamics (10 points)

An economy consists of two types of workers: those that are employed (in state 1) and those that are unemployed (in state 2). Let x_{1t} denote the fraction of the population that is employed during period t, let x_{2t} denote the fraction of the population that is unemployed during period t, and let x_t denote the vector

$$x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}.$$

Suppose that each worker who is employed during period t faces a 10 percent chance of becoming unemployed during period $t+1$, while each worker who is unemployed during period t has a 40 percent chance of finding a job during period $t+1$. Then if the population is large, the evolution of x_t is governed by the system of difference equations

$$x_{t+1} = Mx_t,$$

where

$$M = \begin{bmatrix} 0.90 & 0.40 \\ 0.10 & 0.60 \end{bmatrix}.$$

a) As a first step in analyzing this model, find the eigenvalues of the matrix M.

b) Next, find the eigenvectors corresponding to the eigenvectors that you identified above.

c) Use your results from above to obtain the general solution to the system of difference equations $x_{t+1} = Mx_t$.

d) Now find the particular solution to the system under the assumption that the population initially consists of equal numbers of employed and unemployed workers, so that

$$x_0 = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}.$$

e) Use your results from above to show that as t becomes arbitrarily large, the economy will converge from x_0 to a steady state in which the fractions of employed and unemployed workers are constant.

f) In the steady state, what fraction of the population is employed? What fraction of the population is unemployed?

6 Nonlinear Difference Equations: Capital Accumulation and Economic Growth (5 points)

Consider an economy in which time is discrete and the horizon infinite, so that periods can be indexed by $t = 0, 1, 2, ...$ During each period $t = 0, 1, 2, ...$, output y_t is produced with capital k_t according to the production function

$$y_t = Ak_t^\alpha,$$

where, by assumption, the parameters A and α satisfy $A > 0$ and $1 > \alpha > 0$.

Now make two simplifying assumptions. First, assume that capital depreciates completely between periods. Second, assume that in this economy, agents save and invest a constant fraction s, $1 > s > 0$, of output. Under these two additional assumptions, the capital stock evolves according to the nonlinear difference equation

$$k_{t+1} = sAk_t^\alpha$$

for all $t = 0, 1, 2, ...$ That is, the capital stock during period $t + 1$ equals the amount saved during period t, which in turn just equals the constant savings rate s times the amount of output Ak_t^α.

a) Find the stationary solutions to this nonlinear difference equation.

b) Now draw a diagram that illustrates the asymptotic stability or instability of these stationary solutions.

Solutions to Final Exam

Economics 720: Mathematics for Economists

Fall 2002

1 Dynamic Optimization: Optimal Growth

a) The Lagrangian is

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_{t+1}[k_t + f(k_t) - \delta k_t - c_t - k_{t+1}].$$

The first-order condition for c_t, $t = 0, 1, 2, ...$, is

$$\beta^t u'(c_t) - \lambda_{t+1} = 0.$$

The first-order condition for k_t, $t = 1, 2, 3, ...$, is

$$\lambda_{t+1}[1 + f'(k_t) - \delta] - \lambda_t = 0.$$

The binding constraint, $t = 0, 1, 2, ...$, is

$$k_t + f(k_t) - \delta k_t - c_t - k_{t+1} = 0.$$

b) The Hamiltonian is

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t u(c_t) + \pi_{t+1}[f(k_t) - \delta k_t - c_t].$$

The first-order condition for c_t is

$$\beta^t u'(c_t) - \pi_{t+1} = 0.$$

The difference equations for π_t and k_t are

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}[f'(k_t) - \delta]$$

and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = f(k_t) - \delta k_t - c_t.$$

c) The Bellman equation is

$$v(k_t;t) = \max_{c_t} u(c_t) + \beta v[k_t + f(k_t) - \delta k_t - c_t; t+1].$$

The first-order condition for c_t is

$$u'(c_t) - \beta v'(k_{t+1}; t+1) = 0.$$

The envelope condition for k_t is

$$v'(k_t;t) = \beta[1 + f'(k_t) - \delta]v'(k_{t+1}; t+1).$$

The binding constraint is

$$k_{t+1} = k_t + f(k_t) - \delta k_t - c_t.$$

Notice that the equations that you derived in parts (a), (b), and (c) of this question coincide, as of course they must, with

$$\lambda_{t+1} = \pi_{t+1} = \beta^{t+1} v'(k_{t+1}; t+1).$$

2 Stochastic Dynamic Programming: Saving with a Serially Correlated Random Return

a) The Bellman equation is

$$v(A_t, R_t) = \max_{s_t} \ln(A_t - s_t) + \beta E_t v(R_{t+1}s_t, R_{t+1}).$$

b) Using the guess for the form of the value function, and the facts about s_t and R_{t+1}, the Bellman equation can be written

$$F\ln(A_t) + G\ln(R_t) + H = \max_{e_t} \ln(A_t - s_t) + \beta F \ln(s_t) + \beta(F+G)\rho\ln(R_t) + \beta H.$$

The first-order condition for s_t is

$$-\frac{1}{A_t - s_t} + \frac{\beta F}{s_t} = 0,$$

while the envelope condition for A_t is

$$\frac{F}{A_t} = \frac{1}{A_t - s_t}.$$

c) The first-order condition implies that

$$s_t = \left(\frac{\beta F}{1+\beta F}\right) A_t.$$

d) The envelope condition now implies that

$$FA_t - F\left(\frac{\beta F}{1+\beta F}\right)A_t = A_t$$

or

$$F\left[1-\left(\frac{\beta F}{1+\beta F}\right)\right] = 1$$

or

$$F = 1 + \beta F$$

or

$$F = \frac{1}{1-\beta}.$$

e) Return to

$$s_t = \left(\frac{\beta F}{1+\beta F}\right)A_t$$

and substitute in the solution for F to obtain

$$s_t = \beta A_t.$$

Next, substitute this solution for s_t into the binding constraint to obtain

$$A_{t+1} = R_{t+1}s_t = \beta R_{t+1}A_t.$$

Given the initial condition A_0 and a realized path for R_{t+1}, these last two equations can be used to construct the optimal paths for s_t, $t = 0, 1, 2, ...$, and A_t, $t = 1, 2, 3, ...$

3 A Scalar Differential Equation: General Solution and Phase Diagrams

a) The general solution is

$$y(t) = ke^{at},$$

where different values of the parameter k correspond, for instance, to different initial conditions $y(0) = y_0 = k$.

b) Stationary solutions take the form

$$y(t) = y$$

and

$$\dot{y}(t) = 0,$$

where the constant y must satisfy

$$0 = ay.$$

Since a is, by assumption, nonzero, it is clear from this last equation that the unique stationary solution is

$$y(t) = 0.$$

c) In this case, with $a < 0$, the phase diagram reveals that the stationary solution is asymptotically stable.

d) In this case, with $a > 0$, the phase diagram reveals that the stationary solution is unstable.

4 A System of Linear Differential Equations

a) By definition, an eigenvalue r of A must solve the characteristic equation

$$\begin{aligned}
0 &= \det(A - rI) \\
&= \det\left(\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \right) \\
&= \det \begin{bmatrix} -1-r & 3 \\ -2 & 4-r \end{bmatrix} \\
&= (-1-r)(4-r) + 6 \\
&= -4 - 4r + r + r^2 + 6 \\
&= r^2 - 3r + 2 \\
&= (r-1)(r-2).
\end{aligned}$$

Evidently, the two eigenvalues are $r_1 = 1$ and $r_2 = 2$.

b) Again by definition, the eigenvector v_1 corresponding to the eigenvalue $r_1 = 1$ must satisfy

$$(A - r_1 I)v_1 = 0$$

or

$$\begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$-2v_{11} + 3v_{12} = 0.$$

Evidently, v_1 must be of the form

$$v_1 = \begin{bmatrix} v_{11} \\ (2/3)v_{11} \end{bmatrix}.$$

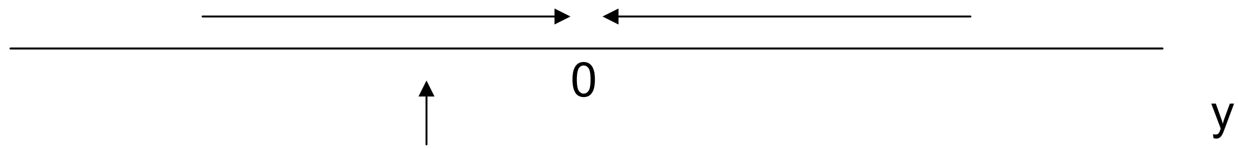
A particularly simple choice for v_1 is therefore

$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Similarly, the eigenvector v_2 corresponding to the eigenvalue $r_2 = 2$ must satisfy

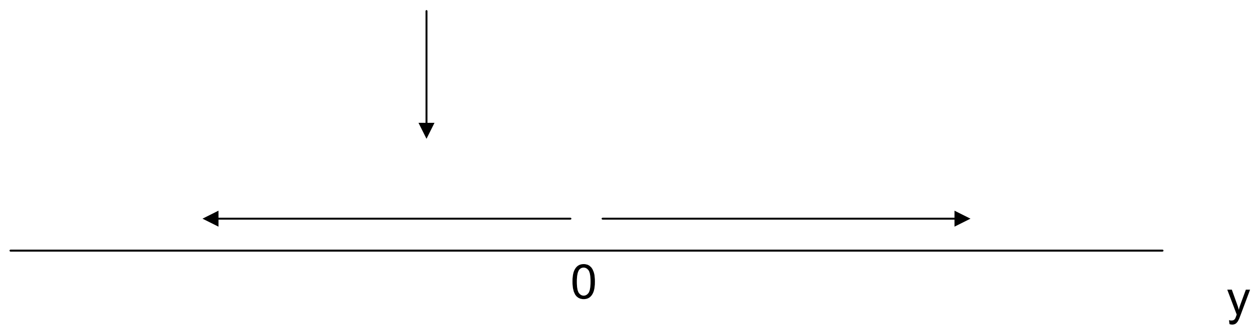
$$(A - r_2 I)v_2 = 0$$

Phase diagrams for $y'(t) = ay(t)$:



When $a < 0$, $y'(t) < 0$ if $y(t) > 0$ and $y'(t) > 0$ if $y(t) < 0$.
The stationary solution $y(t) = 0$ is asymptotically stable.

When $a > 0$, $y'(t) > 0$ if $y(t) > 0$ and $y'(t) < 0$ if $y(t) < 0$.
The stationary solution $y(t) = 0$ is unstable.



or

$$\begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$-3v_{21} + 3v_{22} = 0$$

and

$$-2v_{21} + 2v_{22} = 0$$

Evidently, v_2 must be of the form

$$v_2 = \begin{bmatrix} v_{21} \\ v_{21} \end{bmatrix}.$$

A particularly simple choice for v_2 is therefore

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

c) Form the matrix P using the eigenvectors v_1 and v_2 as its columns:

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}.$$

Form the matrix D using the eigenvalues r_1 and r_2 as its diagonal elements:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since the eigenvectors v_1 and v_2 are linearly independent, P is nonsingular and

$$P^{-1}AP = D.$$

d) For this system of linear differential equations, the general solution takes the form

$$x_t = k_1 e^{r_1 t} v_1 + k_2 e^{r_2 t} v_2.$$

Plugging in the specific values of r_1, r_2, v_1, and v_2 yields

$$x_t = k_1 e^t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + k_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

e) A stationary solution to the system takes the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$$

and

$$\dot{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0,$$

where the constants x_1 and x_2 must satisfy

$$0 = Ax.$$

Since A is nonsingular, the unique stationary solution must have $x = 0$, that is,

$$x_1(t) = 0 \text{ and } x_2(t) = 0.$$

Referring back to the general solution

$$x(t) = k_1 e^t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + k_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we can see that even for small values of k_1 and k_2, corresponding to initial conditions $x_1(0)$ and $x_2(0)$ close to the stationary solution,

$$\lim_{t \to \infty} x(t) = \infty,$$

implying that the stationary solution is unstable. This last result follows from the fact that the eigenvalues r_1 and r_2 are both positive.

f) We already know that the general solution takes the form

$$x(t) = k_1 e^t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + k_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If we are also given the initial conditions $x_1(0) = 2$ and $x_2(0) = 1$, then we must have

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = x(0) = k_1 e^0 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + k_2 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Evidently, the parameters k_1 and k_2 must satisfy

$$2 = 3k_1 + k_2$$

and

$$1 = 2k_1 + k_2.$$

Solve the first of these two equations for k_2:

$$k_2 = 2 - 3k_1.$$

Now substitute this result into the second equation and solve for k_1:

$$1 = 2k_1 + 2 - 3k_1$$

or

$$k_1 = 1.$$

Finally, return to the solution for k_2 to obtain

$$k_2 = 2 - 3k_1 = -1.$$

Evidently, the unique functions $x_1(t)$ and $x_2(t)$ that satisfy the system of differential equations and the initial conditions are

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^t \begin{bmatrix} 3 \\ 2 \end{bmatrix} - e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

5 Linear Difference Equations: Employment and Unemployment Dynamics

a) By definition, r is an eigenvalue of M if it solves the characteristic equation

$$\begin{aligned}
0 &= \det(M - rI) \\
&= \det \begin{bmatrix} 0.90 - r & 0.40 \\ 0.10 & 0.60 - r \end{bmatrix} \\
&= (0.90 - r)(0.60 - r) - (0.40)(0.10) \\
&= 0.54 - 0.90r - 0.60r + r^2 - 0.04 \\
&= r^2 - 1.5r + 0.5 \\
&= (r - 1)(r - 0.5).
\end{aligned}$$

Evidently, the two eigenvalues are $r_1 = 1$ and $r_2 = 0.5$.

b) Again by definition, the eigenvector v_1 corresponding to the eigenvalue $r_1 = 1$ must satisfy

$$\begin{bmatrix} 0.90 - 1 & 0.40 \\ 0.10 & 0.60 - 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which requires that

$$-0.10v_{11} + 0.40v_{12} = 0$$

and

$$0.10v_{11} - 0.40v_{12} = 0$$

or, more simply,

$$v_{12} = (1/4)v_{11}.$$

Evidently, v_1 must be of the form

$$v_1 = \begin{bmatrix} v_{11} \\ (1/4)v_{12} \end{bmatrix}.$$

A simple vector of this form is

$$v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Similarly, the eigenvector v_2 corresponding to the eigenvalue $r_2 = 0.5$ must satisfy

$$\begin{bmatrix} 0.90 - 0.50 & 0.40 \\ 0.10 & 0.60 - 0.50 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which requires that

$$0.40v_{21} + 0.40v_{22} = 0$$

and

$$0.10v_{21} + 0.10v_{22} = 0$$

or, more simply,

$$v_{22} = -v_{21}.$$

Evidently, v_2 must be of the form

$$v_2 = \begin{bmatrix} v_{21} \\ -v_{22} \end{bmatrix}.$$

A simple vector of this form is

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

c) The general solution to this system of difference equations takes the form

$$x_t = k_1 r_1^t v_1 + k_2 r_t^2 v_2,$$

where r_1 and r_2 are the eigenvalues of M and v_1 and v_2 are the corresponding eigenvectors. Since $r_1^t = 1$ for all $t = 0, 1, 2, ...$, the general solution to the difference equation for x_t is therefore

$$x_t = k_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + k_2 (0.5)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

d) Given the initial condition

$$x_0 = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix},$$

the general solution from above requires that

$$\begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix} = k_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or

$$0.50 = 4k_1 + k_2$$

and

$$0.50 = k_1 - k_2.$$

Use the second of these two equations to solve for k_2 in terms of k_1:

$$k_2 = k_1 - 0.50.$$

Now substitute this result into the first equation, and solve for k_1:

$$0.50 = 4k_1 + k_2 = 4k_1 + k_1 - 0.50$$

$$1 = 5k_1$$

$$k_1 = 0.20.$$

Finally, substitute this solution for k_1 back into the equation for k_2 to obtain

$$k_2 = k_1 - 0.50 = 0.20 - 0.50 = -0.30.$$

Thus, starting from the initial condition given above, the evolution of x_t is determined by the particular solution

$$x_t = 0.20 \begin{bmatrix} 4 \\ 1 \end{bmatrix} - (0.30)(0.5)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

e) Since

$$\lim_{t\to\infty}(0.5)^t = 0,$$

the particular solution from above implies that starting from x_0,

$$\lim_{t\to\infty} x_t = 0.20 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix}.$$

Evidently, the economy converges to a steady state, in which the fractions of agents employed and unemployed are constant.

f) The last result from above indicates that in the steady state, 80 percent of all workers are employed and 20 percent of all workers are unemployed.

6 Nonlinear Difference Equations: Capital Accumulation and Economic Growth

a) A stationary solution must have

$$k_t = k \text{ and } k_{t+1} = k,$$

where the constant k satisfies

$$k = sAk^\alpha.$$

Evidently, there are two stationary solutions, one with

$$k_t = 0$$

for all $t = 0, 1, 2, ...$, and the other with

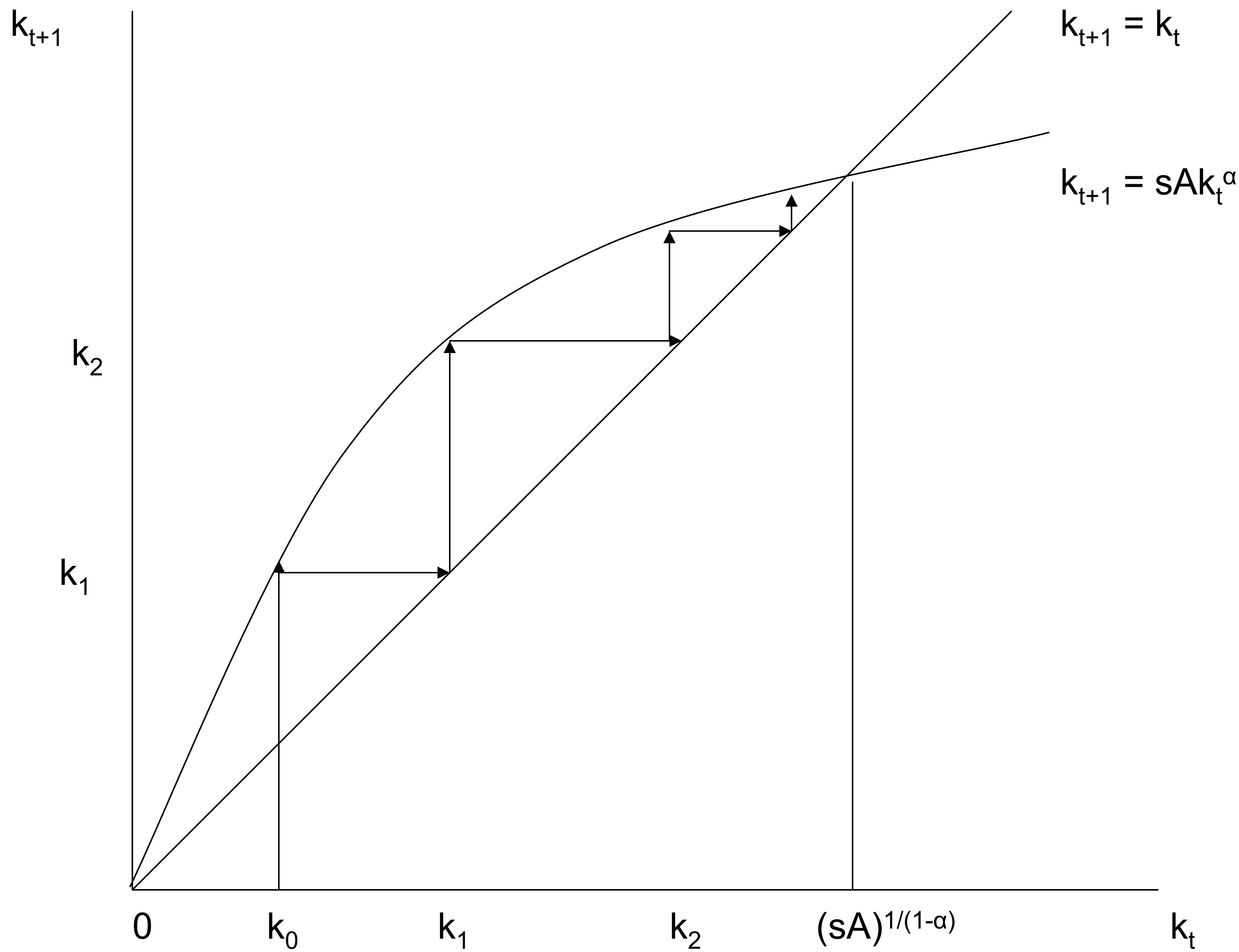
$$k = (sA)^{1/(1-\alpha)} > 0$$

for all $t = 0, 1, 2, ...$

b) A diagram with k_{t+1} on the y-axis and k_t on the x-axis reveals that the stationary solution $k_t = 0$ is unstable, whereas the stationary solution $k_t = (sA)^{1/(1-\alpha)} > 0$ is asymptotically stable. These results follow mainly from the fact that the expression on the right-hand side of the difference equation

$$k_{t+1} = sAk_t^\alpha$$

is strictly increasing and strictly concave when viewed as a function of k_t, given the assumptions that $1 > s > 0$, $A > 0$, and $1 > \alpha > 0$.



k_{t+1}
$k_{t+1} = k_t$
$k_{t+1} = sAk_t^{\alpha}$
k_2
k_1
0
k_0
k_1
k_2
$(sA)^{1/(1-\alpha)}$
k_t

Final Exam

Economics 720: Mathematics for Economists

Fall 2003

This exam has five questions on seven pages; before you begin, please check to make sure that your copy has all five questions and all seven pages. The five questions will receive equal weight in determining your overall exam score. Since this is a two-hour exam, I suggest that you spend approximately (120 minutes)/(5 questions) = 24 minutes on each question.

1 Applying the Kuhn-Tucker Theorem: The Linear Expenditure System

This question asks you to derive a version of the linear expenditure system by applying the Kuhn-Tucker theorem to a consumer's static optimization problem.

The consumer likes two goods: c_1 and c_2. The consumer's utility function takes the Stone-Geary form

$$U(c_1, c_2) = (c_1 - x_1)^\alpha (c_2 - x_2)^{1-\alpha},$$

where $x_1 > 0$ and $x_2 > 0$ are positive parameters that measure the subsistence levels of the two goods and where α is parameter that lies between zero and one: $0 < \alpha < 1$. The consumer seeks to maximize this utility function subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2,$$

where I denotes the consumer's income and p_1 and p_2 are the prices of the two goods. To insure that this problem has a well-defined solution, it is necessary to assume that the consumer's income I is large enough to allow him or her to purchase and consume amounts of each good that exceed the subsistence levels; this extra assumption can be written as

$$I > p_1 x_1 + p_2 x_2.$$

The consumer's problem can now be stated as

$$\max_{c_1, c_2} (c_1 - x_1)^\alpha (c_2 - x_2)^{1-\alpha} \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

a) As a first step in applying the Kuhn-Tucker theorem, define the Lagrangian for this problem.

b) Now write down the complete set conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve the consumer's problem together with the associated value of the Lagrange multiplier.

c) Use your results from part (b) above to obtain a mathematical expression that restates the familiar graphical condition: the optimizing consumer acts so as to equate the marginal rate of substitution between the two goods (the slope of the indifference curve) to the ratio of their prices (the slope of the budget constraint).

d) Finally, use your results from parts (b) and (c) above to show how the consumer's optimal *expenditures* $p_1c_1^*$ and $p_2c_2^*$ on each good depend on the model's parameters: x_1, x_2, α, I, p_1, and p_2. HINT: There are a variety of ways to do this, but perhaps the easiest is to start with your result from part (c) above, and use it to obtain an expression linking optimal expenditures $p_2c_2^*$ on good two to optimal expenditures $p_1c_1^*$ on good one. Then use this result, together with the budget constraint (which binds at the optimum), to solve for $p_1c_1^*$ in terms of the model's parameters. Finally, substitute this solution for $p_1c_1^*$ back into your expression for $p_2c_2^*$ to solve for $p_2c_2^*$ in terms of the model's parameters.

e) Using your results from part (d) above, answer the following three questions.

i) Does the optimal $p_1c_1^*$ rise, fall, or remain unchanged when income I rises?

ii) Does the optimal $p_1c_1^*$ rise, fall, or remain unchanged when the price p_1 of good one rises?

iii) Does the optimal $p_1c_1^*$ rise, fall, or remain unchanged when the price p_2 of good two rises?

2 The Maximum Principle in Discrete Time: Life Cycle Saving and Borrowing Constraints

This problem asks you to use the maximum principle to study two versions of a model of life cycle saving: one that allows the consumer to borrow and the other that imposes an additional constraint that prevents the consumer from borrowing.

Consider, therefore, a consumer who is employed for periods $t = 0, 1, ..., T$. During each period of employment, the consumer receives constant labor income w. Let k_t denote the consumer's stock of assets at the beginning of period t, and assume that $k_0 = 0$, so that the consumer begins his or her career with no assets.

In the first version of this model, the consumer will be allowed to borrow by choosing negative values for k_t for any $t = 1, 2, ..., T$. However, the consumer must eventually save for retirement: k_{T+1} must satisfy

$$k_{T+1} \geq k^* > 0,$$

where k^* denotes the amount of saving needed for retirement.

If r denotes the constant interest rate, then the consumer's stock of assets evolves according to

$$w + rk_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, ..., T$, where c_t denotes consumption at each date $t = 0, 1, ..., T$. The consumer's utility function is

$$\sum_{t=0}^{T} \beta^t \ln(c_t),$$

where the discount factor lies between zero and one: $0 < \beta < 1$.

a) In this first version of the model, where borrowing is permitted, the consumer chooses sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ to maximize the utility function

$$\sum_{t=0}^{T} \beta^t \ln(c_t),$$

subject to the constraints

$$w + rk_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, ..., T$,

$$k_0 = 0,$$

and

$$k_{T+1} \geq k^*.$$

As a first step in using the maximum principle, define the Hamiltonian corresponding to this problem.

b) Now write down the first-order condition for c_t and the pair of difference equations for π_t and k_t that, according to the maximum principle, characterize the solution to the consumer's dynamic optimization problem.

c) Use your results from above to derive an expression for the optimal growth rate of consumption c_t/c_{t-1} in terms of the model's parameters.

d) Next, consider a second version of this model, in which the consumer is prevented from borrowing through the imposition of the additional constraints

$$k_{t+1} \geq 0$$

for all $t = 0, 1, ..., T$, which require the stock of assets to always be nonnegative. Note that since the constraint governing the evolution of k_t will always bind at the optimum, these borrowing constraints can conveniently be rewritten as

$$k_t + w + rk_t - c_t \geq 0$$

for all $t = 0, 1, ..., T$. Thus, in this second version of the model, the consumer chooses sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ to maximize the utility function

$$\sum_{t=0}^{T} \beta^t \ln(c_t),$$

subject to the constraints

$$w + rk_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, ..., T$,

$$k_t + w + rk_t - c_t \geq 0$$

for all $t = 0, 1, .., T$,

$$k_0 = 0,$$

and

$$k_{T+1} \geq k^*.$$

As a first step in using the maximum principle to analyze this second version of the model, define the Hamiltonian corresponding to the consumer's problem.

e) Now write down the first-order condition for c_t and the pair of difference equations for π_t and k_t that, according to the maximum principle, characterize the solution to the consumer's dynamic optimization problem.

f) Finally, use your results from above to derive a *lower bound* on the optimal growth rate of consumption c_t/c_{t-1} that depends on the model's parameters. What impact does the imposition of the borrowing constraint have on the optimal growth rate of consumption?

3 The Maximum Principle in Continuous Time: Optimal Monetary Policy

This problem asks you to use the maximum principle to characterize optimal monetary policy in an economy where there is Phillips-curve trade-off between inflation and unemployment.

More specifically, the Phillips curve in this economy is described by the equation

$$\dot{p}(t) = -k[u(t) - \bar{u}],$$

where $p(t)$ denotes the inflation rate, $u(t)$ denotes the unemployment rate, $k > 0$ measures the slope of the Phillips curve, and $\bar{u} > 0$ is the "natural rate of unemployment." Hence, this equation indicates that inflation falls when the unemployment rate is above the natural rate and inflation rises when the unemployment rate is below the natural rate.

The central bank in this economy attempts to maximize a social welfare function that penalizes deviations of the unemployment rate $u(t)$ from the natural rate $\bar{u}$ and deviations of the inflation rate $p(t)$ from some target rate $\bar{p} > 0$. More specifically, over the infinite horizon, the central bank attempts to maximize

$$-(1/2)\int_0^\infty e^{-\rho t}\{[u(t)-\bar{u}]^2 + [p(t)-\bar{p}]^2\}dt,$$

where $\rho > 0$ measures the discount rate. Note that the minus sign out in front of this objective function is essential to capture the idea that deviations of unemployment or inflation from the central bank's targets ought to be penalized instead of rewarded. Meanwhile, the 1/2 out in front of the objective function is there to make the algebra a little easier.

Thus, the central bank in this economy chooses continuously differentiable functions $u(t)$ and $p(t)$ for all $t \in [0, \infty)$ to maximize the objective function

$$-(1/2)\int_0^\infty e^{-\rho t}\{[u(t)-\bar{u}]^2 + [p(t)-\bar{p}]^2\}dt$$

subject to the constraints

$$p(0) \text{ given}$$

and

$$\dot{p}(t) = -k[u(t)-\bar{u}]$$

for all $t \in [0, \infty)$.

a) As a first step in applying the maximum principle, define the Hamiltonian for this problem. HINT: In this problem, where both positive and negative deviations of inflation $p(t)$ from the central bank's target $\bar{p}$ are penalized, it is necessary to view the constraint $\dot{p}(t) = -k[u(t)-\bar{u}]$ as an equality constraint. Nevertheless, if you define the Hamiltonian as if the constraint could be written as

$$-k[u(t)-\bar{u}] \geq \dot{p}(t),$$

you'll be led to the correct set of optimality conditions below.

b) Next, write down the first-order condition for $u(t)$ and the pair of differential equations for $\pi(t)$ and $p(t)$ that, according to the maximum principle, characterize the solution to the central bank's problem.

c) Your results from part (b) above form a system of three equations in the three unknowns $u(t)$, $p(t)$, and $\pi(t)$. The first two of these objects have straightforward economic interpretations: $u(t)$ measures the unemployment rate during period t and $p(t)$ measures the inflation rate during period t. But the third unknown, $\pi(t)$, was introduced only in defining the Hamiltonian and lacks a completely straightforward economic interpretation. With these ideas in mind, use your results from above to reduce the system of optimality conditions down to a pair of just two differential equations involving the two economic variables $u(t)$ and $p(t)$.

d) Now use your results from part (c) above to draw a phase diagram, with inflation p on the x-axis and unemployment u on the y-axis, that illustrates the qualitative properties of the solution to the central bank's dynamic optimization problem.

e) Finally, use the phase diagram to answer the following questions:

   i) What are the steady-state values of inflation and unemployment in this model?

   ii) Starting from an initial condition $p(0)$ for inflation that is *above* the steady state, is it optimal for the central bank to reduce inflation down to its steady-state value? If so, does the central bank accomplish this goal by raising unemployment above the natural rate or by pushing unemployment below its natural rate?

   iii) Starting from an initial condition $p(0)$ for inflation that is *below* the steady state, is it optimal for the central bank to increase inflation up to its steady-state value? If so, does the central bank accomplish this goal by raising unemployment above the natural rate or by pushing unemployment below its natural rate?

4 Dynamic Programming Under Certainty: Cake Eating

This question asks you to use dynamic programming to characterize the optimal strategy for eating a cake that spoils, or depreciates away, over time.

Time is discrete and the horizon is infinite, so that periods are indexed by $t = 0, 1, 2, ...$ Let k_t denote the fraction of the cake that remains uneaten at the beginning of each period $t = 0, 1, 2, ...$ and let c_t denote the fraction of the cake that is eaten during period t. Assume, too, that the fraction δ, $0 < \delta < 1$, of the remaining cake spoils or depreciates away during each period $t = 0, 1, 2, ...$ Then

$$k_0 = 1$$

(the consumer starts off with one full cake) and

$$(1 - \delta)k_t - c_t \geq k_{t+1}$$

for all $t = 0, 1, 2, ...$ Subject to these constraints, the consumer chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

with constant discount factor $0 < \beta < 1$.

a) Write down the Bellman equation for this problem.

b) Now guess that the value function takes the time-invariant form

$$v(k_t) = E + F \ln(k_t),$$

where E and F are unknown constants. Using this guess, derive the first-order condition for the control variable c_t and the envelope condition for the state variable k_t.

c) Use your results from part (b) above to show how the optimal choice of how much cake to eat during period t, c_t, depends on the amount of cake remaining at the beginning of period t, k_t, and on the parameters δ and β.

d) Finally, use your results from above to show how k_{t+1}, the amount of cake left at the beginning of period $t+1$, depends on k_t, the amount of cake left at the beginning of period t, as well as the parameters δ and β.

5 Stochastic Dynamic Programming: Optimal Growth

This problem asks you to use dynamic programming to solve a stochastic version of the optimal growth model, in which an explicit solution for the value function can be found.

Suppose that the social planner or representative consumer maximizes the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints k_0 given and

$$z_t k_t^{\alpha} \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, ...$, where c_t denotes consumption during period t, k_t denotes the capital stock at the beginning of period t, and z_t represents a shock to the productivity of capital. The value of z_t is known when c_t and k_{t+1} are chosen during period t, but the value of z_{t+1} is random and satisfies $E_t \ln(z_{t+1}) = 0$. The model's parameters α and β both lie between zero and one: $0 < \alpha < 1$ and $0 < \beta < 1$.

a) Write down the Bellman equation for this problem.

b) Now guess that the value function takes the form

$$v(k_t, z_t) = E + F \ln(k_t) + G \ln(z_t),$$

where E, F, and G are unknown constants. Using this guess, derive the first order condition for the control variable c_t and the envelope condition for the state variable k_t. When you do this, you may find it helpful to recall that, by assumption, $E_t \ln(z_{t+1}) = 0$. You might also find it helpful to notice that since k_{t+1} is known during period t,

$$E_t \ln(k_{t+1}) = E_t \ln(z_t k_t^{\alpha} - c_t) = \ln(z_t k_t^{\alpha} - c_t) = \ln(k_{t+1}).$$

c) Use your results from part (b) above to show how the optimal choice of c_t depends on the capital stock k_t and the productivity shock z_t at t as well as the model's parameters α and β.

d) Finally, use your results from above to show how the capital stock k_{t+1} at $t+1$ depends on the capital stock k_t and the productivity shock z_t at t as well as the model's parameters α and β.

Solutions to Final Exam

Economics 720: Mathematics for Economists

Fall 2003

1 Applying the Kuhn-Tucker Theorem: The Linear Expenditure System

a) The Lagragian for the consumer's problem is defined as

$$L(c_1, c_2, \lambda) = (c_1 - x_1)^\alpha (c_2 - x_2)^{1-\alpha} + \lambda(I - p_1 c_1 - p_2 c_2).$$

b) According to the Kuhn-Tucker theorem, if c_1^* and c_2^* are the values of c_1 and c_2 that solve the consumer's problem, then there exists a value λ^* of the Lagrange multiplier λ such that the following conditions are satisfied: the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*) = \alpha(c_1^* - x_1)^{\alpha-1}(c_2^* - x_2)^{1-\alpha} - \lambda^* p_1 = 0$$

for c_1^* and

$$L_2(c_1^*, c_2^*, \lambda^*) = (1-\alpha)(c_1^* - x_1)^\alpha (c_2^* - x_2)^{-\alpha} - \lambda^* p_2 = 0$$

for c_2^*, the constraint

$$L_3(c_1^*, c_2^*, \lambda^*) = I - p_1 c_1^* - p_2 c_2^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0.$$

c) Divide the first-order condition for c_1^* by the first-order condition for c_2^* to obtain

$$\frac{\alpha(c_1^* - x_1)^{\alpha-1}(c_2^* - x_2)^{1-\alpha}}{(1-\alpha)(c_1^* - x_1)^\alpha (c_2^* - x_2)^{-\alpha}} = \frac{p_1}{p_2},$$

which restates the familiar graphical condition that says that the optimizing consumer acts so as to equate the marginal rate of substitution between the two goods to the ratio of their prices.

d) To solve for expenditures on each good, start with the optimality condition from part (c) above, which can be simplified to read

$$\frac{\alpha(c_2^* - x_2)}{(1-\alpha)(c_1^* - x_1)} = \frac{p_1}{P_2},$$

and use to it obtain the expression

$$p_2 c_2^* = \left(\frac{1-\alpha}{\alpha}\right) p_1 c_1^* - \left(\frac{1-\alpha}{\alpha}\right) x_1 p_1 + x_2 p_2.$$

Next, substitute this expression into the budget constraint, which is binding at the optimum, to solve for optimal expenditures on good one:

$$p_1 c_1^* = \alpha I + (1-\alpha) x_1 p_1 - \alpha x_2 p_2.$$

And finally, substitute this solution for $p_1 c_1^*$ back into the previous expression for $p_2 c_2^*$ to solve for optimal expenditures on good two:

$$p_2 c_2^* = (1-\alpha) I - (1-\alpha) x_1 p_1 + \alpha x_2 p_2.$$

e) The solutions for the optimal expenditures reveal that:

i) Expenditures $p_1 c_1^*$ on good one rise when income I rises.

ii) Expenditures $p_1 c_1^*$ on good one rise when the price p_1 of good one rises.

iii) Expenditures $p_1 c_1^*$ on good one fall when the price p_2 of good two rises.

2 The Maximum Principle in Discrete Time: Life Cycle Saving and Borrowing Constraints

a) The Hamiltonian for the first problem, where borrowing is permitted, is given by

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(w + rk_t - c_t).$$

b) According to the maximum principle, the solution to the consumer's problem must satisfy the first-order condition

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

for all $t = 0, 1, ..., T$ and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1} r$$

for all $t = 1, 2, ..., T$ and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = w + rk_t - c_t$$

for all $t = 0, 1, ..., T$.

c) Rewrite the difference equation for π_t as

$$(1+r)\pi_{t+1} = \pi_t.$$

Next, consider the first-order condition for c_t, which implies that

$$\pi_{t+1} = \frac{\beta^t}{c_t}$$

and

$$\pi_t = \frac{\beta^{t-1}}{c_{t-1}}.$$

Combining these three results yields the expression for the optimal growth rate of consumption:

$$c_t/c_{t-1} = \beta(1+r).$$

d) The Hamiltonian for the second problem, where borrowing is not permitted, is given by

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(w + rk_t - c_t) \text{ subject to } k_t + w + rk_t - c_t \geq 0.$$

e) By the Kuhn-Tucker theorem,

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(w + rk_t - c_t) + \lambda_t(k_t + w + rk_t - c_t),$$

where λ_t denotes the Lagrange multiplier on the borrowing constraint for period t. Thus, according to the maximum principle, the solution to the consumer's problem must satisfy the first-order condition

$$\frac{\beta^t}{c_t} - \pi_{t+1} - \lambda_t = 0$$

for all $t = 0, 1, ...T$ and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}r - \lambda_t(1+r)$$

for all $t = 1, 2, ..., T$ and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = w + rk_t - c_t$$

for all $t = 0, 1, ..., T$.

f) Rewrite the difference equation for π_t as

$$(1+r)(\pi_{t+1} + \lambda_t) = \pi_t.$$

Next, consider the first-order condition for c_t, which implies that

$$\frac{\beta^t}{c_t} = \pi_{t+1} + \lambda_t$$

and

$$\frac{\beta^{t-1}}{c_{t-1}} = \pi_t + \lambda_{t-1} \geq \pi_t,$$

where the last inequality in the expression for β^{t-1}/c_{t-1} follows from the Kuhn-Tucker condition the requires λ_{t-1} to be nonnegative. Combining these results yields

$$(1+r)\left(\frac{\beta^t}{c_t}\right) \leq \frac{\beta^{t-1}}{c_{t-1}},$$

which places a lower bound on the optimal growth rate for consumption:

$$c_t/c_{t-1} \geq \beta(1+r).$$

Evidently, the imposition of the borrowing constraint tends to increase the optimal growth rate of consumption. This makes sense: since borrowing allows the consumer to choose a higher value of c_{t-1} in exchange for a lower value of c_t, borrowing reduces the growth rate of consumption. Equivalently, the imposition of the borrowing constraint tends to increase the growth rate of consumption.

3 The Maximum Principle in Continuous Time: Optimal Monetary Policy

a) The Hamiltonian for the central bank's problem is given by

$$H(p(t),\pi(t);t) = \max_{u(t)} -(1/2)e^{-\rho t}\{[u(t)-\bar{u}]^2 + [p(t)-\bar{p}]^2\} - \pi(t)k[u(t)-\bar{u}].$$

b) According to the maximum principle, the solution to the central bank's problem must satisfy the first-order condition

$$-e^{-\rho t}[u(t)-\bar{u}] - \pi(t)k = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_p(p(t),\pi(t);t) = e^{-\rho t}[p(t)-\bar{p}]$$

and

$$\dot{p}(t) = H_\pi(p(t),\pi(t);t) = -k[u(t)-\bar{u}]$$

for all $t \in [0,\infty)$.

c) Rewrite the first-order condition as

$$\pi(t) = -(1/k)e^{-\rho t}[u(t)-\bar{u}]$$

and differentiate both sides with respect to t to obtain

$$\dot{\pi}(t) = \rho(1/k)e^{-\rho t}[u(t)-\bar{u}] - (1/k)e^{-\rho t}\dot{u}(t).$$

Next, substitute this expression into the differential equation for $\pi(t)$ to obtain

$$\rho(1/k)e^{-\rho t}[u(t) - \bar{u}] - (1/k)e^{-\rho t}\dot{u}(t) = e^{-\rho t}[p(t) - \bar{p}]$$

or, more simply,

$$\dot{u}(t) = \rho[u(t) - \bar{u}] - k[p(t) - \bar{p}].$$

Combine this last equation with

$$\dot{p}(t) = -k[u(t) - \bar{u}]$$

to obtain a system of two differential equations involving $u(t)$ and $p(t)$.

d) First, consider the differential equation for $p(t)$, which implies that:

$\dot{p}(t) = 0$ if $u(t) = \bar{u}$
$\dot{p}(t) < 0$ if $u(t) > \bar{u}$
$\dot{p}(t) > 0$ if $u(t) < \bar{u}$

Next, consider the differential equation for $u(t)$, which implies that:

$\dot{u}(t) = 0$ if $u(t) = \bar{u} + (k/\rho)[p(t) - \bar{p}]$
$\dot{u}(t) > 0$ if $u(t) > \bar{u} + (k/\rho)[p(t) - \bar{p}]$
$\dot{u}(t) < 0$ if $u(t) < \bar{u} + (k/\rho)[p(t) - \bar{p}]$

These conditions can be illustrated on a phase diagram.

e) The phase diagram reveals that:

i) The steady-state rate of inflation is $\bar{p}$ and the steady-state rate of unemployment is $\bar{u}$.

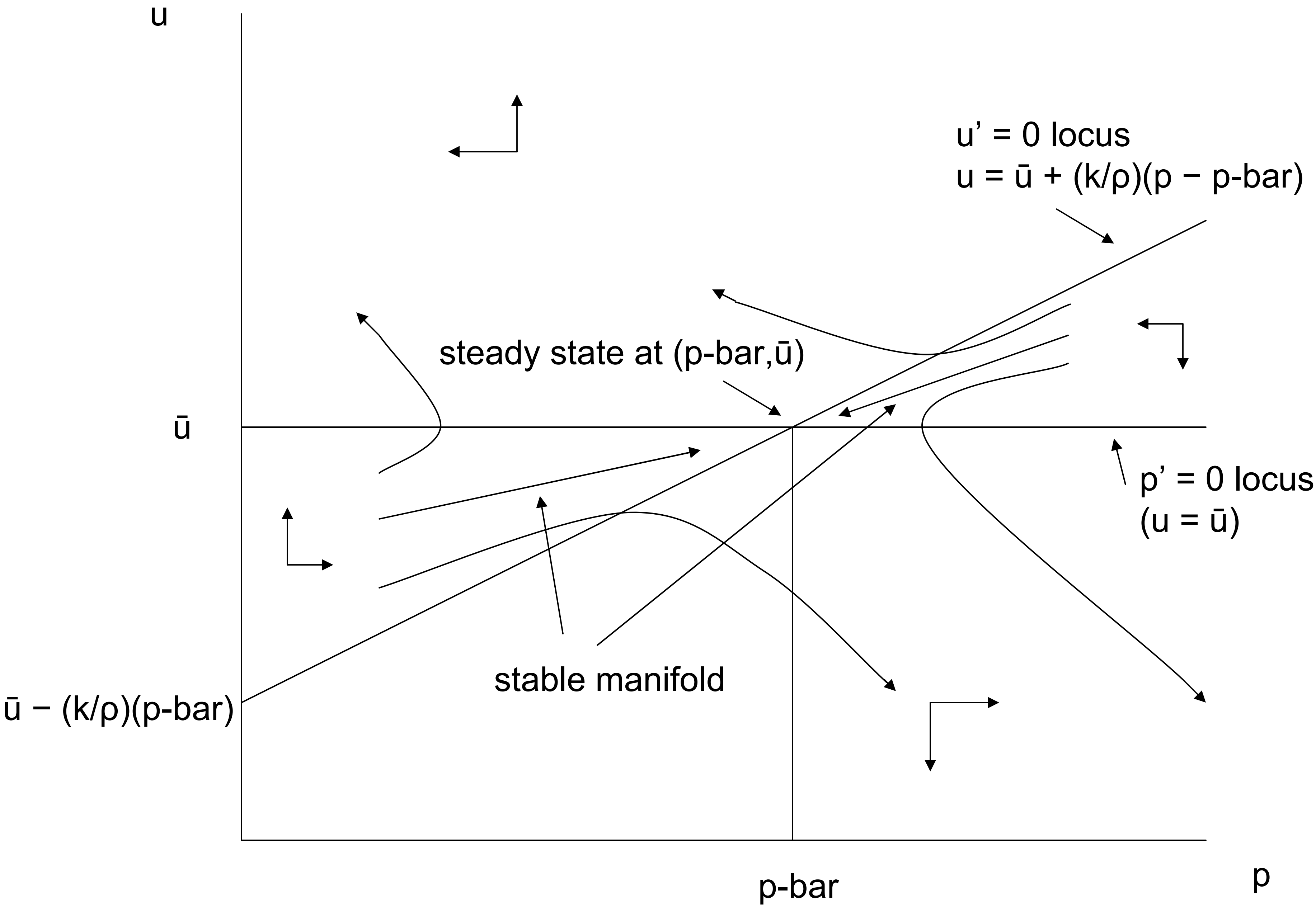
ii) Starting from an initial inflation rate $p(0)$ that is above the steady steady state, it is optimal for the central bank to lower inflation to $\bar{p}$ by choosing values for unemployment that are above the natural rate.

iii) Starting from an initial inflation rate $p(0)$ that is below the steady state, it is optimal for the central bank to raise inflation to $\bar{p}$ by choosing values for unemployment that are below the natural rate.

4 Dynamic Programming Under Certainty: Cake Eating

a) The Bellman equation for the consumer's problem is

$$v(k_t; t) = \max_{c_t} \ln(c_t) + \beta v[(1-\delta)k_t - c_t; t+1].$$


u
u’ = 0 locus
u = ū + (k/ρ)(p − p-bar)
steady state at (p-bar,ū)
ū
p’ = 0 locus
(u = ū)
stable manifold
ū − (k/ρ)(p-bar)
p-bar
p

b) Using the conjectured form for the value function, the Bellman equation becomes

$$E + F\ln(k_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F \ln[(1-\delta)k_t - c_t].$$

The first-order condition for c_t is

$$\frac{1}{c_t} - \frac{\beta F}{(1-\delta)k_t - c_t} = 0,$$

and the envelope condition for k_t is

$$\frac{F}{k_t} = \frac{\beta(1-\delta)F}{(1-\delta)k_t - c_t}.$$

c) The first-order condition for c_t implies that

$$(1-\delta)k_t - c_t = \beta F c_t$$

or

$$c_t = \left(\frac{1-\delta}{1+\beta F}\right) k_t.$$

Substitute this result into the envelope condition for k_t to obtain

$$F(1-\delta)k_t - F\left(\frac{1-\delta}{1+\beta F}\right) k_t = \beta(1-\delta)F k_t$$

or, more simply,

$$\frac{1}{1+\beta F} = 1 - \beta.$$

And substitute this result back into the previous expression for c_t to obtain

$$c_t = (1-\delta)(1-\beta)k_t,$$

which shows how the optimal choice of c_t depends on k_t, δ, and β.

d) Substitute the solution for c_t into the binding constraint

$$k_{t+1} = (1-\delta)k_t - c_t$$

to obtain

$$k_{t+1} = (1-\delta)\beta k_t,$$

which shows how the optimal choice of k_{t+1} depends on k_t, δ, and β. In fact, given the initial condition $k_0 = 1$, the optimal sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ can be constructed using

$$c_t = (1-\delta)(1-\beta)k_t$$

and

$$k_{t+1} = (1-\delta)\beta k_t$$

for all $t = 0, 1, 2, \ldots$.

5 Stochastic Dynamic Programming: Optimal Growth

a) The Bellman equation for the social planner's problem is

$$v(k_t, z_t) = \max_{c_t} \ln(c_t) + \beta E_t v(z_t k_t^\alpha - c_t, z_{t+1}).$$

b) Using the conjectured form for the value function and the facts that $E_t \ln(k_{t+1}) = \ln(k_{t+1})$ and $E_t \ln(z_{t+1}) = 0$, the Bellman equation becomes

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(z_t k_t^\alpha - c_t).$$

The first-order condition for c_t is

$$\frac{1}{c_t} - \frac{\beta F}{z_t k_t^\alpha - c_t} = 0,$$

and the envelope condition for k_t is

$$\frac{F}{k_t} = \frac{\alpha\beta F z_t k_t^{\alpha-1}}{z_t k_t^\alpha - c_t}.$$

c) The first-order condition implies that

$$c_t = \left(\frac{1}{1+\beta F}\right) z_t k_t^\alpha.$$

Substitute this expression into the envelope condition to obtain

$$\frac{1}{1+\beta F} = 1 - \alpha\beta.$$

And substitute this result back into the previous expression for c_t to obtain

$$c_t = (1-\alpha\beta) z_t k_t^\alpha,$$

which reveals that it is optimal for consumption c_t to be a fixed fraction $1-\alpha\beta$ of output $z_t k_t^\alpha$.

d) Substitute the solution for c_t into the binding constraint to obtain

$$k_{t+1} = \alpha\beta z_t k_t^\alpha,$$

which shows how the optimal choice for k_{t+1} depends on k_t, z_t, α, and β.

Final Exam

Economics 720: Mathematics for Economists

Fall 2005

This exam has six questions on seven pages; before you begin, please check to make sure that your copy has all six questions and all seven pages. The six questions will receive equal weight in determining your overall exam score. Since this is a two-hour exam, I suggest that you spend approximately (120 minutes)/(6 questions) = 20 minutes on each question.

1 Quasi-Linear Preferences

This question asks you to apply the Kuhn-Tucker theorem to characterize the solution to a consumer's maximization problem where the utility function takes the special *quasi-linear* form. Suppose, in particular, that there are two goods, x and y, which sell at prices p and q. The consumer seeks to maximize his or her utility, given by

$$U(x, y) = \ln(x) + y$$

subject to the budget constraint

$$I \geq px + qy,$$

where I denotes the consumer's income. Also, the consumer's choices of x and y must be nonnegative, as dictated by the nonnegativity constraints

$$x \geq 0$$

and

$$y \geq 0.$$

a) Define the Lagrangian for this consumer's problem: choose x and y to maximize utility, subject to the budget and nonnegativity constraints. As we discussed in class, there are a number of different ways to incorporate nonnegativity constraints into the definition of the Lagrangian, so you can choose whichever way is easiest for you: all will eventually lead you to the same results provided you write the associated optimality conditions in the appropriate way in part (b) below.

b) Let x^* and y^* denote the values of x and y that solve this problem. Write down the full set of first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by x^*, y^*, and the associated values of the Lagrange multipliers.

c) In order to more sharply characterize the solution to the consumer's problem, it is helpful to make the following observations. First, since the utility function is strictly increasing in both x and y, the budget constraint $I \geq px + qy$ will always bind at the optimum. Second, the logarithmic form for utility from good x implies that the marginal utility of consuming this good becomes infinite as the quantity consumed approaches zero. Hence, the consumer will always choose a positive amount of good x and the constraint $x \geq 0$ will not bind at the optimum. Third, the linear form for utility from good y implies that the marginal utility of consuming this good is constant and finite. It is possible, therefore, for the constraint $y \geq 0$ to bind at the optimum at least for certain values of the parameters I, p, and q. In light of these observations, use your results from part (b) above to answer: under what conditions on the parameters I, p, and q will the consumer find it optimal to choose $y^* = 0$? What is the optimal value for x^*, expressed in terms of the parameters I, p, and q, in this first case?

d) Similarly, use your results from part (b) to answer: under what conditions on the parameters I, p, and q will the consumer find it optimal to choose $y^* > 0$? What are the optimal values for x^* and y^*, expressed again in terms of the parameters I, p, and q, in this second case?

e) Finally, use your results from parts (c) and (d) above to answer: which of the two goods, x or y, might more appropriately be called a "luxury good," in the sense that it is purchased only by higher-income consumers?

2 Investment with Adjustment Costs

This question asks you to use the maximum principle to characterize the optimal investment strategy of a firm that faces a quadratic cost of adjusting its capital stock. Suppose, in particular, that time is continuous and the horizon infinite, so that periods can be indexed by $t \in [0, \infty)$. During each period t, the firm produces output $Y(t)$ with capital $K(t)$ according to the production function

$$Y(t) = K(t)^{\alpha},$$

where $1 > \alpha > 0$. By investing $I(t)$ units of output at t, the firm increases its capital stock according to

$$I(t) \geq \dot{K}(t),$$

but also incurs a quadratic adjustment cost given by

$$(\phi/2)[I(t)]^2,$$

where $\phi > 0$. In this problem, the firm's choice of $I(t)$ can be positive, in which case the firm is installing new capital, or negative, in which case the firm is selling off existing capital; but either way, the formulation implies that it will incur adjustment costs in making these changes. The firm sells off whatever output remains after its investment choice is made; its profits during period t are therefore

$$K(t)^{\alpha} - I(t) - (\phi/2)[I(t)]^2.$$

Finally, let $r > 0$ denote the constant discount rate. The firm's problem can now be stated as: choose continuously differentiable functions $I(t)$ and $K(t)$ for $t \in [0, \infty)$ to maximize the discounted value of profits,

$$\int_0^\infty e^{-rt}\{K(t)^\alpha - I(t) - (\phi/2)[I(t)]^2\},$$

subject to the capital accumulation constraint $I(t) \geq \dot{K}(t)$, which must hold for all $t \in [0, \infty)$, taking the initial capital stock $K(0) = K_0 > 0$ as given.

a) Define the Hamiltonian for the firm's problem. Again as we discussed in class, there are a number of ways to do this, but so long as you write down the corresponding optimality conditions in the appropriate way, you'll be led to the same solution below.

b) Write down the first order condition for the firm's optimal choice of $I(t)$ in each period $t \in [0, \infty)$, together with the pair of differential equations that according to the maximum principle help characterize the solution to the firm's problem.

c) Your answers from part (b) above form a system of three equations in three unknowns. Use one of these equations to eliminate one of the variables—the additional variable that you introduced into the problem when defining the Hamiltonian—to rewrite the system as one involving two equations in the two unknowns: investment $I(t)$ and the capital stock $K(t)$.

d) Use your results from part (c) above to calculate the steady-state values of investment and capital, that is, the constant values of $I(t) = I^*$ and $K(t) = K^*$ that prevail when $\dot{I}(t) = 0$ and $\dot{K}(t) = 0$.

e) Finally, use your results from above to draw a phase diagram that illustrates the following property of the solution to the firm's problem: starting from any value of the initial capital stock K_0, there is a unique value of $I(0)$ such that starting from $K(0) = K_0$ and $I(0)$, the firm's investment $I(t)$ and capital stock $K(t)$ converge to their steady-state values. In drawing this phase diagram, it may be helpful to note that while investment $I(t)$ may take on either positive or negative values, depending on whether the firm is accumulating new capital or selling off existing capital, the capital stock $K(t)$ must always remain positive.

3 Optimal Resource Depletion

This problem asks you to use the maximum principle to derive a system of differential equations that characterize the optimal consumption of an exhaustible resource and then to use the guess-and-verify method to solve that system of differential equations. Once again, let time be continuous and the horizon infinite, so that periods can be indexed by $t \in [0, \infty)$. Let $x(t)$ denote the stock of an exhaustible resource (like oil or coal) that remains available during period t, and let $c(t)$ denote the amount of this resource that is consumed during period t. Since no new units of the resource are ever created, the amount consumed simply subtracts from the available stock according to

$$-c(t) \geq \dot{x}(t)$$

for all $t \in [0, \infty)$. The optimization problem then involves choosing continuously differentiable functions $c(t)$ and $x(t)$ for $t \in [0, \infty)$ to maximize utility from consuming the resource over the infinite horizon, given by

$$\int_0^\infty e^{-\rho t} \ln(c(t))dt,$$

where $\rho > 0$ is the discount rate, subject to the constraint $-c(t) \geq \dot{x}(t)$ for all $t \in [0, \infty)$, taking as given the level of the initial resource stock $x(0) = x_0 > 0$.

a) Define the Hamiltonian for this problem, using $\pi(t)$ to denote the multiplier corresponding to the constraint $-c(t) \geq \dot{x}(t)$ for period t.

b) Next, write down the first-order condition for $c(t)$ and the pair of differential equations for $\dot{\pi}(t)$ and $\dot{x}(t)$ that, according to the maximum principle, help characterize the solution to this problem.

c) Probably the easiest way to solve the pair of differential equations that you derived in part (b) above is to use the guess-and-verify method. So guess that these differential equations have general solutions of the form

$$\pi(t) = \pi$$

and

$$x(t) = \left(\frac{1}{\pi\rho}\right) e^{-\rho t} + k,$$

where π and k are two constants that remain to be determined, then verify that these two functions do in fact satisfy the differential equations that you derived in part (b) above.

d) The maximum principle also gives us two boundary conditions that must be satisfied by the functions $\pi(t)$ and $x(t)$: the initial condition

$$x(0) = x_0 \text{ given}$$

and the terminal, or transversality, condition

$$\lim_{T\to\infty} \pi(T)x(T) = 0.$$

Use these boundary conditions to find specific values for the two unknown constants π and k that enter into the general solutions for $\pi(t)$ and $x(t)$ that you found in part (c) above.

e) Finally, combine your specific solutions to the differential equations and boundary conditions with the first-order condition for $c(t)$ to obtain an expression that shows how the optimal choice of $c(t)$ depends on the initial resource stock x_0 and the discount rate ρ.

4 Saving Under Certainty

This problem asks you to use dynamic programming to characterize the optimizing behavior of an infinitely-lived consumer who acts under conditions of perfect foresight or complete certainty. Let time be discrete; since the horizon is infinite, time periods are indexed by $t = 0, 1, 2, ...$ Let A_t denote this consumer's bank account balance at the beginning of each period $t = 0, 1, 2, ...$ The initial condition A_0 is given, but for $t = 0, 1, 2, ...$ the consumer can choose negative values for A_t, that is, borrowing is permitted. During each period the consumer receives a constant level of labor income w. Let c_t denote consumption and s_t denote savings; these variables are linked via the constraint

$$A_t + w \geq c_t + s_t$$

for all $t = 0, 1, 2, ...$ Savings earn interest at the constant rate r, so that

$$(1 + r)s_t \geq A_{t+1}$$

must also hold for all $t = 0, 1, 2, ...$ The consumer's utility function is

$$\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t u(A_t + w - s_t),$$

where the second expression for utility uses the fact that so long as u is strictly increasing, the constraint linking c_t and s_t will always bind (as will the constraint $(1 + r)s_t \geq A_{t+1}$). The consumer's problem can now be stated as: choose sequences $\{s_t\}_{t=0}^{\infty}$ and $\{A_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(A_t + w - s_t)$$

subject to the constraints A_0 given and $(1 + r)s_t \geq A_{t+1}$ for all $t = 0, 1, 2, ...$

a) Write down the Bellman equation for this problem.

b) Next, write down the first-order condition for the control variable and the envelope condition for the state variable.

c) A problem with your answers from part (b) above is that both make reference to an unknown function: the derivative of the value function that you introduced into the problem when setting up the Bellman equation. Combine your two equations from part (b) above to obtain a single optimality condition that only involves these objects with direct economic interpretations: consumption c_t and c_{t+1} during periods t and $t + 1$ and the parameters β and r. In doing this, you may find it helpful to also use the binding constraints $A_t + w = c_t + s_t$ and $(1 + r)s_t = A_{t+1}$.

d) Suppose that the consumer's single-period utility function takes the logarithmic form,

$$u(c_t) = \ln(c_t).$$

Under this additional assumption, use your result from part (c) above to obtain an expression for the optimal growth rate of consumption, c_{t+1}/c_t, in terms of the parameters β and r.

e) Finally, use your result from part (d) above to answer the following question: will a more patient consumer choose a faster or slower growth rate for consumption?

5 Optimal Stochastic Growth

This problem asks you to use dynamic programming to solve a stochastic version of the optimal growth model, for which an explicit solution for the value function can be found. In the model, time is discrete and the horizon is infinite, so that periods are indexed by $t = 0, 1, 2, \ldots.$ Suppose that the representative consumer chooses contingency plans for consumption c_t for $t = 0, 1, 2, \ldots$ and the capital stock k_t for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints k_0 given and

$$z_t k_t^\alpha \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, \ldots$, where $z_t k_t^\alpha$ measures output produced during period t, so that z_t represents a shock to the productivity of capital. The value of z_t is known when c_t and k_{t+1} are chosen during period t, but the value of z_{t+1} is random and satisfies $E_t \ln(z_{t+1}) = 0$. The model's parameters α and β both lie between zero and one: $0 < \alpha < 1$ and $0 < \beta < 1$.

a) Write down the Bellman equation for this problem.

b) Now guess that the value function takes the form

$$v(k_t, z_t) = E + F \ln(k_t) + G \ln(z_t),$$

where E, F, and G are unknown constants. Using this guess, derive the first order condition for the control variable c_t and the envelope condition for the state variable k_t. When you do this, you may find it helpful to recall that, by assumption, $E_t \ln(z_{t+1}) = 0$. You might also find it helpful to notice that since k_{t+1} is known during period t,

$$E_t \ln(k_{t+1}) = E_t \ln(z_t k_t^\alpha - c_t) = \ln(z_t k_t^\alpha - c_t) = \ln(k_{t+1}).$$

c) Use your results from part (b) above to show how the optimal choice of c_t depends on the capital stock k_t and the productivity shock z_t as well as the model's parameters α and β.

d) Use your results from above to show how the capital stock k_{t+1} at $t+1$ depends on the capital stock k_t and the productivity shock z_t as well as the model's parameters α and β.

e) Suppose that, just by chance, the productivity shocks z_0 and z_1 during periods $t = 0$ and $t = 1$ both turn out to equal one: $z_0 = 1$ and $z_1 = 1$. Under these additional assumptions, use your results from parts (c) and (d) above to derive expressions that show how the optimal consumption choices for periods $t = 0$ and $t = 1$, c_0 and c_1, depend on the initial capital stock k_0 and the model's parameters α and β.

6 Scalar Differential Equations

a) Write down the general solution to the scalar differential equation

$$\dot{y}(t) = -y(t).$$

b) Write down the unique solution to the initial value problem consisting of the same differential equation

$$\dot{y}(t) = -y(t)$$

from part (a) above and the initial condition

$$y(0) = y_0 \text{ given.}$$

c) Find the stationary solutions to the scalar differential equation

$$\dot{y}(t) = y(t)[1 - y(t)^2].$$

d) Now draw a phase diagram to illustrate the properties of all of the solutions to the same differential equation

$$\dot{y}(t) = y(t)[1 - y(t)^2]$$

from part (c) above.

e) Which stationary solutions to the differential equation

$$\dot{y}(t) = y(t)[1 - y(t)^2]$$

from parts (c) and (d) above are asymptotically stable? Which are unstable?

Solutions to Final Exam

Economics 720: Mathematics for Economists

Fall 2005

1 Quasi-Linear Preferences

The consumer's problem is to choose quantities of the two goods x and y to maximize

$$U(x, y) = \ln(x) + y$$

subject to the budget constraint

$$I \geq px + qy$$

and the nonnegativity constraints

$$x \geq 0$$

and

$$y \geq 0.$$

a) There are a number of ways to define the Lagrangian for this problem, depending on how one chooses to treat the nonnegativity constraints; here, I will work with the formulation that explicitly incorporates them into the Lagrangian, defined as

$$L(x, y, \lambda, \mu, \phi) = \ln(x) + y + \lambda(I - px - qy) + \mu x + \phi y.$$

b) Given the way the Lagrangian has been defined above, the Kuhn-Tucker theorem implies that the values x^* and y^* that solve the consumer's problem, together with the associated values λ^*, μ^*, and ϕ^* of the three multipliers, must satisfy the first-order conditions

$$L_1(x^*, y^*, \lambda^*, \mu^*, \phi^*) = 1/x^* - \lambda^* p + \mu^* = 0 \tag{1}$$

and

$$L_2(x^*, y^*, \lambda^*, \mu^*, \phi^*) = 1 - \lambda^* q + \phi^* = 0, \tag{2}$$

the constraints

$$L_3(x^*, y^*, \lambda^*, \mu^*, \phi^*) = I - px^* - qy^* \geq 0, \tag{3}$$

$$L_4(x^*, y^*, \lambda^*, \mu^*, \phi^*) = x^* \geq 0, \tag{4}$$

and

$$L_5(x^*, y^*, \lambda^*, \mu^*, \phi^*) = y^* \geq 0, \tag{5}$$

the nonnegativity conditions

$$\lambda^* \geq 0, \tag{6}$$

$$\mu^* \geq 0, \tag{7}$$

and

$$\phi^* \geq 0, \tag{8}$$

and the complementary slackness conditions

$$\lambda^*(I - px^* - qy^*) = 0, \tag{9}$$

$$\mu^* x^* = 0, \tag{10}$$

and

$$\phi^* y^* = 0. \tag{11}$$

c) The form of the utility function implies that the budget constraint will always bind. The form of the utility function also implies that $x^* > 0$ and hence, by (10), that $\mu^* = 0$ as well. Suppose first that $y^* = 0$. In this case, (1)-(3) and (8) require that

$$1/x^* - \lambda^* p = 0,$$

$$1 - \lambda^* q + \phi^* = 0,$$

$$I - px^* = 0,$$

and

$$\phi^* \geq 0$$

must hold. The binding budget constraint implies that the consumer spends all of his or her income on good x:

$$x^* = I/p.$$

The first-order condition for x^* then implies that

$$\lambda^* = p/x^* = 1/I.$$

The first-order condition for y^* and the nonnegativity condition for ϕ^* then require that

$$0 \leq \phi^* = \lambda^* q - 1 = q/I - 1$$

or more simply that

$$q \geq I.$$

Thus, whenever $q \geq I$ it is optimal for the consumer to choose $x^* = I/p$ and $y^* = 0$.

d) Suppose now that $y^* > 0$. In this case, (11) implies that $\phi^* = 0$, so that (1)-(3) require that

$$1/x^* - \lambda^* p = 0,$$

$$1 - \lambda^* q = 0,$$

and

$$I - px^* - qy^* = 0.$$

Note that in this case, the first-order condition for y^* immediately implies that

$$\lambda^* = 1/q.$$

Hence, the first-order condition for x^* implies that

$$x^* = q/p$$

and the budget constraint implies that

$$y^* = (I - q)/q,$$

which is strictly positive as required if

$$I > q.$$

Thus, whenever $I > q$ it is optimal for the consumer to choose $x^* = q/p$ and $y^* = (I - p)/q$.

e) The results from parts (c) and (d) above imply that only consumers with income I larger than q will purchase good y. In this sense, y is best regarded as the "luxury" good in this example.

2 Investment with Adjustment Costs

The firm's problem is to choose continuously differentiable functions $I(t)$ and $K(t)$ for $t \in [0, \infty)$ to maximize the discounted value of profits,

$$\int_0^\infty e^{-rt}\{K(t)^\alpha - I(t) - (\phi/2)[I(t)]^2\},$$

subject to the capital accumulation constraint $I(t) \geq \dot{K}(t)$, which must hold for all $t \in [0, \infty)$, taking the initial capital stock $K(0) = K_0 > 0$ as given.

a) The Hamiltonian for this problem can be written in present value form as

$$H(k(t), \pi(t); t) = \max_{I(t)} e^{-rt}\{K(t)^\alpha - I(t) - (\phi/2)[I(t)]^2\} + \pi(t)I(t).$$

b) According to the maximum principle, the firm's optimal choices must satisfy the first-order condition

$$-e^{-rt}[1 + \phi I(t)] + \pi(t) = 0 \tag{12}$$

as well as the pair of differential equations

$$\dot{\pi}(t) = -H_k(k(t), \pi(t); t) = -\alpha e^{-rt}K(t)^{\alpha-1} \tag{13}$$

and

$$\dot{K}(t) = H_\pi(k(t), \pi(t); t) = I(t) \tag{14}$$

for all $t \in [0, \infty)$.

c) Rewrite (12) as

$$\pi(t) = e^{-rt}[1 + \phi I(t)]$$

and differentiate both sides with respect to t to obtain

$$\dot{\pi}(t) = -re^{-rt}[1 + \phi I(t)] + \phi e^{-rt}\dot{I}(t).$$

Substitute this last result into (13) to obtain

$$\phi e^{-rt}\dot{I}(t) = re^{-rt}[1 + \phi I(t)] - \alpha e^{-rt}K(t)^{\alpha-1}$$

or more simply

$$\dot{I}(t) = (1/\phi)\{r[1 + \phi I(t)] - \alpha K(t)^{\alpha-1}\}. \tag{15}$$

Together, (14) and (15) form a system of two differential equations in the two unknowns $I(t)$ and $K(t)$.

d) Equations (14) and (15) implies that when $\dot{I}(t) = 0$ and $\dot{K}(t) = 0$, $I(t) = I^*$ and $K(t) = K^*$ where

$$I^* = 0$$

and

$$K^* = (r/\alpha)^{1/(\alpha-1)}.$$

e) Equations (14) and (15) also imply that

$$\dot{K}(t) = 0 \text{ whenever } I(t) = 0,$$

$$\dot{K}(t) > 0 \text{ whenever } I(t) > 0,$$

$$\dot{K}(t) < 0 \text{ whenever } I(t) < 0,$$

$$\dot{I}(t) = 0 \text{ whenever } I(t) = \left(\frac{\alpha}{\phi r}\right)K(t)^{\alpha-1} - \frac{1}{\phi},$$

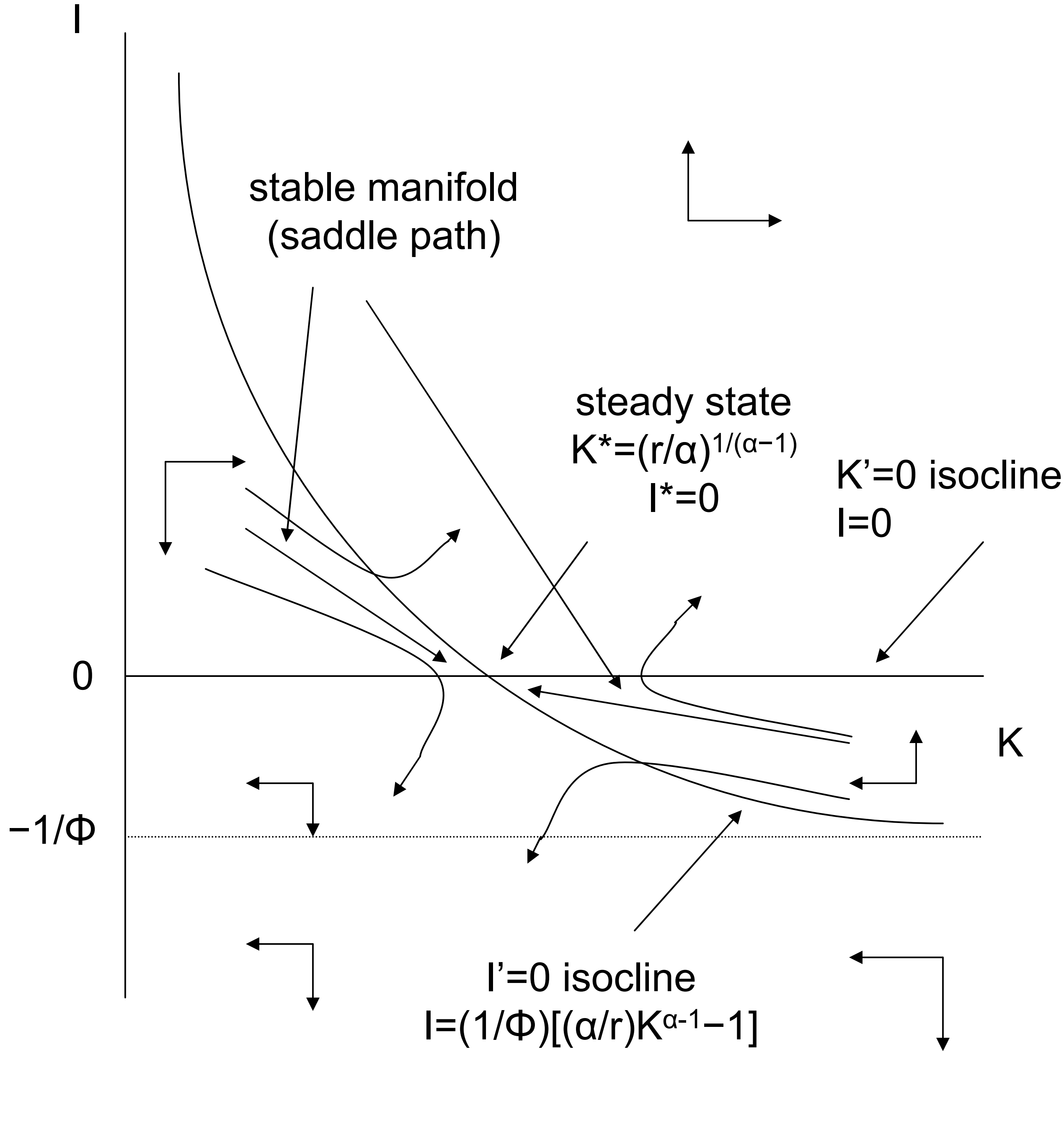
$$\dot{I}(t) = 0 \text{ whenever } I(t) > \left(\frac{\alpha}{\phi r}\right)K(t)^{\alpha-1} - \frac{1}{\phi},$$

and

$$\dot{I}(t) < 0 \text{ whenever } I(t) < \left(\frac{\alpha}{\phi r}\right)K(t)^{\alpha-1} - \frac{1}{\phi}.$$

Using these observations, it is possible to draw a phase diagram that illustrates that starting from any value of the initial capital stock K_0, there is a unique value of $I(0)$ such that starting from $K(0) = K_0$ and $I(0)$, the firm's investment $I(t)$ and capital stock $K(t)$ converge to their steady-state values.

For each value of K_0, there is a unique value of I_0 that leads the system to the steady state.



3 Optimal Resource Depletion

The problem is to choose continuously differentiable functions $c(t)$ and $x(t)$ for $t \in [0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t))dt$$

subject to the constraint $-c(t) \geq \dot{x}(t)$ for all $t \in [0, \infty)$, taking as given the level of the initial resource stock $x(0) = x_0 > 0$.

a) The Hamiltonian for this problem can be written as

$$H(x(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) - \pi(t)c(t).$$

b) According to the maximum principle, the optimal choices must satisfy the first-order condition

$$\frac{e^{-\rho t}}{c(t)} = \pi(t) \tag{16}$$

as well as the pair of differential equations

$$\dot{\pi}(t) = -H_x(x(t), \pi(t); t) = 0 \tag{17}$$

and

$$\dot{x}(t) = H_\pi(x(t), \pi(t); t) = -c(t) = -\frac{e^{-\rho t}}{\pi(t)} \tag{18}$$

for all $t \in [0, \infty)$.

c) Probably the easiest way to solve the pair of differential equations (17) and (18) is by the guess-and-verify method. Differentiating both sides of

$$\pi(t) = \pi$$

with respect to t yields

$$\dot{\pi}(t) = 0,$$

verifying that (17) is satisfied by this choice. Differentiating both sides of

$$x(t) = \left(\frac{1}{\pi\rho}\right) e^{-\rho t} + k$$

with respect to t yields

$$\dot{x}(t) = -\left(\frac{1}{\pi}\right) e^{-\rho t},$$

verifying that (18) is satisfied as well.

d) The maximum principle also states that the solution to the problem must satisfy the initial condition

$$x(0) = x_0 \text{ given}$$

and the terminal, or transversality, condition

$$\lim_{T\to\infty} \pi(T)x(T) = 0.$$

These boundary conditions pin down values for the unknown constants π and k that appear in the general solutions for $\pi(t)$ and $x(t)$. In particular, the initial condition requires that

$$x_0 = x(0) = \left(\frac{1}{\pi\rho}\right)e^{-\rho 0} + k = \frac{1}{\pi\rho} + k,$$

While the terminal condition requires that

$$0 = \lim_{T\to\infty} \pi(T)x(T) = \lim_{T\to\infty} \pi\left[\left(\frac{1}{\pi\rho}\right)e^{-\rho t} + k\right] = \lim_{T\to\infty}\left(\frac{1}{\rho}\right)e^{-\rho t} + \pi k = \pi k.$$

Rewrite the first of these two equations as

$$k = x_0 - \frac{1}{\pi\rho}$$

and substitute it into the second to obtain

$$0 = \pi x_0 - \frac{1}{\rho}.$$

Hence

$$\pi = \frac{1}{\rho x_0}$$

and

$$k = 0.$$

e) The results from parts (c) and (d) from above imply that

$$\pi(t) = \pi = \frac{1}{\rho x_0}$$

for all $t \in [0, \infty)$. Substitute this result into the first-order condition (16) to obtain the solution

$$c(t) = \frac{e^{-\rho t}}{\pi(t)} = \rho x_0 e^{-\rho t}. \tag{19}$$

Intuitively, (19) implies that consumption during period t is higher when the initial resource stock x_0 is larger and when the discount rate ρ is larger, that is, when the planner cares less about the future. In addition, consumption declines over time and converges gradually to zero as the resource stock is depleted over the infinite horizon.

4 Saving Under Certainty

The problem is to choose sequences $\{s_t\}_{t=0}^{\infty}$ and $\{A_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(A_t + w - s_t)$$

subject to the constraints A_0 given and $(1+r)s_t \geq A_{t+1}$for all $t = 0, 1, 2, ...$

a) The Bellman equation for this problem is

$$v(A_t; t) = \max_{s_t} u(A_t + w - s_t) + \beta v[(1+r)s_t; t+1].$$

b) The first order condition for the control variable s_t is

$$-u(A_t + w - s_t) + \beta(1+r)v'[(1+r)s_t; t+1] = 0. \tag{20}$$

The envelope condition for the state variable A_t is

$$v'(A_t; t) = u'(A_t + w - s_t). \tag{21}$$

c) Use the constraints to rewrite (20) and (21) more simply as

$$u'(c_t) = \beta(1+r)v'(A_{t+1}; t+1) \tag{22}$$

and

$$v'(A_t; t) = u'(c_t). \tag{23}$$

Since (23) must hold for all $t = 0, 1, 2, ...$, it also implies that

$$v'(A_{t+1}; t+1) = u'(c_{t+1}).$$

Substitute this last condition into (22) to obtain

$$u'(c_t) = \beta(1+r)u'(c_{t+1}), \tag{24}$$

a restatement of the optimality conditions (20) and (21) that makes reference only to objects with direct economic interpretations.

d) The additional assumption that utility is logarithmic implies that (24) can be specialized to

$$c_{t+1}/c_t = \beta(1+r). \tag{25}$$

e) A more patient consumer puts a higher weight β on future consumption relative to current consumption. Hence, according to (25), more patient consumers will choose faster growth rates for consumption.

5 Optimal Stochastic Growth

The problem is to choose contingency plans for c_t, $t = 0, 1, 2, ...$, and k_t, $t = 1, 2, 3, ...$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints k_0 given and $z_t k_t^\alpha \geq c_t + k_{t+1}$ for all $t = 0, 1, 2, ...$

a) The Bellman equation for this problem is

$$v(k_t, z_t) = \max_{c_t} \ln(c_t) + \beta E_t v(z_t k_t^\alpha - c_t, z_{t+1}).$$

b) With the suggested guess, the Bellman equation becomes

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(z_t k_t^\alpha - c_t),$$

after also using the results that, by assumption, $E_t \ln(z_{t+1}) = 0$ and $E_t \ln(z_t k_t^\alpha - c_t) = \ln(z_t k_t^\alpha - c_t)$. The first-order condition for the control variable c_t is then

$$\frac{1}{c_t} - \frac{\beta F}{z_t k_t^\alpha - c_t} = 0, \tag{26}$$

and the envelope condition for the state variable k_t is

$$\frac{F}{k_t} = \frac{\alpha\beta F z_t k_t^{\alpha-1}}{z_t k_t^\alpha - c_t}. \tag{27}$$

c) The first-order condition (26) implies

$$c_t = \left(\frac{1}{1+\beta F}\right) z_t k_t^\alpha \tag{28}$$

and, in light of this result, the envelope condition (27) implies

$$\frac{1}{1+\beta F} = 1 - \alpha\beta. \tag{29}$$

Substituting (29) back into (28) yields the desired solution for c_t:

$$c_t = (1-\alpha\beta) z_t k_t^\alpha. \tag{30}$$

d) Finally, in light of (30), the binding constraint yields the desired solution for k_{t+1}:

$$k_{t+1} = z_t k_t^\alpha - c_t = \alpha\beta z_t k_t^\alpha \tag{31}$$

e) Given k_0, (30) implies that if $z_0 = 1$, then

$$c_0 = (1-\alpha\beta) k_0^\alpha.$$

Likewise, (31) implies that

$$k_1 = \alpha\beta k_0^\alpha.$$

Substituting this last result into (30) then implies that if $z_1 = 1$ as well, then

$$c_1 = (1-\alpha\beta)(\alpha\beta k_0^\alpha)^\alpha.$$

6 Scalar Differential Equations

a) The general solution to the scalar differential equation

$$\dot{y}(t) = -y(t)$$

can be found by the guess-and-verify method. Guess that the general solution is

$$y(t) = ke^{-t},$$

where $k \in \mathbf{R}$. To verify that this guess works, simply note that it implies

$$\dot{y}(t) = -ke^{-t} = -y(t)$$

as required.

b) To find the unique solution to the initial value problem consisting of the same differential equation

$$\dot{y}(t) = -y(t)$$

from part (a) above and the initial condition

$$y(0) = y_0 \text{ given,}$$

take the general solution $y(t) = ke^{-t}$ from above and impose the initial condition to pin down k:

$$y_0 = y(0) = ke^{-0} = k.$$

Evidently, the unique solution is

$$y(t) = y_0 e^{-t}.$$

c) The stationary solutions to the scalar differential equation

$$\dot{y}(t) = y(t)[1 - y(t)^2]$$

take the form

$$y(t) = c$$

for all t and hence

$$\dot{y}(t) = 0.$$

Hence, the stationary solutions can be found be solving for the values of c that satisfy

$$0 = c(1 - c^2),$$

namely, $c = -1$, $c = 0$, and $c = 1$. Evidently, there are three stationary solutions

$$y(t) = -1, \ y(t) = 0, \text{ and } y(t) = 1.$$

d) The analysis from part (c) above has already told us that the differential equation

$$\dot{y}(t) = y(t)[1 - y(t)^2]$$

implies that

$$\dot{y}(t) = 0 \text{ if } y(t) = -1,\ y(t) = 0, \text{ or } y(t) = 1.$$

Note that the equation also implies that

$$\dot{y}(t) > 0 \text{ if } y(t) > 0 \text{ and } 1 - y(t)^2 > 0, \text{ that is, if } 1 > y(t) > 0,$$

$$\dot{y}(t) > 0 \text{ if } y(t) < 0 \text{ and } 1 - y(t)^2 < 0, \text{ that is, if } -1 > y(t),$$

$$\dot{y}(t) < 0 \text{ if } y(t) > 0 \text{ and } 1 - y(t)^2 < 0, \text{ that is, if } y(t) > 1,$$

and

$$\dot{y}(t) < 0 \text{ if } y(t) < 0 \text{ and } 1 - y(t)^2 > 0, \text{ that is, if } -1 > y(t) > 0.$$

A phase diagram that illustrates these properties of the differential equation can be used to highlight the features of solutions starting from any initial condition $y(0) = y_0$.

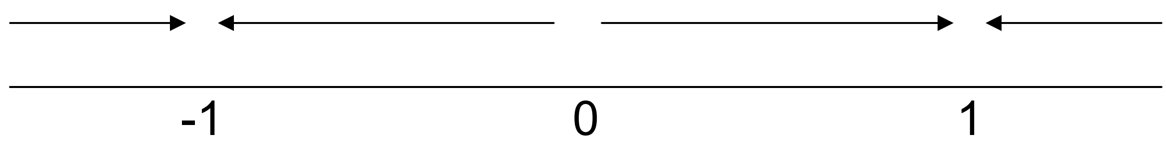
e) In addition, the phase diagram reveals that the stationary solutions

$$y(t) = -1 \text{ and } y(t) = 1$$

are asymptotically stable, while the stationary solution

$$y(t) = 0$$

is unstable.



$y'(t) = y(t)[1-y(t)^2]$

The phase diagram reveals that:

If $y_0 < -1$, y converges to -1.

If $y_0 = -1$, y remains at -1.

If $-1 < y_0 < 0$, y converges to -1.

If $y_0 = 0$, y remains at 0.

If $0 < y_0 < 1$, y converges to 1.

If $y_0 = 1$, y remains at 1.

If $1 < y_0$, y converges to 1.

Final Exam

Economics 720: Mathematics for Economists

Fall 2006

This exam has six questions on eight pages; before you begin, please make sure that your copy has all six questions and all eight pages. The six questions will receive equal weight in determining your overall exam score. Since this is a 110-minute exam, I suggest that you spend approximate (110 minutes)/(6 questions) $\approx$ 18 minutes on each question.

1 Expenditure Minimization

A consumer has preferences over two goods, as described by the Cobb-Douglas utility function

$$U(c_1, c_2) = c_1^a c_2^{1-a},$$

where c_1 denotes consumption of good 1, c_2 denotes consumption of good 2, and a is a parameter that lies between zero and one: $0 < a < 1$. Assume that the consumer can purchase as many or as few units of each good as he or she chooses in perfectly competitive markets at the strictly positive prices $p_1 > 0$ for good 1 and $p_2 > 0$ for good 2.

Now suppose that this consumer chooses $c_1 \geq 0$ and $c_2 \geq 0$ in an attempt to minimize the total expenditures $p_1c_1 + p_2c_2$ required to attain a level of utility that meets or exceeds some prespecified level $\bar{U}$. That is, suppose that the consumer solves

$$\min_{c_1,c_2} p_1c_1 + p_2c_2 \text{ subject to } c_1^a c_2^{1-a} \geq \bar{U}.$$

a) Define (set up) the Lagrangian for this expenditure minimization problem. In doing this, it may be helpful for you to note that the form of the utility function implies that the nonnegativity constraints $c_1 \geq 0$ and $c_2 \geq 0$ will not bind at the optimum; hence, it is safe to ignore these constraints in setting up the Lagrangian and in working through the rest of the analysis below.

b) Let c_1^* and c_2^* denote the values of c_1 and c_2 that solve this problem. Write down the full set of first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by c_1^*, c_2^*, and the associated value of the Lagrange multiplier.

c) Next, use these Kuhn-Tucker conditions to find solutions for c_1^*, c_2^*, and the associated value of the Lagrange multiplier in terms of the model's four parameters: p_1, p_2, $\bar{U}$, and a. [*Note:* these solutions for c_1^* and c_2^* define the *Hicksian* or *compensated* demand functions for the two goods.]

d) Now define the *minimum expenditure* function $E(p_1, p_2, \bar{U})$ as the minimized value of the objective function, taken subject to the constraint; that is, define

$$E(p_1, p_2, \bar{U}) = \min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } c_1^a c_2^{1-a} \geq \bar{U}.$$

Use your results from above to obtain expressions for the derivatives of this minimum expenditure function, $E_1(p_1, p_2, \bar{U})$, $E_2(p_1, p_2, \bar{U})$, and $E_3(p_1, p_2, \bar{U})$, in terms of the parameters p_1, p_2, $\bar{U}$, and a.

e) What do your results from above tell you about the relationship between the Hicksian demand functions for the two consumption goods and the derivatives of the minimum expenditure function? What economic interpretation does your results provide of the Lagrange multiplier in this constrained minimization problem?

2 Utility Maximization

Consider the same consumer as in question 1 from above, with preferences over the two goods described by the Cobb-Douglas utility function

$$U(c_1, c_2) = c_1^a c_2^{1-a},$$

where c_1 denotes consumption of good 1, c_2 denotes consumption of good 2, and a is a parameter that lies between zero and one: $0 < a < 1$. Assume again that the consumer can purchase as many or as few units of each good as he or she chooses in perfectly competitive markets at the strictly positive prices $p_1 > 0$ for good 1 and $p_2 > 0$ for good 2.

Now, however, suppose that this consumer chooses $c_1 \geq 0$ and $c_2 \geq 0$ in an attempt to maximize his or her utility, subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2,$$

where I denotes the consumer's fixed level of income. That is, suppose that the consumer solves

$$\max_{c_1, c_2} c_1^a c_2^{1-a} \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

a) Define (set up) the Lagrangian for this utility maximization problem. In doing this, it may be helpful for you to note that the form of the utility function implies that the nonnegativity constraints $c_1 \geq 0$ and $c_2 \geq 0$ will not bind at the optimum; hence, it is safe to ignore these constraints in setting up the Lagrangian and in working through the rest of the analysis below.

b) Let c_1^* and c_2^* denote the values of c_1 and c_2 that solve this problem. Write down the full set of first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by c_1^*, c_2^*, and the associated value of the Lagrange multiplier.

c) Next, use these Kuhn-Tucker conditions to find solutions for c_1^*, c_2^*, and the associated value of the Lagrange multiplier in terms of the model's four parameters: p_1, p_2, I, and a. [*Note:* these solutions for c_1^* and c_2^* define the *Marshallian* or *Walrasian* demand functions for the two goods.]

d) Now define the *indirect utility* function $V(p_1, p_2, I)$ as the maximized value of the objective function, taken subject to the constraint; that is, define

$$V(p_1, p_2, I) = \max_{c_1, c_2} c_1^a c_2^{1-a} \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

Use your results from above to obtain expressions for the derivatives of this indirect utility function, $V_1(p_1, p_2, I)$, $V_2(p_1, p_2, I)$, and $V_3(p_1, p_2, I)$, in terms of the parameters p_1, p_2, I, and a.

e) What do your results from above tell you about the relationship between the Marshallian demand functions for the two consumption goods and the derivatives of the indirect utility function? What economic interpretation does your results provide of the Lagrange multiplier in this constrained maximization problem?

3 Fishing

Let $y(t)$ denote the number of fish in a lake and assume that in the absence of fishing, the fish population evolves according to the so-called *logistic* model

$$\dot{y}(t) = ay(t)[b - y(t)]$$

for all $t \in [0, \infty)$, where $a > 0$ and $b > 0$ are both positive parameters. Catching $c(t)$ fish yields a consumer utility

$$\int_0^\infty e^{-\rho t} \ln(c(t)) dt$$

over the infinite horizon, where $\rho > 0$ is a positive discount rate, but subtracts from the fish population so that

$$\dot{y}(t) = ay(t)[b - y(t)] - c(t)$$

for all $t \in [0, \infty)$. In working through the analysis below, it may be helpful for you to assume that the model's parameters also satisfy $ab > \rho$.

Hence, the consumer must choose continuously differentiable functions $c(t)$ for $t \in [0, \infty)$ and $y(t)$ for all $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t)) dt$$

subject to the constraints

$$ay(t)[b - y(t)] - c(t) \geq \dot{y}(t)$$

for all $t \in [0, \infty)$, taking the initial stock of fish $y(0) > 0$ as given.

a) As a first step in using the maximum principle to characterize the solution to the consumer's problem, define (set up) the Hamiltonian for this problem.

b) Now write down the first-order condition and the pair of differential equations that, according to the maximum principle, are necessary conditions for the values of $c(t)$ and $y(t)$ that solve the consumer's problem.

c) Your answer to part (b) from above should take the form of a system of three equations in three unknowns. Use one of these equations to eliminate one of the variables—the additional variable that you introduced into the problem when defining the Hamiltonian—to rewrite the system as one involving two equations in two unknowns: consumption $c(t)$ and the fish population $y(t)$.

d) Use your results from part (c) above to find solutions for the optimal steady-state values c^* and y^* of consumption and the fish population size in terms of the model's parameters: a, b, and ρ.

e) Finally, use your results from above to draw a phase diagram that illustrates the following property of the solution to the consumer's problem: starting from any value $y(0) > 0$ for the initial fish population size, there is a unique value of $c(0)$ such that starting from $y(0)$ and $c(0)$, optimally-chosen consumption $c(t)$ and the population size $y(t)$ converge to their steady-state values c^* and y^*. In drawing this phase diagram, it may be helpful to note that although we did not specifically impose the nonnegativity constraints $c(t) \geq 0$ and $y(t) \geq 0$ for all $t \in [0, \infty)$ in setting up the consumer's dynamic optimization problem, the form of the utility function (which implies that the marginal utility of consumption becomes arbitrarily large as consumption approaches zero) and the law of motion for the fish population (which implies that no new fish are born when the fish population is fully depleted) imply that these constraints will never bind when the variables are chosen optimally.

4 Investment with Adjustment Costs

During each period $t \in [0, \infty)$, a firm produces output $Y(t)$ with capital $K(t)$ according to the production function

$$Y(t) = K(t)^\alpha,$$

where $0 < \alpha < 1$. Capital depreciates at the constant rate δ, where $0 < \delta < 1$; hence, by investing $I(t)$ units of output at t, the firm augments its capital stock according to

$$I(t) - \delta K(t) \geq \dot{K}(t).$$

but at the same time also incurs a quadratic adjustment cost given by

$$(\phi/2)[I(t)]^2,$$

where $\phi > 0$. In this problem, the firm's choice of $I(t)$ can be positive, in which case the firm is installing new capital, or negative, in which case the firm is selling off existing capital;

but either way, the formulation implies that it will incur adjustment costs in making these changes. The firm sells off whatever output remains after its investment choice is made; its profits during period t are therefore

$$K(t)^{\alpha} - I(t) - (\phi/2)[I(t)]^2.$$

Finally, let $r > 0$ denote the constant discount rate.

The firm's problem can now be stated as: choose continuously differentiable functions $I(t)$ for $t \in [0, \infty)$ and $K(t)$ for $t \in (0, \infty)$ to maximize the discounted value of profits,

$$\int_0^{\infty} e^{-rt}\{K(t)^{\alpha} - I(t) - (\phi/2)[I(t)]^2\}dt,$$

subject to the capital accumulation constraints

$$I(t) - \delta K(t) \geq \dot{K}(t)$$

for all $t \in [0, \infty)$, taking the initial capital stock $K(0) > 0$ as given.

a) As a first step in using the maximum principle to characterize the solution to the firm's problem, define (set up) the Hamiltonian for this problem.

b) Now write down the first-order condition and the pair of differential equations that, according to the maximum principle, are necessary conditions for the values of $I(t)$ and $K(t)$ that solve the consumer's problem.

c) Your answer to part (b) from above should take the form of a system of three equations in three unknowns. Use one of these equations to eliminate one of the variables—the additional variable that you introduced into the problem when defining the Hamiltonian—to rewrite the system as one involving two equations in two unknowns: investment $I(t)$ and the capital stock $K(t)$.

d) Using your results from part (c) above, write down a set of two equations that determine the optimal steady-state values I^* and K^*. *NOTE:* In this case, it may be too difficult to actually solve these two equations for the steady-state values of I^* and K^*, so DON'T BOTHER TO DO THIS. Instead, just write down the two equations that determine I^* and K^*.

e) Finally, use your results from above to draw a phase diagram that illustrates the following property of the solution to the firm's problem: starting from any value of the initial capital stock $K(0) > 0$, there is a unique value of $I(0)$ such that starting from $K(0)$ and $I(0)$, optimally-chosen investment $I(t)$ and capital stock $K(t)$ converge to their steady-state values K^* and I^*. In drawing this phase diagram, it may be helpful to note that while investment $I(t)$ may take on either positive or negative values, depending on whether the firm is accumulating new capital or selling off existing capital, the capital stock $K(t)$ must always remain positive when the variables are chosen optimally.

5 Human Capital and Growth

For $t = 0, 1, 2, ...$, let h_t denote the level of skill, or stock of human capital, attained by a representative consumer who splits his or her time between two activities: studying to increase his or her stock of human capital or working to produce output. More specifically, by allocating the fraction u_t to his or her time to studying, the consumer adds to his or her stock of human capital according to

$$h_{t+1} = (1 + \gamma u_t)h_t,$$

where $\gamma > 0$ is a positive parameter. Note that this specification implies that when $u_t = 0$, so that no time is spent studying, $h_{t+1} = h_t$, so that human capital remains unchanged. When $u_t = 1$, so that all available time is spent studying, $h_{t+1} = (1 + \gamma)h_t$, so that human capital grows at the fastest possible rate γ. For values of u_t between zero and one, human capital grows at a rate that lies between zero and γ. By allocating the remaining fraction $1 - u_t$ of his or her time to working, the consumer produces y_t units of output according to the technology described by the production function

$$y_t = [(1 - u_t)h_t]^\alpha,$$

where α lies between zero and one: $0 < \alpha < 1$. This specification implies that more output can be obtained when either: (i) the consumer spends a larger fraction $1 - u_t$ of his or her time working or (ii) the consumer is more skilled, that is, has a larger stock of human capital h_t.

Let the consumer have preferences over consumption c_t at each date $t = 0, 1, 2, ...$, as described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where the discount factor β also lies between zero and one: $0 < \beta < 1$. If output cannot be stored, the optimizing consumer will always choose $c_t = y_t = [(1 - u_t)h_t]^\alpha$ or, taking logs, $\ln(c_t) = \alpha \ln(1 - u_t) + \alpha \ln(h_t)$. In working through the analysis that follows, it will be helpful to assume that the parameters γ and β are such that $\beta(1 + \gamma) > 1$; the results that you will derive below imply that when this condition is satisfied, it will always be optimal for the consumer to choose a value of u_t that satisfies $0 < u_t < 1$, so that in formulating the consumer's problem it will not be necessary to impose nonnegativity constraints on the times spent studying or working.

Hence, the consumer's problem can now be formulated as one of choosing sequences $\{u_t\}_{t=0}^\infty$ and $\{h_t\}_{t=1}^\infty$ to maximize

$$\sum_{t=0}^{\infty} \beta^t [\alpha \ln(1 - u_t) + \alpha \ln(h_t)]$$

subject to the constraints

$$(1 + \gamma u_t)h_t \geq h_{t+1}$$

for all $t = 0, 1, 2, ...$, taking the initial stock of human capital h_0 as given.

a) Write down the Bellman equation for the consumer's problem, using h_t as the state variable and u_t as the control variable.

b) Next, guess that the value function takes the time-invariant form

$$v(h_t) = E + F\ln(h_t),$$

where E and F are constants to be determined. Using this guess, derive the first-order and envelope conditions that characterize the optimal choices of u_t and h_t.

c) Use your results from part (b) above to obtain an expression that shows how the optimal choice of u_t depends on the model's parameters: γ, α, and β.

d) Use your results from above to obtain an expression that shows how the optimal growth rate of human capital h_{t+1}/h_t depends on the model's parameters: γ, α, and β.

e) Finally, use your results from above to obtain an expression that shows how the optimal growth rate of output and consumption $y_{t+1}/y_t = c_{t+1}/c_t$ depends on the model's parameters: γ, α, and β.

6 Monopolistic Price Adjustment

A monopolist faces the downward-sloping demand curve

$$d_t = p_t^{-\varepsilon}$$

in each period $t = 0, 1, 2, ...$, where d_t denotes demand for the firm's output, p_t denotes the price that the firm sets for each unit of output, and the parameter $\varepsilon > 1$ measures the absolute value of the constant price elasticity of demand. Suppose that the firm can produce each unit of output at the constant marginal cost $c > 0$. Suppose also that it faces the quadratic cost

$$\frac{\phi}{2}\left(\frac{p_t}{p_{t-1}} - 1\right)^2$$

of adjusting its price between periods; this cost of adjustment can either be interpreted literally as reflecting the administrative costs within the firm of changing its price or more broadly as reflecting the negative impact on the firm resulting from consumers' resistance to and dissatisfaction with rapid price changes. The firm's total profits during each period t then equal revenues from goods sold (price times quantity demanded $d_t p_t = p_t^{1-\varepsilon}$) minus costs of goods sold (marginal cost times quantity demanded $cd_t = cp_t^{-\varepsilon}$) minus costs of adjustment; if the firm discounts future profits at the constant rate $r > 0$, then the present value of profits over the infinite horizon is measured by

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left[p_t^{1-\varepsilon} - cp_t^{-\varepsilon} - \frac{\phi}{2}\left(\frac{p_t}{p_{t-1}} - 1\right)^2\right].$$

This model lacks what might clearly be identified as "stock" and "flow" variables. However, the firm's problem can be solved using dynamic programming methods by employing the following notational trick. For all $t = 0, 1, 2, \ldots$, let y_t denote "last period's price during period t." Then $y_t = p_{t-1}$, $y_{t+1} = p_t$, and so on. Now the firm can be viewed as solving the problem: maximize by choice of sequences $\{p_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ the objective function

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[p_t^{1-\varepsilon} - cp_t^{-\varepsilon} - \frac{\phi}{2}\left(\frac{p_t}{y_t} - 1\right)^2\right]$$

subject to the "binding constraints"

$$y_{t+1} = p_t$$

for all $t = 0, 1, 2, \ldots$, taking the initial condition $y_0 = p_{-1}$ as given.

a) Write down the Bellman equation for the firm's problem, using y_t as the state variable and p_t as the control variable. In doing this, you may find it helpful to substitute the "binding constraint" $y_{t+1} = p_t$ into the expression on the right-hand side of the Bellman equation, so that the optimization problem on the right-hand side of the Bellman equation becomes an unconstrained maximization problem.

b) Next, write down the first-order and envelope conditions that characterize the optimal choices of p_t and y_t.

c) Together with the "binding constraint" $y_{t+1} = p_t$, your answer from part (b) above should take the form of a system of three equations in three unknowns. Use one of these equations to eliminate one of the unknowns—the unknown value function that you introduced into the problem when setting up the Bellman equation—to rewrite the system as one involving two equations in two unknowns: p_t and y_t.

d) Now use the "binding constraints" $y_{t+1} = p_t$ and $y_t = p_{t-1}$ to rewrite your answer to part (c) from above in the form of a single equation in a single unknown: the firm's optimal price p_t.

e) Finally, let p^* denote the firm's optimal price in a steady state where $p_{t+1} = p_t = p_{t-1} = p^*$ for all $t = 0, 1, 2, \ldots$. Use your answer from part (d) above to solve for p^* in terms of the model's parameters: ε, c, ϕ, and r. How does the monopolist's steady-state "markup" of price over marginal cost, as measured by the ratio p^*/c, change as the absolute value of the elasticity of demand ε rises?

Solutions to Final Exam

Economics 720: Mathematics for Economists

Fall 2006

1 Expenditure Minimization

The consumer solves

$$\min_{c_1,c_2} p_1c_1 + p_2c_2 \text{ subject to } c_1^a c_2^{1-a} \geq \bar{U}.$$

a) There are a number of ways to define the Lagrangian for this expenditure minimization problem, but one way is to subtract the product of the multiplier λ and the constraint from the objective function to obtain

$$L(c_1, c_2, \lambda) = p_1c_1 + p_2c_2 - \lambda(c_1^a c_2^{1-a} - \bar{U}).$$

b) With the Lagrangian defined as above, the Kuhn-Tucker theorem states that if c_1^* and c_2^* solve the consumer's problem, then there exists a value λ^* of λ such that, together, c_1^*, c_2^*, and λ^* satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*) = p_1 - a\lambda^*(c_1^*)^{a-1}(c_2^*)^{1-a} = 0 \tag{1}$$

and

$$L_2(c_1^*, c_2^*, \lambda^*) = p_2 - (1-a)\lambda^*(c_1^*)^a(c_2^*)^{-a} = 0, \tag{2}$$

the constraint

$$(c_1^*)^a(c_2^*)^{1-a} - \bar{U} \geq 0, \tag{3}$$

the nonnegativity condition

$$\lambda^* \geq 0, \tag{4}$$

and the complementary slackness condition

$$\lambda^*[(c_1^*)^a(c_2^*)^{1-a} - \bar{U}] = 0. \tag{5}$$

c) Again, there are many ways of using these Kuhn-Tucker conditions to find solutions for c_1^*, c_2^*, and λ^* in terms of the parameters p_1, p_2, $\bar{U}$, and a, but one way starts by rearranging the first-order condition (1) to read

$$\lambda^* = \frac{p_1}{a(c_1^*)^{a-1}(c_2^*)^{1-a}}. \tag{6}$$

Since the objects in both the numerator and the denominator on the right-hand side of (6) are all strictly positive, the multiplier λ^* must also be strictly positive; the complementary slackness condition (5) then implies that the constraint (3) must bind at the optimum. Now use (6) to substitute out for λ^* in (2) and rearrange to obtain

$$c_2^* = \left(\frac{1-a}{a}\right)\left(\frac{p_1}{p_2}\right)c_1^*. \tag{7}$$

Substitute this last result into the binding constraint to obtain the desired solution for c_1^*:

$$c_1^* = \left(\frac{a}{1-a}\right)^{1-a}\left(\frac{p_2}{p_1}\right)^{1-a}\bar{U}. \tag{8}$$

Substitute this expression back into (7) to obtain the desired solution for c_2^*:

$$c_2^* = \left(\frac{1-a}{a}\right)^{a}\left(\frac{p_1}{p_2}\right)^{a}\bar{U}. \tag{9}$$

Finally, substitute (8) and (9) back into (6) to obtain the desired solution for λ^*:

$$\lambda^* = a^{-a}(1-a)^{-(1-a)}p_1^a p_2^{1-a}. \tag{10}$$

d) Now define the minimum expenditure function $E(p_1, p_2, \bar{U})$ as

$$E(p_1, p_2, \bar{U}) = \min_{c_1, c_2} p_1c_1 + p_2c_2 \text{ subject to } c_1^a c_2^{1-a} \geq \bar{U}.$$

Related, let (8)-(10) define the functions $c_1^*(p_1, p_2, \bar{U})$, $c_2^*(p_1, p_2, \bar{U})$, and $\lambda^*(p_1, p_2, \bar{U})$, describing how optimal consumptions and the associated value of the Lagrange multipliers change as the parameters p_1, p_2, and $\bar{U}$ vary. Then, in light of the complementary slackness condition (5), the minimum expenditure function can be evaluated as

$$\begin{aligned} E(p_1, p_2, \bar{U}) \;&=\; p_1c_1^*(p_1, p_2, \bar{U}) + p_2c_2^*(p_1, p_2, \bar{U}) \\ &\quad -\lambda^*(p_1, p_2, \bar{U})\left\{[c_1^*(p_1, p_2, \bar{U})]^a[c_2^*(p_1, p_2, \bar{U})]^{1-a} - \bar{U}\right\}. \end{aligned}$$

The envelope theorem implies that in differentiating both sides of this last expression, it is safe to ignore the dependence of c_1^*, c_2^*, and λ^* on p_1, p_2, and $\bar{U}$ and write

$$E_1(p_1, p_2, \bar{U}) = c_1^*(p_1, p_2, \bar{U}) = \left(\frac{a}{1-a}\right)^{1-a}\left(\frac{p_2}{p_1}\right)^{1-a}\bar{U}, \tag{11}$$

$$E_2(p_1, p_2, \bar{U}) = c_2^*(p_1, p_2, \bar{U}) = \left(\frac{1-a}{a}\right)^{a}\left(\frac{p_1}{p_2}\right)^{a}\bar{U}, \tag{12}$$

and

$$E_3(p_1, p_2, \bar{U}) = \lambda^*(p_1, p_2, \bar{U}) = a^{-a}(1-a)^{-(1-a)}p_1^a p_2^{1-a}. \tag{13}$$

e) Equations (11) and (12) indicate that the Hicksian demand function for each good coincides with the derivative of the minimum expenditure function with respect to the price of that good; these results resemble those given to us by Sheppard's lemma, which says that for the cost minimizing firm, the conditional factor demand functions coincide with the derivatives of the minimum cost with respect to the relevant factor prices. And since $E_3(p_1, p_2, \bar{U})$ measures the additional expenditure that the optimizing consumer would require to attain a slightly higher level of utility, (13) provides an economic interpretation of the Lagrange multiplier as the "marginal cost" of utility.

2 Utility Maximization

Now the consumer solves

$$\max_{c_1, c_2} c_1^a c_2^{1-a} \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

a) Once again there are a number of ways to define the Lagrangian for this utility maximization problem, but one way is to add the product of the multiplier λ and the constraint to the objective function to obtain

$$L(c_1, c_2, \lambda) = c_1^a c_2^{1-a} + \lambda(I - p_1 c_1 - p_2 c_2).$$

b) With the Lagrangian defined as above, the Kuhn-Tucker theorem states that if c_1^* and c_2^* solve the consumer's problem, then there exists a value λ^* of λ such that, together, c_1^*, c_2^*, and λ^* satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*) = a(c_1^*)^{a-1}(c_2^*)^{1-a} - \lambda^* p_1 = 0 \tag{14}$$

and

$$L_2(c_1^*, c_2^*, \lambda^*) = (1-a)(c_1^*)^a (c_2^*)^{-a} - \lambda^* p_2 = 0, \tag{15}$$

the constraint

$$I - p_1 c_1^* - p_2 c_2^* \geq 0, \tag{16}$$

the nonnegativity condition

$$\lambda^* \geq 0, \tag{17}$$

and the complementary slackness condition

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0. \tag{18}$$

c) Again, there are many ways of using these Kuhn-Tucker conditions to find solutions for c_1^*, c_2^*, and λ^* in terms of the parameters p_1, p_2, I, and a, but one way starts by rearranging the first-order condition (14) to read

$$\lambda^* = \frac{a(c_1^*)^{a-1}(c_2^*)^{1-a}}{p_1}. \tag{19}$$

Since the objects in both the numerator and the denominator on the right-hand side of (19) are all strictly positive, the multiplier λ^* must also be strictly positive; the complementary slackness condition (18) then implies that the constraint (16) must bind at the optimum. Now use (19) to substitute out for λ^* in (15) and rearrange to obtain

$$c_2^* = \left(\frac{1-a}{a}\right)\left(\frac{p_1}{p_2}\right)c_1^*. \tag{20}$$

Substitute this last result into the binding constraint to obtain the desired solution for c_1^*:

$$c_1^* = \frac{aI}{p_1}. \tag{21}$$

Substitute this expression back into (20) to obtain the desired solution for c_2^*:

$$c_2^* = \frac{(1-a)I}{p_2}. \tag{22}$$

Finally, substitute (21) and (22) back into (6) to obtain the desired solution for λ^*:

$$\lambda^* = a^a(1-a)^{1-a}p_1^{-a}p_2^{-(1-a)}. \tag{23}$$

d) Now define the indirect utility function $V(p_1, p_2, I)$ as

$$V(p_1, p_2, I) = \max_{c_1,c_2} c_1^a c_2^{1-a} \text{ subject to } I \geq p_1c_1 + p_2c_2.$$

Related, let (21)-(23) define the functions $c_1^*(p_1, p_2, I)$, $c_2^*(p_1, p_2, I)$, and $\lambda^*(p_1, p_2, I)$, describing how optimal consumptions and the associated value of the Lagrange multipliers change as the parameters p_1, p_2, and I vary. Then, in light of the complementary slackness condition (18), the indirect utility function can be evaluated as

$$\begin{aligned} V(p_1, p_2, I) &= [c_1^*(p_1, p_2, I)]^a[c_2^*(p_1, p_2, I)]^{1-a} \\ &\quad +\lambda^*(p_1, p_2, I)[I - p_1c_1^*(p_1, p_2, I) - p_2c_2^*(p_1, p_2, I)]. \end{aligned}$$

The envelope theorem implies that in differentiating both sides of this last expression, it is safe to ignore the dependence of c_1^*, c_2^*, and λ^* on p_1, p_2, and I and write

$$\begin{aligned} V_1(p_1, p_2, I) &= -\lambda^*(p_1, p_2, I)c_1^*(p_1, p_2, I) \\ &= -a^a(1-a)^{1-a}p_1^{-a}p_2^{-(1-a)}(aI/p_1) \\ &= -a^{1+a}(1-a)^{1-a}p_1^{-(1+a)}p_2^{-(1-a)}I, \end{aligned} \tag{24}$$

$$\begin{aligned} V_2(p_1, p_2, I) &= -\lambda^*(p_1, p_2, I)c_2^*(p_1, p_2, I) \\ &= -a^a(1-a)^{1-a}p_1^{-a}p_2^{-(1-a)}[(1-a)I/p_2] \\ &= -a^a(1-a)^{2-a}p_1^{-a}p_2^{-(2-a)}I, \end{aligned} \tag{25}$$

and

$$V_3(p_1, p_2, I) = \lambda^*(p_1, p_2, I) = a^a(1-a)^{1-a}p_1^{-a}p_2^{-(1-a)}. \tag{26}$$

e) Equations (24)-(26) indicate that the Marshallian demand functions and the derivatives of the indirect utility function are related via

$$c_1^*(p_1, p_2, I) = -\frac{V_1(p_1, p_2, I)}{V_3(p_1, p_2, I)}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{V_2(p_1, p_2, I)}{V_3(p_1, p_2, I)}.$$

These last two expressions are statements of *Roy's identity*, which like Shephard's lemma turns out to be a special case or application of the envelope theorem. And since $V_3(p_1, p_2, I)$ measures the additional utility the optimizing consumer can attain if provided with a slightly higher level of income, (26) also provides an economic interpretation of the Lagrange multiplier as the "marginal utility" of income.

3 Fishing

The consumer must choose continuously differentiable functions $c(t)$ for $t \in [0, \infty)$ and $y(t)$ for all $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t))dt$$

subject to the constraints

$$ay(t)[b - y(t)] - c(t) \geq \dot{y}(t)$$

for all $t \in [0, \infty)$, taking the initial stock of fish $y(0) > 0$ as given.

a) The Hamiltonian for this problem can be defined in present value terms as

$$H(y(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)\{ay(t)[b - y(t)] - c(t)\}.$$

b) According to the maximum principle, the values of $c(t)$ and $y(t)$ that solve the consumer's problem and the associated value of $\pi(t)$ must satisfy the first-order condition

$$\frac{e^{-\rho t}}{c(t)} - \pi(t) = 0 \tag{27}$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_y(y(t), \pi(t); t) = -\pi(t)[ab - 2ay(t)] \tag{28}$$

and

$$\dot{y}(t) = H_\pi(y(t), \pi(t); t) = ay(t)[b - y(t)] - c(t) \tag{29}$$

for all $t \in [0, \infty)$.

c) Rewrite (27) as

$$e^{-\rho t} = \pi(t)c(t)$$

and differentiate both sides with respect to t to obtain

$$-\rho e^{-\rho t} = \dot{\pi}(t)c(t) + \pi(t)\dot{c}(t).$$

Now use (27) and (28) to rewrite this last result as

$$-\rho\pi(t)c(t) = -\pi(t)[ab - 2ay(t)]c(t) + \pi(t)\dot{c}(t)$$

or, after dividing through by $\pi(t)$ and isolating $\dot{c}(t)$ on the left-hand side,

$$\dot{c}(t) = c(t)[ab - 2ay(t) - \rho]. \tag{30}$$

Together, (29) and (30) form a system of two equations in two unknowns: consumption $c(t)$ and the fish population $y(t)$.

d) In a steady state where $c(t) = c^*$ and $y(t) = y^*$, $\dot{c}(t) = 0$ and $\dot{y}(t) = 0$. Substituting these steady-state conditions into (29) and (30) yields

$$0 = ay^*(b - y^*) - c^*$$

and

$$0 = c^*(ab - 2ay^* - \rho).$$

Use the second of these two conditions to solve for

$$y^* = \frac{ab - \rho}{2a}.$$

Then substitute this solution into the first condition to find

$$c^* = a\left(\frac{ab - \rho}{2a}\right)\left(b - \frac{ab - \rho}{2a}\right) = \frac{a^2b^2 - \rho^2}{4a}.$$

e) Consider first (30), which implies that

$$\dot{c}(t) = 0 \text{ when } y(t) = y^*,$$

$$\dot{c}(t) < 0 \text{ when } y(t) > y^*,$$

and

$$\dot{c}(t) > 0 \text{ when } y(t) < y^*.$$

Consider next (29), which implies that

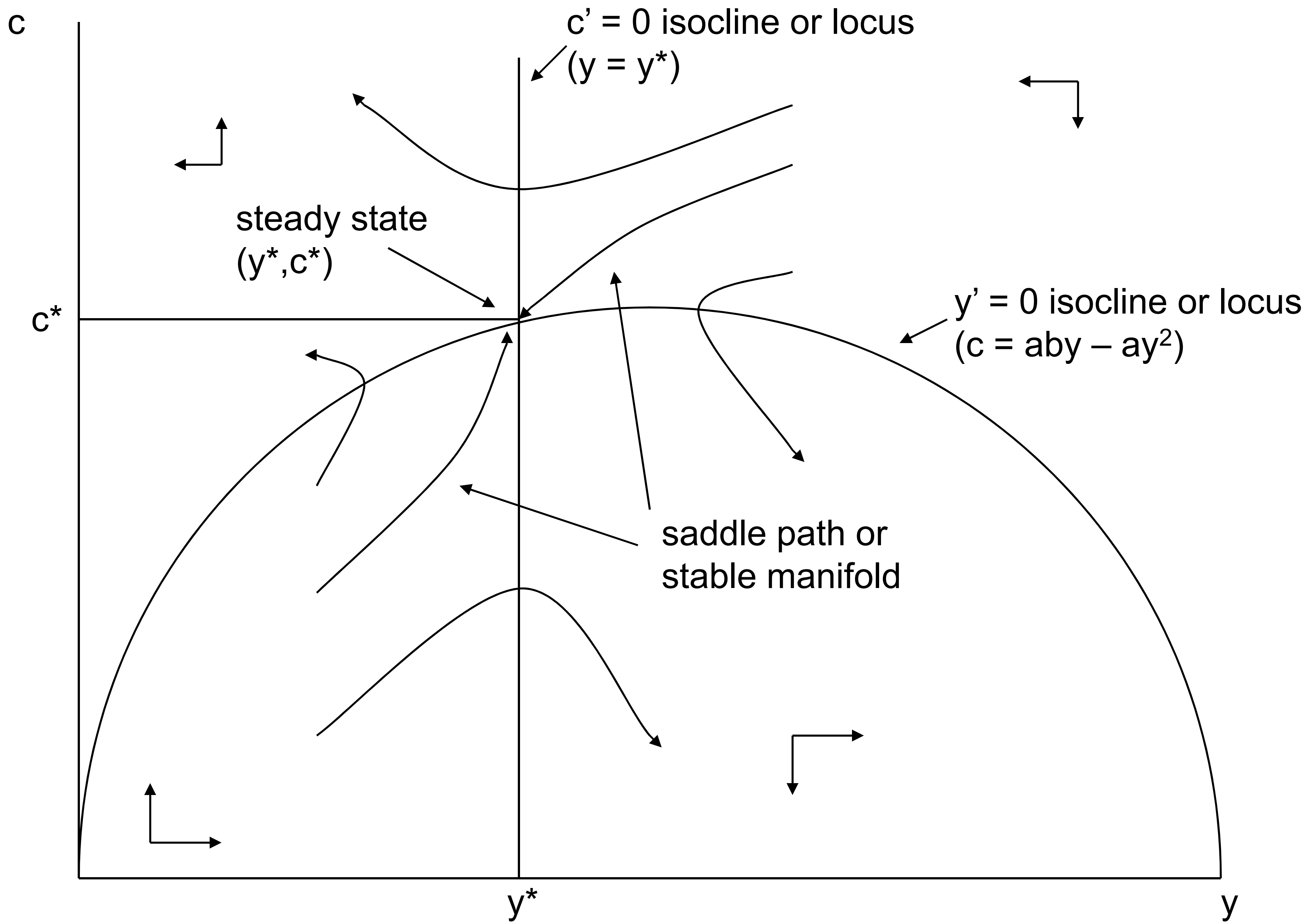
$$\dot{y}(t) = 0 \text{ when } c(t) = aby(t) - ay(t)^2,$$

$$\dot{y}(t) < 0 \text{ when } c(t) > aby(t) - ay(t)^2,$$

and

$$\dot{y}(t) > 0 \text{ when } c(t) < aby(t) - ay(t)^2.$$

The phase diagram that reflects these implications of (29) and (30) also reveals that starting from any value $y(0) > 0$ for the initial fish population size, there is a unique value of $c(0)$ such that starting from $y(0)$ and $c(0)$, optimally-chosen consumption $c(t)$ and the population size $y(t)$ converge to their steady-state values c^* and y^*.



c
c' = 0 isocline or locus
(y = y*)
steady state
(y*,c*)
c*
y' = 0 isocline or locus
(c = aby – ay²)
saddle path or
stable manifold
y*
y

4 Investment with Adjustment Costs

The firm chooses continuously differentiable functions $I(t)$ for $t \in [0,\infty)$ and $K(t)$ for $t \in (0,\infty)$ to maximize the discounted value of profits,

$$\int_0^\infty e^{-rt}\{K(t)^\alpha - I(t) - (\phi/2)[I(t)]^2\}dt,$$

subject to the capital accumulation constraints

$$I(t) - \delta K(t) \geq \dot{K}(t)$$

for all $t \in [0,\infty)$, taking the initial capital stock $K(0) > 0$ as given.

a) The Hamiltonian for this problem can be defined in present value terms as

$$H(K(t),\pi(t);t) = \max_{I(t)} e^{-rt}\{K(t)^\alpha - I(t) - (\phi/2)[I(t)]^2\} + \pi(t)[I(t) - \delta K(t)].$$

b) According to the maximum principle, the values of $I(t)$ and $K(t)$ that solve the firm's problem and the associated value of $\pi(t)$ must satisfy the first-order condition

$$-e^{-rt}[1+\phi I(t)] + \pi(t) = 0 \tag{31}$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_K(K(t),\pi(t);t) = -e^{-rt}\alpha K(t)^{\alpha-1} + \pi(t)\delta \tag{32}$$

and

$$\dot{K}(t) = H_\pi(K(t),\pi(t);t) = I(t) - \delta K(t) \tag{33}$$

for all $t \in [0,\infty)$.

c) Rewrite (31) as

$$\pi(t) = e^{-rt}[1+\phi I(t)]$$

and differentiate both sides with respect to t to obtain

$$\dot{\pi}(t) = -re^{-rt}[1+\phi I(t)] + e^{-rt}\phi\dot{I}(t).$$

Now substitute this last result together with (31) into (32) to obtain

$$-re^{-rt}[1+\phi I(t)] + e^{-rt}\phi\dot{I}(t) = -e^{-rt}\alpha K(t)^{\alpha-1} + \delta e^{-rt}[1+\phi I(t)]$$

or, after dividing through by e^{-rt} and isolating $\dot{I}(t)$ on the left-hand side,

$$\dot{I}(t) = (1/\phi)\{(\delta+r)[1+\phi I(t)] - \alpha K(t)^{\alpha-1}\} \tag{34}$$

Together, (33) and (34) form a system of two equations in two unknowns: investment $I(t)$ and the capital stock $K(t)$.

d) In a steady state where $I(t) = I^*$ and $K(t) = K^*$, $\dot{I}(t) = 0$ and $\dot{K}(t) = 0$. Substituting these steady-state conditions into (33) and (34) yields

$$I^* = \delta K^*$$

and

$$(\delta + r)(1 + \phi I^*) = \alpha (K^*)^{\alpha - 1}.$$

These last two equations determine the optimal steady-state values I^* and K^*.

e) Consider first (33), which implies that

$$\dot{K}(t) = 0 \text{ when } I(t) = \delta K(t),$$

$$\dot{K}(t) > 0 \text{ when } I(t) > \delta K(t),$$

and

$$\dot{K}(t) < 0 \text{ when } I(t) < \delta K(t).$$

Consider next (34), which implies that

$$\dot{I}(t) = 0 \text{ when } I(t) = \frac{1}{\phi}\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha - 1} - \frac{1}{\phi},$$

$$\dot{I}(t) > 0 \text{ when } I(t) > \frac{1}{\phi}\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha - 1} - \frac{1}{\phi},$$

and

$$\dot{I}(t) < 0 \text{ when } I(t) < \frac{1}{\phi}\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha - 1} - \frac{1}{\phi}.$$

A phase diagram that reflects these implications of (33) and (34) also reveals that starting from any value of the initial capital stock $K(0) > 0$, there is a unique value of $I(0)$ such that starting from $K(0)$ and $I(0)$, optimally-chosen investment $I(t)$ and capital stock $K(t)$ converge to their steady-state values K^* and I^*.

5 Human Capital and Growth

The consumer chooses sequences $\{u_t\}_{t=0}^{\infty}$ and $\{h_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t [\alpha \ln(1 - u_t) + \alpha \ln(h_t)]$$

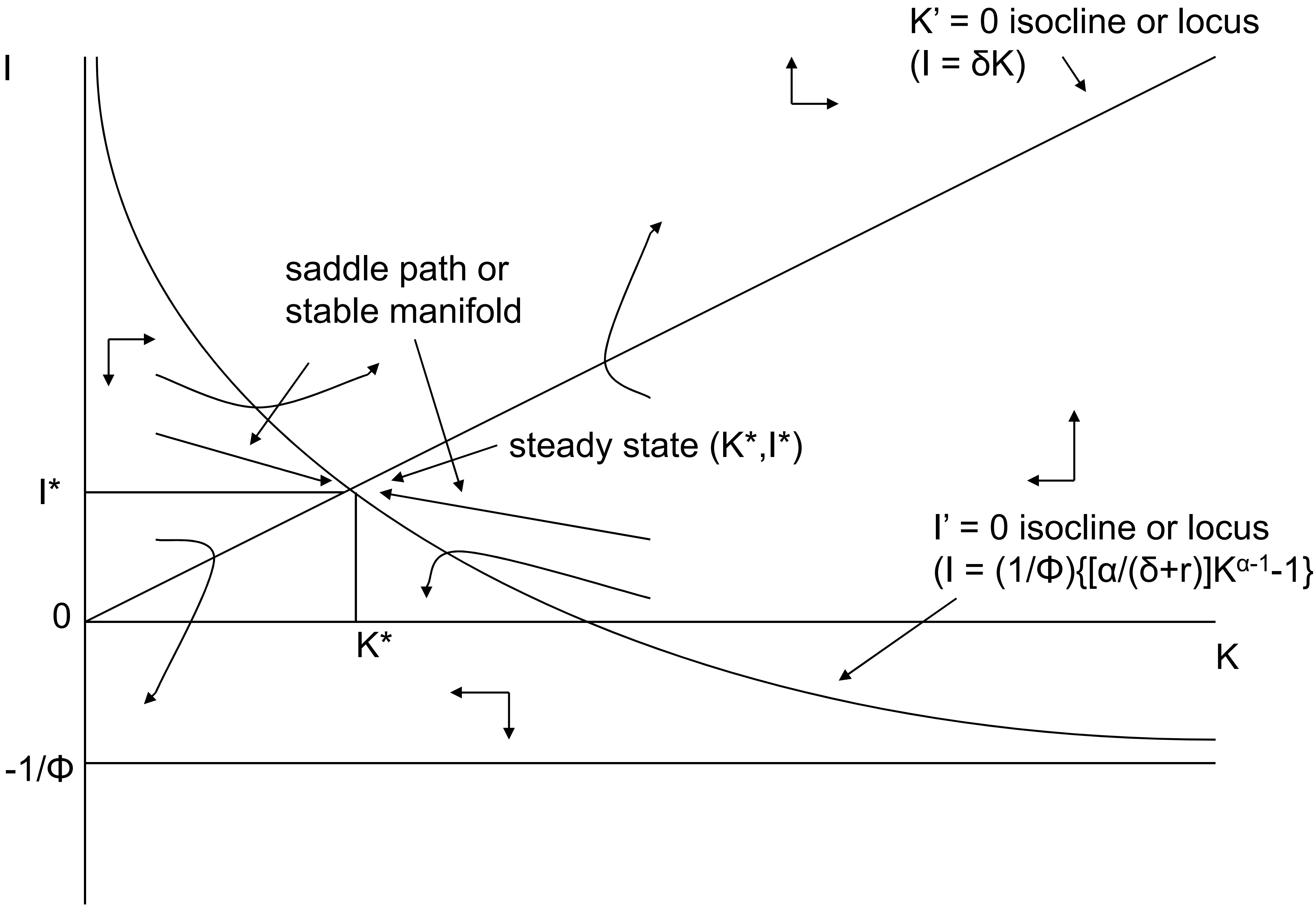
subject to the constraints

$$(1 + \gamma u_t) h_t \geq h_{t+1}$$

for all $t = 0, 1, 2, ...$, taking the initial stock of human capital h_0 as given.

a) The Bellman equation for the consumer's problem is

$$v(h_t; t) = \max_{u_t} \alpha \ln(1 - u_t) + \alpha \ln(h_t) + \beta v[(1 + \gamma u_t) h_t; t + 1].$$



I
K' = 0 isocline or locus
(I = δK)
saddle path or
stable manifold
steady state (K*,I*)
I*
0
K*
K
I' = 0 isocline or locus
(I = (1/Φ){[α/(δ+r)]K^α-1-1})
-1/Φ

b) Using the conjectured form of the value function, the Bellman equation becomes

$$E + F\ln(h_t) = \max_{u_t} \alpha\ln(1-u_t) + \alpha\ln(h_t) + \beta E + \beta F\ln(1+\gamma u_t) + \beta F\ln(h_t).$$

The first-order condition for u_t is then

$$-\frac{\alpha}{1-u_t} + \frac{\beta F\gamma}{1+\gamma u_t} = 0 \tag{35}$$

while the envelope condition for h_t is

$$\frac{F}{h_t} = \frac{\alpha}{h_t} + \frac{\beta F}{h_t}. \tag{36}$$

c) For this problem, the envelope condition (36) yields an immediate solution for F:

$$F = \frac{\alpha}{1-\beta}.$$

Substitute this solution into (35) to obtain the desired solution for u_t:

$$u_t = \frac{\beta\gamma + \beta - 1}{\gamma} \tag{37}$$

for all $t = 0, 1, 2, \ldots.$ Evidently, it is optimal to choose a constant value of u_t. This constant value of u_t is strictly less than one since $\gamma > 0$ and $1 > \beta$, and this constant value of u_t is strictly positive since $\beta(1+\gamma) > 1$.

d) Using the binding constraint and the solution (37) for u_t,

$$h_{t+1}/h_t = 1 + \gamma u_t = \beta(1+\gamma) \tag{38}$$

for all $t = 0, 1, 2, \ldots.$ Evidently, it is optimal to have human capital grow at a constant rate, which is positive since $\beta(1+\gamma) > 1$ and that is larger when β is larger, so that the consumer is more patient, and when γ increases, so time spent studying yields a larger payoff in terms of human capital growth.

e) Since

$$y_t = c_t = [(1-u_t)h_t]^\alpha,$$

(37) and (38) imply that

$$c_{t+1}/c_t = (h_{t+1}/h_t)^\alpha = [\beta(1+\gamma)]^\alpha \tag{39}$$

for all $t = 0, 1, 2, \ldots.$ Since time spent producing $1 - u_t$ is constant and since human capital growth h_{t+1}/h_t is constant, consumption and output growth c_{t+1}/c_t are constant as well.

6 Monopolistic Price Adjustment

The firm can be viewed as choosing sequences $\{p_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ to maximize the objective function

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t\left[p_t^{1-\varepsilon}-cp_t^{-\varepsilon}-\frac{\phi}{2}\left(\frac{p_t}{y_t}-1\right)^2\right]$$

subject to the "binding constraints"

$$y_{t+1}=p_t$$

for all $t=0,1,2,...$, taking the initial condition $y_0=p_{-1}$ as given.

a) The Bellman equation for the firm's problem is

$$v(y_t;t)=\max_{p_t} p_t^{1-\varepsilon}-cp_t^{-\varepsilon}-\frac{\phi}{2}\left(\frac{p_t}{y_t}-1\right)^2+\left(\frac{1}{1+r}\right)v(p_t;t+1).$$

b) The first-order condition for p_t is

$$(1-\varepsilon)p_t^{-\varepsilon}+\varepsilon cp_t^{-\varepsilon-1}-\phi\left(\frac{p_t}{y_t}-1\right)\frac{1}{y_t}+\left(\frac{1}{1+r}\right)v'(p_t;t+1)=0, \tag{40}$$

and the envelope condition for y_t is

$$v'(y_t;t)=\phi\left(\frac{p_t}{y_t}-1\right)\frac{p_t}{y_t^2}. \tag{41}$$

c) Substitute the "binding constraint" $y_{t+1}=p_t$ into the last term on the left-hand side of (40) to obtain

$$(1-\varepsilon)p_t^{-\varepsilon}+\varepsilon cp_t^{-\varepsilon-1}-\phi\left(\frac{p_t}{y_t}-1\right)\frac{1}{y_t}+\left(\frac{1}{1+r}\right)v'(y_{t+1};t+1)=0.$$

Then consider (41), which since it must hold for all $t=0,1,2,...$ implies that

$$v'(y_{t+1};t+1)=\phi\left(\frac{p_{t+1}}{y_{t+1}}-1\right)\frac{p_{t+1}}{y_{t+1}^2}.$$

Substitute the second of these last two equations into the first to obtain

$$(1-\varepsilon)p_t^{-\varepsilon}+\varepsilon cp_t^{-\varepsilon-1}-\phi\left(\frac{p_t}{y_t}-1\right)\frac{1}{y_t}+\left(\frac{1}{1+r}\right)\phi\left(\frac{p_{t+1}}{y_{t+1}}-1\right)\frac{p_{t+1}}{y_{t+1}^2}=0. \tag{42}$$

Now (42) and the binding constraint form a system of two equations in two unknowns, p_t and y_t, that makes no reference to the unknown value function v.

d) Substitute $y_{t+1}=p_t$ and $y_t=p_{t-1}$ into (42) to obtain

$$(1-\varepsilon)p_t^{-\varepsilon}+\varepsilon cp_t^{-\varepsilon-1}-\phi\left(\frac{p_t}{p_{t-1}}-1\right)\frac{1}{p_{t-1}}+\left(\frac{1}{1+r}\right)\phi\left(\frac{p_{t+1}}{p_t}-1\right)\frac{p_{t+1}}{p_t^2}=0, \tag{43}$$

which is now a single equation in a single unknown: the firm's optimal price p_t.

e) In a steady state where $p_{t+1} = p_t = p_{t-1} = p^*$ for all $t = 0, 1, 2, ...$, (43) implies that

$$(1 - \varepsilon)(p^*)^{-\varepsilon} + \varepsilon c(p^*)^{-\varepsilon-1} = 0.$$

Use this last equation to find the desired solution for p^*:

$$p^* = \left(\frac{\varepsilon}{\varepsilon - 1}\right) c. \tag{44}$$

Equation (44) restates a familiar result from microeconomics, which says that an optimizing monopolist should set its steady-state markup p^*/c equal to $\varepsilon/(\varepsilon - 1)$, where ε measures the absolute value of the elasticity of demand. According to (44) and consistent with intuition, the markup falls as ε rises, that is, as demand becomes more elastic with respect to price changes.

Midterm Exam

Economics 720: Mathematics for Economists

Fall 2007

This exam has five questions on five pages; before you begin, please make sure that your copy has all five questions and all five pages. The five questions will receive equal weight in determining your overall exam score. Since this is a 75-minute exam, I suggest that you spend approximately (75 minutes/5 questions) = 15 minutes on each question.

1 Cost Minimization

A firm produces output y with capital k and labor l according to the Cobb-Douglas specification

$$k^a l^b \geq y,$$

where the parameters a and b both lie, individually, between zero and one and sum to a number less than one, so that production exhibits decreasing returns to scale: $0 < a < 1$, $0 < b < 1$, and $0 < a + b < 1$.

Suppose that the firm chooses capital and labor inputs in order to minimize the total cost $rk + wl$ of producing at least $\bar{y}$ units of output, where $r > 0$ is the rental rate for capital and $w > 0$ is the wage rate for labor. That is, suppose that the firm solves

$$\min_{k,l} rk + wl \text{ subject to } k^a l^b \geq \bar{y}.$$

a) Define (set up) the Lagrangian for this cost minimization problem. In doing this, you can ignore the nonnegativity constraints $k \geq 0$ and $l \geq 0$ for the capital and labor inputs, as the form of the production function implies that these constraints will not bind at the optimum.

b) Let k^* and l^* denote the values of k and l that solve this problem. Write down the full set of first-order conditions, constraints, conditions on the sign of the Lagrange multiplier, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by k^*, l^*, and the associated value of the Lagrange multiplier.

c) Next, use these Kuhn-Tucker conditions to find solutions for k^*, l^*, and the associated value of the Lagrange multiplier in terms of the model's five parameters: r, w, $\bar{y}$, a, and b. [*Note:* these solutions for k^* and l^* define the firm's conditional factor demand curves for capital and labor.]

d) Now define the minimum cost function $C(r, w, \bar{y})$ as the minimized value of the objective function, taken subject to the constraint:

$$C(r, w, \bar{y}) = \min_{k,l} rk + wl \text{ subject to } k^a l^b \geq \bar{y}.$$

Use your results from above to obtain expressions for the derivatives of the minimum cost function, $C_1(r, w, \bar{y})$, $C_2(r, w, \bar{y})$, and $C_3(r, w, \bar{y})$, in terms of the parameters r, w, $\bar{y}$, a, and b.

e) What do your results from above tell you about the relationship between the conditional factor demand curves and the derivatives of the minimum cost function? What economic interpretation does your results provide for the Lagrange multiplier in this constrained minimization problem?

2 Profit Maximization

Consider the same firm as in question 1 from above, producing output y with capital k and labor l according to the Cobb-Douglas specification

$$k^a l^b \geq y,$$

where the parameters a and b both lie, individually, between zero and one and sum to a number less than one, so that production exhibits decreasing returns to scale: $0 < a < 1$, $0 < b < 1$, and $0 < a + b < 1$.

Assume, as above, that the firm faces the rental rate $r > 0$ for capital and the wage rate $w > 0$ for labor. Now, however, suppose that the firm maximizes profits $py - rk - wl$, where $p > 0$ denotes its output price, subject to the constraint imposed by its production possibilities. That is, suppose that the firm solves

$$\max_{y,k,l} py - rk - wl \text{ subject to } k^a l^b \geq y.$$

a) Define (set up) the Lagrangian for this profit maximization problem. In doing this, you can ignore the nonnegativity constraints $y \geq 0$, $k \geq 0$, and $l \geq 0$ for output and the capital and labor inputs, as the set-up of the problem implies these constraints will not bind at the optimum.

b) Let y^*, k^*, and l^* denote the values of y, k, and l that solve this problem. Write down the full set of first-order conditions, constraints, conditions on the sign of the Lagrange multiplier, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by y^*, k^*, l^*, and the associated value of the Lagrange multiplier.

c) Next, use these Kuhn-Tucker conditions to find solutions for y^*, k^*, l^*, and the associated value of the Lagrange multiplier in terms of the model's five parameters: p, r, w, a, and b. [*Note:* this solution for y^* defines the firm's supply function, and these solutions for k^* and l^* define the firm's factor demand curves for capital and labor.]

d) Now define the profit function $\Pi(p, r, w)$ as the maximized value of the objective function, taken subject to the constraint:

$$\Pi(p, r, w) = \max_{y,k,l} py - rk - wl \text{ subject to } k^a l^b \geq y.$$

Use your results from above to obtain expressions for the derivatives of the profit function, $\Pi_1(p, r, w)$, $\Pi_2(p, r, w)$, and $\Pi_3(p, r, w)$, in terms of the parameters p, r, w, a, and b.

e) What do your results from above tell you about the relationship between the supply and factor demand curves and the derivatives of the profit function?

3 A Dynamic Optimization Problem in Discrete Time

Consider a discrete-time dynamic optimization problem in which the terminal date is $T = 2$, so that the time periods are $t = 0$, $t = 1$, and $t = 2$. Let z_t, $t = 0, 1, 2$, denote the value of a flow variable during each period t, and let y_t, $t = 0, 1, 2, 3$, denote the value of a stock variable at the beginning of each period t, so that in particular y_0 is the initial value of the stock and y_3 is the terminal value of the stock.

Suppose that in this case, the objective function and the constraints cannot be written in an additively time-separable way, so that the objective function

$$F(y_0, z_0, y_1, z_1, y_2, z_2)$$

depends simultaneously on variables from all periods $t = 0, 1, 2$ and the constraint

$$c \geq G(y_0, z_0, y_1, z_1, y_2, z_2, y_3)$$

depends simultaneously on variables from all periods $t = 0, 1, 2$ as well as the terminal value of the stock y_3.

Hence, the problem can be stated formally as: choose values z_0, z_1, and z_2 for the flow variable and values y_1, y_2, and y_3 for the stock variable to maximize the objective function

$$F(y_0, z_0, y_1, z_1, y_2, z_2)$$

subject to the constraints

$$c \geq G(y_0, z_0, y_1, z_1, y_2, z_2, y_3),$$

$$y_0 \text{ given},$$

and

$$y_3 \geq y^*,$$

where c in the first constraint is a constant parameter and y^* represents a lower bound that is imposed on the terminal value of the stock.

a) Since this problem lacks the additively time-separable structure that is required for an application of the maximum principle, the best way to solve it is to use the Kuhn-Tucker theorem. Accordingly, begin by defining (setting up) the Lagrangian for this problem.

b) Next, write down the set of first-order conditions that must be satisfied by the values of z_0, z_1, z_2, y_1, y_2, and y_3 that solve the problem, together with the associated values of the multipliers that you introduced into the problem above, when writing down the Lagrangian.

c) Write down the constraints that must be satisfied by the values of z_0, z_1, z_2, y_1, y_2, and y_3 that solve the problem.

d) Write down the nonnegativity or nonpositivity constraints that must be satisfied by the values of the Lagrange multipliers.

e) Write down the complementary slackness conditions associated with each constraint and its Lagrange multiplier.

4 The Maximum Principle in Discrete Time

Consider the same fairly general discrete-time dynamic optimization problem that we studied in class. The horizon is finite, so that periods are indexed by $t = 0, 1, ..., T$. There is one stock variable y_t and one flow variable z_t. Thus, sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ must be chosen to maximize the objective function

$$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t),$$

with $1 \geq \beta > 0$, subject to the constraints

$$y_t + Q(y_t, z_t; t) \geq y_{t+1} \text{ for all } t = 0, 1, ..., T,$$

$$c \geq G(y_t, z_t; t) \text{ for all } t = 0, 1, ..., T,$$

$$y_0 \text{ given,}$$

and

$$y_{T+1} \geq y^*.$$

a) Define the Hamiltonian for this problem.

b) Write down the first-order condition for the static optimization problem that appears in the definition of the Hamiltonian.

c) Write down the pair of difference equations that, according to the Maximum Principle, must be satisfied by the solution to the original, dynamic optimization problem (here, as in class, you can assume that the constraint governing the evolution of the stock variable always binds, as it will in virtually all economic applications).

d) Write down a fourth equation that, together with the three equations you derived to in parts (b) and (c) above, can be used to help find the sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ that solve the original dynamic optimization problem.

e) Write down the pair of boundary conditions that, according to the Maximum Principle, must also be satisfied by the solution to the original, dynamic optimization problem.

5 Optimal Growth in Discrete Time

Consider an infinite-horizon economy in which output is produced with capital during each period $t = 0, 1, 2, ...$ according to the production function

$$y_t = F(k_t) = k_t^\alpha,$$

where the parameter α lies between zero and one: $0 < \alpha < 1$. Let a representative household's consumption during each period $t = 0, 1, 2, ...$ be denoted by c_t, and let the parameter δ, with $0 < \delta \leq 1$, denote the rate at which physical capital depreciates between periods. Then, the capital stock k_{t+1} at the beginning of period $t + 1$ equals the capital stock at the beginning of period t, plus the amount of new output k_t^α produced during period t, less the amount of capital δk_t that depreciates away during period t, less the amount of output c_t consumed during period t; or, allowing for the free disposal of capital:

$$k_t + k_t^\alpha - \delta k_t - c_t \geq k_{t+1}$$

for all $t = 0, 1, 2, ...$

In this economy, a benevolent social planner takes the initial capital stock k_0 as given, and chooses sequences $\{c_t\}_{t=0}^\infty$ and $\{k_t\}_{t=1}^\infty$ to maximize the representative household's utility, given by the additively-time separable and logarithmic specification

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

subject to the constraints

$$k_t + k_t^\alpha - \delta k_t - c_t \geq k_{t+1}$$

for all $t = 0, 1, 2, ...$ and

$$k_0 \text{ given.}$$

a) As a first step in solving this problem using the maximum principle, begin by defining (setting up) the Hamiltonian.

b) Next, write down the first-order condition for the static optimization problem that appears in the definition of the Hamiltonian.

c) Write down the pair of difference equations that, according to the Maximum Principle, must be satisfied by the solution to the social planner's original, dynamic optimization problem.

d) Now assume that $\delta = 1$, so that the capital stock depreciates fully between periods, and use this assumption to simplify the three equations that you derived in parts (b) and (c) above.

e) Your three equations from part (d) above should involve three unknowns: the values of c_t and k_t that solve the social planner's problem and the value of the additional variable that you introduced into the problem in part (a) from above when setting up the Hamiltonian. Use one of the equations as a solution for the additional variable, and rewrite the system as one involving just two equations in the two variables with direct economic interpretations: c_t and k_t.

Solutions to Midterm Exam

Economics 720: Mathematics for Economists

Fall 2007

1 Cost Minimization

The firm solves

$$\min_{k,l} rk + wl \text{ subject to } k^a l^b \geq \bar{y}.$$

a) There are a number of ways to define the Lagrangian for this cost minimization problem, but one way is to subtract the product of the multiplier λ and the constraint from the objective function to obtain

$$L(k, l, \lambda) = rk + wl - \lambda(k^a l^b - \bar{y}).$$

b) With the Lagrangian defined as above, the Kuhn-Tucker theorem states that if k^* and l^* solve the firm's problem, then there exists a value λ^* of λ such that, together, k^*, l^*, and λ^* satisfy the first-order conditions

$$L_1(k^*, l^*, \lambda^*) = r - a\lambda^*(k^*)^{a-1}(l^*)^b = 0 \tag{1}$$

and

$$L_2(k^*, l^*, \lambda^*) = w - b\lambda^*(k^*)^a(l^*)^{b-1} = 0, \tag{2}$$

the constraint

$$(k^*)^a(l^*)^b \geq \bar{y}, \tag{3}$$

the nonnegativity condition

$$\lambda^* \geq 0, \tag{4}$$

and the complementary slackness condition

$$\lambda^*[(k^*)^a(l^*)^b - \bar{y}] = 0. \tag{5}$$

c) Again, there are many ways of using these Kuhn-Tucker conditions to find solutions for k^*, l^*, and λ^* in terms of the parameters r, w, $\bar{y}$, a, and b, but one way starts by rearranging the first-order condition (1) to read

$$\lambda^* = \frac{r}{a(k^*)^{a-1}(l^*)^b}. \tag{6}$$

Since the objects in both the numerator and the denominator on the right-hand side of (6) are all strictly positive, the multiplier λ^* must also be strictly positive; the complementary slackness condition (5) then implies that the constraint (3) must bind at the optimum. Now use (6) to substitute out for λ^* in (2) and rearrange to obtain

$$l^* = \left(\frac{br}{aw}\right) k^*. \tag{7}$$

Substitute this last result into the binding constraint to obtain the desired solution for k^*:

$$k^* = \left(\frac{aw}{br}\right)^{b/(a+b)} (\bar{y})^{1/(a+b)}. \tag{8}$$

Substitute this expression back into (7) to obtain the desired solution for l^*:

$$l^* = \left(\frac{br}{aw}\right)^{a/(a+b)} (\bar{y})^{1/(a+b)}. \tag{9}$$

Finally, substitute (8) and (9) back into (6) to obtain the desired solution for λ^*:

$$\lambda^* = \left(\frac{r}{a}\right)^{a/(a+b)} \left(\frac{w}{b}\right)^{b/(a+b)} (\bar{y})^{(1-a-b)/(a+b)}. \tag{10}$$

d) Now define the minimum cost function $C(r, w, \bar{y})$ as

$$C(r, w, \bar{y}) = \min_{k,l} rk + wl \text{ subject to } k^a l^b \geq \bar{y}.$$

Related, let (8)-(10) define the functions $k^*(r, w, \bar{y})$, $l^*(r, w, \bar{y})$, and $\lambda^*(r, w, \bar{y})$ describing how the optimal capital and labor inputs and the associated value of the Lagrange multiplier depend on the parameters r, w, and $\bar{y}$. Then, in light of the complementary slackness condition (5), the minimum cost function can be evaluated as

$$C(r, w, \bar{y}) = rk^*(r, w, \bar{y}) + wl^*(r, w, \bar{y}) - \lambda^*(r, w, \bar{y})\{[k^*(r, w, \bar{y})]^a [l^*(r, w, \bar{y})]^b - \bar{y}\}.$$

The envelope theorem implies that in differentiating both sides of this last expression, it is safe to ignore the dependence of k^*, l^*, and λ^* on r, w, and $\bar{y}$ and write

$$C_1(r, w, \bar{y}) = k^*(r, w, \bar{y}) = \left(\frac{aw}{br}\right)^{b/(a+b)} (\bar{y})^{1/(a+b)}, \tag{11}$$

$$C_2(r, w, \bar{y}) = l^*(r, w, \bar{y}) = \left(\frac{br}{aw}\right)^{a/(a+b)} (\bar{y})^{1/(a+b)}, \tag{12}$$

and

$$C_3(r, w, \bar{y}) = \lambda^*(r, w, \bar{y}) = \left(\frac{r}{a}\right)^{a/(a+b)} \left(\frac{w}{b}\right)^{b/(a+b)} (\bar{y})^{(1-a-b)/(a+b)}. \tag{13}$$

e) Equations (11) and (12) reveal that the derivatives of the minimum cost function with respect to the rental rate for capital and the wage rate for labor coincide with the conditional factor demand curves for capital and labor: they provide a statement of Shephard's lemma. Equation (13) provides an interpretation of the Lagrange multiplier as a measure of marginal cost at the optimum.

2 Profit Maximization

Now the firm solves

$$\max_{y,k,l} py - rk - wl \text{ subject to } k^a l^b \geq y.$$

a) Once again there are a number of ways to define the Lagrangian for this profit maximization problem, but one way is to add the product of the multiplier λ and the constraint to the objective function to obtain

$$L(y, k, l, \lambda) = py - rk - wl + \lambda(k^a l^b - y).$$

b) With the Lagrangian defined as above, the Kuhn-Tucker theorem states that if y^*, k^*, and l^* solve the firm's problem, then there exists a value λ^* of λ such that, together, y^*, k^*, $l^*$and λ^* satisfy the first-order conditions

$$L_1(y^*, k^*, l^*, \lambda^*) = p - \lambda^* = 0. \tag{14}$$

$$L_2(y^*, k^*, l^*, \lambda^*) = -r + a\lambda^*(k^*)^{a-1}(l^*)^b = 0 \tag{15}$$

and

$$L_3(y^*, k^*, l^*, \lambda^*) = -w + b\lambda^*(k^*)^a(l^*)^{b-1} = 0 \tag{16}$$

the constraint

$$(k^*)^a(l^*)^b \geq y^*, \tag{17}$$

the nonnegativity condition

$$\lambda^* \geq 0, \tag{18}$$

and the complementary slackness condition

$$\lambda^*[(k^*)^a(l^*)^b - y^*] = 0. \tag{19}$$

c) Again, there are many ways of using these Kuhn-Tucker conditions to find solutions for y^*, k^*, l^*, and λ^* in terms of the model's five parameters: p, r, w, a, and b, but one way starts by observing from (14) that

$$\lambda^* = p, \tag{20}$$

which provides the desired solution for λ^*. Since the output price p is strictly positive, (20) implies that λ^* is also strictly positive; the complementary slackness condition (19) then implies that the constraint (17) must bind at the optimum. Use (20) to substitute out for λ^* in (15), and rearrange to obtain

$$l^* = \left(\frac{r}{ap}\right)^{1/b} (k^*)^{(1-a)/b}. \tag{21}$$

Substitute this last result into (16) and use (14) again to obtain

$$k^* = p^{1/(1-a-b)} \left(\frac{a}{r}\right)^{(1-b)/(1-a-b)} \left(\frac{b}{w}\right)^{b/(1-a-b)}, \tag{22}$$

which provides the desired solution for k^*. Substitute (22) back into (21) to obtain the desired solution for l^*:

$$l^* = p^{1/(1-a-b)} \left(\frac{a}{r}\right)^{a/(1-a-b)} \left(\frac{b}{w}\right)^{(1-a)/(1-a-b)}. \tag{23}$$

Finally, substitute (22) and (23) into the binding constraint to obtain the desired solution for y^*:

$$y^* = p^{(a+b)/(1-a-b)} \left(\frac{a}{r}\right)^{a/(1-a-b)} \left(\frac{b}{w}\right)^{b/(1-a-b)}. \tag{24}$$

d) Now define the profit function $\Pi(p, r, w)$ as

$$\Pi(p, r, w) = \max_{y,k,l} py - rk - wl \text{ subject to } k^a l^b \geq y.$$

Related, let (20) and (22)-(24) define the functions $y^*(p, r, w)$, $k^*(p, r, w)$, $l^*(p, r, w)$, and $\lambda^*(p, r, w)$ describing how the optimal level of output, the optimal capital and labor inputs, and the associated value of the Lagrange multiplier depend on the parameters p, r, and w. Then, in light of the complementary slackness condition (19), the profit function can be evaluated as

$$\begin{aligned} \Pi(p, r, w) &= py^*(p, r, w) - rk^*(p, r, w) - wl^*(p, r, w) \\ &\quad + \lambda^*(p, r, w)\{[k^*(p, r, w)]^a[l^*(p, r, w)]^b - y^*(p, r, w)\}. \end{aligned}$$

The envelope theorem implies that in differentiating both sides of this last expression, it is safe to ignore the dependence of y^*, k^*, l^*, and λ^* on p, r, and w and write

$$\Pi_1(p, r, w) = y^*(p, r, w) = p^{(a+b)/(1-a-b)} \left(\frac{a}{r}\right)^{a/(1-a-b)} \left(\frac{b}{w}\right)^{b/(1-a-b)}, \tag{25}$$

$$\Pi_2(p, r, w) = -k^*(p, r, w) = -p^{1/(1-a-b)} \left(\frac{a}{r}\right)^{(1-b)/(1-a-b)} \left(\frac{b}{w}\right)^{b/(1-a-b)}, \tag{26}$$

and

$$\Pi_3(p, r, w) = -l^*(p, r, w) = -p^{1/(1-a-b)} \left(\frac{a}{r}\right)^{a/(1-a-b)} \left(\frac{b}{w}\right)^{(1-a)/(1-a-b)}. \tag{27}$$

e) Equations (25)-(27) provide a statement of Hotelling's lemma. According to (25), the derivative of the profit function with respect to the output price p coincides with the supply function. According to (26) and (27), the derivative of the profit function with respect to the two factor prices r and w coincide with the factor demand curves for capital and labor, multiplied by -1, since an increase in these factor prices leads to a decrease in profits.

3 A Dynamic Optimization Problem in Discrete Time

The problem can be stated as

$$\max_{z_0,z_1,z_2,y_1,y_2,y_3} F(y_0,z_0,y_1,z_1,y_2,z_2)$$
$$\text{subject to } c \geq G(y_0,z_0,y_1,z_1,y_2,z_2,y_3),\ y_0 \text{ given, and } y_3 \geq y^*.$$

a) There are a number of ways to define the Lagrangian for this constrained maximization problem, but one way is as

$$\begin{aligned} L(z_0,z_1,z_2,y_1,y_2,y_3,\lambda,\phi) &= F(y_0,z_0,y_1,z_1,y_2,z_2) \\ &\quad +\lambda[c-G(y_0,z_0,y_1,z_1,y_2,z_2,y_3)]+\phi(y_3-y^*). \end{aligned}$$

b) Since there are six choice variables, there are six first-order conditions, each obtained by differentiating the Lagrangian through by one of the choice variables and equating to zero:

$$L_1(z_0^*,z_1^*,z_2^*,y_1^*,y_2^*,y_3^*,\lambda^*,\phi^*) = F_2(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*)-\lambda^* G_2(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*,y_3^*) = 0,$$
$$L_2(z_0^*,z_1^*,z_2^*,y_1^*,y_2^*,y_3^*,\lambda^*,\phi^*) = F_4(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*)-\lambda^* G_4(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*,y_3^*) = 0,$$
$$L_3(z_0^*,z_1^*,z_2^*,y_1^*,y_2^*,y_3^*,\lambda^*,\phi^*) = F_6(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*)-\lambda^* G_6(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*,y_3^*) = 0,$$
$$L_4(z_0^*,z_1^*,z_2^*,y_1^*,y_2^*,y_3^*,\lambda^*,\phi^*) = F_3(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*)-\lambda^* G_3(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*,y_3^*) = 0,$$
$$L_5(z_0^*,z_1^*,z_2^*,y_1^*,y_2^*,y_3^*,\lambda^*,\phi^*) = F_5(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*)-\lambda^* G_5(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*,y_3^*) = 0,$$

and

$$L_6(z_0^*,z_1^*,z_2^*,y_1^*,y_2^*,y_3^*,\lambda^*,\phi^*) = -\lambda^* G_7(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*,y_3^*)+\phi^* = 0$$

c) The optimal choices must also satisfy the constraints

$$L_7(z_0^*,z_1^*,z_2^*,y_1^*,y_2^*,y_3^*,\lambda^*,\phi^*) = c-G(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*,y_3^*) \geq 0,$$
$$L_8(z_0^*,z_1^*,z_2^*,y_1^*,y_2^*,y_3^*,\lambda^*,\phi^*) = y_3^*-y^* \geq 0,$$

and

$$y_0 \text{ given.}$$

d) And with the Lagrangian defined as above, the Kuhn-Tucker theorem implies that the multipliers must satisfy the nonnegativity constraints

$$\lambda^* \geq 0$$

and

$$\phi^* \geq 0.$$

e) Finally, the Kuhn-Tucker theorem implies that the complementary slackness conditions

$$\lambda^*[c-G(y_0,z_0^*,y_1^*,z_1^*,y_2^*,z_2^*,y_3^*)] = 0$$

and

$$\phi^*(y_3^*-y^*) = 0$$

hold at the optimum.

4 The Maximum Principle in Discrete Time

a) Define the Hamiltonian as

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) \text{ subject to } c \geq G(y_t, z_t; t).$$

b) By the Kuhn-Tucker theorem,

$$H(y_t, \pi_{t+1}; t) = \max_{z_t} \beta^t F(y_t, z_t; t) + \pi_{t+1} Q(y_t, z_t; t) + \lambda_t [c - G(y_t, z_t; t)],$$

where z_t satisfies the first-order condition

$$\beta^t F_z(y_t, z_t; t) + \pi_{t+1} Q_z(y_t, z_t; t) - \lambda_t G_z(y_t, z_t; t) = 0. \tag{28}$$

c) According to the Maximum Principle, the solution to the original dynamic optimization problem must satisfy the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_y(y_t, \pi_{t+1}; t) = -[\beta^t F_y(y_t, z_t; t) + \pi_{t+1} Q_y(y_t, z_t; t) - \lambda_t G_y(y_t, z_t; t)] \tag{29}$$

and

$$y_{t+1} - y_t = H_\pi(y_t, \pi_{t+1}; t) = Q(y_t, z_t; t). \tag{30}$$

d) According to the Kuhn-Tucker theorem, the solution to the original dynamic optimization problem must also satisfy the complementary slackness condition

$$\lambda_t [c - G(y_t, z_t; t)] = 0. \tag{31}$$

Equations (28)-(31) form a system of four equations in four unknowns, that helps to identify the values of y_t and z_t that solve the original dynamic optimization problem as well as the associated values of the multipliers π_{t+1} and λ_t.

e) According to the Maximum Principle, the solution to the original dynamic optimization problem must also satisfy two boundary conditions: the initial condition

$$y_0 \text{ given}$$

and the terminal, or transversality, condition

$$\pi_{T+1}(y_{T+1} - y^*) = 0.$$

5 Optimal Growth in Discrete Time

a) Define the Hamiltonian as

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(k_t^\alpha - \delta k_t - c_t).$$

b) By the Kuhn-Tucker theorem, c_t must satisfy the first-order condition

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0. \tag{32}$$

c) According to the Maximum Principle, the solution to the social planner's original, dynamic optimization problem must satisfy the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}(\alpha k_t^{\alpha-1} - \delta) \tag{33}$$

and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = k_t^\alpha - \delta k_t - c_t. \tag{34}$$

d) When $\delta = 1$, (32) remains as before, but (33) and (34) simplify to

$$\pi_t = \alpha \pi_{t+1} k_t^{\alpha-1} \tag{35}$$

and

$$k_{t+1} = k_t^\alpha - c_t. \tag{36}$$

e) A potential problem with the three-equation system formed by (32), (35), and (36) is that it involves the variable π_{t+1}, which lacks an immediate economic interpretation. To eliminate this variable from the system, use the fact that, according to the maximum principle, (32) must hold for all $t = 0, 1, 2, \ldots$ to eliminate

$$\pi_{t+1} = \frac{\beta^t}{c_t}$$

and

$$\pi_t = \frac{\beta^{t-1}}{c_{t-1}}$$

from (35), so that this equation becomes

$$\frac{c_t}{c_{t-1}} = \alpha\beta k_t^{\alpha-1}. \tag{37}$$

Equations (36) and (37) now form a system of two equations in two unknowns: the values of c_t and k_t that solve the social planner's problem. In terms of the economics, (36) just says that when capital depreciates completely between periods, the capital stock k_{t+1} at the beginning of period $t+1$ consists entirely of output $k_t^\alpha - c_t$ that is not consumed during period t. Equation (37) can be rearranged slightly to read

$$\frac{c_t}{\beta c_{t-1}} = \alpha k_t^{\alpha-1}.$$

The left-hand side of this last expression measures the intertemporal marginal rate of substitution, that is, the marginal rate of substitution between consumption during periods t and $t-1$. The right-hand side of this last expression measures the marginal product of capital. Hence, (37) expresses the intertemporal efficiency condition that the intertemporal marginal rate of substitution ought to equal the marginal product of capital.

Final Exam

Economics 720: Mathematics for Economists

Fall 2007

This exam has five questions on six pages; before you begin, please make sure that your copy has all five questions and all six pages. The five questions will receive equal weight in determining your overall exam score. Since this is a 120-minute exam, I suggest that you spend approximately (120 minutes/5 questions) = 24 minutes on each question.

1 The Kuhn-Tucker Theorem

A consumer likes two goods: apples and bananas. Let a denote his or her consumption of apples; let b denote his or her consumption of bananas; and let the function $U(a, b)$ measure the utility that the consumer obtains from eating apples and bananas. Assume that $U(a, b)$ is strictly increasing in both its arguments, so that $U_a(a, b) > 0$ and $U_b(a, b) > 0$ for all values of $a \geq 0$ and $b \geq 0$, where U_a and U_b denote derivatives of U with respect to its first and second arguments. Suppose also that the consumer can purchase as many or as few apples and bananas as he or she likes in perfectly competitive markets at the prices p^a for apples and p^b for bananas. Finally, let I denote the consumer's income. Now the consumer's static utility maximization problem can be stated formally as

$$\max_{a,b} U(a, b) \text{ subject to } I \geq p^a a + p^b b,\ a \geq 0, \text{ and } b \geq 0,$$

where the last two constraints specifically allow for the possibility that the optimizing consumer can be at a corner solution, consuming either no apples or no bananas.

a) Define (write down) the Lagrangian for this problem, allowing for the possibility that the nonnegativity constraint on a or b might bind at the optimum.

b) Write down the first-order conditions for the consumer's problem, again allowing for the possibility that the nonnegativity constraint on a or b might bind at the optimum. NOTE: Your first-order conditions may depend on precisely how you defined the Lagrangian for the problem in part (a), above.

c) Write down the full set of constraints that must be satisfied by the optimal choices of a and b, as well as any additional conditions on the sign (positive or negative) of the Lagrange multiplier or multipliers that you introduced into the problem in part (a) above.

d) Write down the complementary slackness conditions that must be satisfied at the optimum.

e) Use your results from above to answer the following question: under what circumstances will the the optimizing consumer's marginal rate of substitution $U_a(a^*, b^*)/U_b(a^*, b^*)$ between apples and bananas NOT equal the relative price p^a/p^b of apples and bananas?

2 The Maximum Principle in Continuous Time

Consider the continuous-time, dynamic optimization problem of choosing continuously differentiable functions $z(t)$ for $t \in [0, T]$ to describe the evolution of a flow variable and $y(t)$ for $t \in (0, T]$ to describe the evolution of a stock variable to maximize the objective function

$$\int_0^T e^{-\rho t} F(y(t), z(t); t) dt,$$

with $\rho \geq 0$, subject to the constraints

$$Q(y(t), z(t); t) \geq \dot{y}(t) \text{ for all } t \in [0, T],$$

$$c \geq G(y(t), z(t); t) \text{ for all } t \in [0, T],$$

$$y(0) \text{ given}$$

and

$$y(T) \geq y^*.$$

a) Define the Hamiltonian for this problem.

b) Write down the first-order condition for the static optimization problem that appears in the definition of the Hamiltonian.

c) Write down the pair of differential equations that, according to the maximum principle, must be satisfied by the solution to the original, dynamic optimization problem (here, as in class, you can assume that the constraint governing the evolution of the stock variable always binds, as it will in virtually all economic applications).

d) Write down a fourth equation that, together with the three equations you derived to in parts (b) and (c) above, can be used to help find the values of $y(t)$ and $z(t)$ that solve the original dynamic optimization problem.

e) Write down the pair of boundary conditions that, according to the maximum principle, must also be satisfied by the solution to the original, dynamic optimization problem.

3 Optimal Growth in Continuous Time

Consider an infinite-horizon economy in which output is produced with capital according to the production function

$$F(k(t)) = k(t)^{\alpha},$$

where $0 < \alpha < 1$. Let $c(t)$ denote consumption and let $\delta \geq 0$ denote the depreciation rate for capital. Then the capital stock evolves according to

$$k(t)^{\alpha} - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. Then, subject to this constraint and taking the initial value $k(0)$ for the capital stock as given, a benevolent social planner chooses the continuously differential functions $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize a representative consumer's utility, given by the additively time-separable and logarithmic specification

$$\int_0^{\infty} e^{-\rho t} \ln(c(t)) dt,$$

with $\rho \geq 0$.

a) As a first step in solving this problem using the maximum principle, begin by defining (setting up) the Hamiltonian.

b) Next, write down the first-order condition for the static optimization problem that appears in the definition of the Hamiltonian.

c) Write down the pair of difference equations that, according to the maximum principle, must be satisfied by the solution to the social planner's original, dynamic optimization problem.

d) Your three equations from parts (b) and (c) above should involve three unknowns: the values of $c(t)$ and $k(t)$ that solve the social planner's problem and the value of the additional variable that you introduced into the problem in part (a) from above when setting up the Hamiltonian. Rewrite this system as one involving just two equations in the two variables with direct economic interpretations: $c(t)$ and $k(t)$.

e) Your equations from part (d) above imply that the economy has a unique, nontrivial steady state in which $c(t) = c^* > 0$ and $k(t) = k^* > 0$ for all $t \in [0, \infty)$. Use those equations to solve for the steady-state values c^* and k^* of consumption and capital in terms of the model's parameters α, δ, and ρ.

4 Optimal Resource Depletion

Consider a discrete-time, infinite horizon model that characterizes the optimal consumption of an exhaustible resource (like oil or coal). Let time periods be indexed by $t = 0, 1, 2, ...$; and let x_t, $t = 0, 1, 2, ...$, denote the stock of the exhaustible resource that remains at the beginning of period t. Let c_t, $t = 0, 1, 2, ...$, denote the amount of this resource that is consumed during period t. Since no new units of the resource are ever created, the amount consumed simply subtracts from the available stock according to

$$x_t - c_t \geq x_{t+1}$$

for all $t = 0, 1, 2, ...$, where the inequality constraint (which you can assume will always bind at the optimum) simply recognizes that the resource can be freely disposed of. The optimization problem then involves choosing sequences $\{c_t\}_{t=0}^{\infty}$ and $\{x_t\}_{t=1}^{\infty}$ to maximize utility from consuming the resource over the infinite horizon, given by

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where the discount factor lies between zero and one, $0 < \beta < 1$, subject to the constraint $x_t - c_t \geq x_{t+1}$ for all $t = 0, 1, 2, ...$, taking as given the level of the initial resource stock $x_0 > 0$.

a) Define (write down) the Lagrangian for this dynamic optimization problem. Then, write down the first-order conditions and constraints that, according to the Kuhn-Tucker theorem, characterize the solution to the dynamic optimization problem.

b) Define (write down) the Hamiltonian for this dynamic optimization problem. Then, write down the first-order condition and the pair of difference equations that, according to the maximum principle, characterize the solution to the dynamic optimization problem.

c) Write down the Bellman equation for this dynamic optimization problem. Then, write down the first-order condition, the envelope condition, and the constraint that, according to the dynamic programming approach, characterize the solution to the dynamic optimization problem.

d) In each of parts (a), (b), and (c) from above, you should have derived a system of three equations in three unknowns: the values of c_t and x_t that solve the optimal resource depletion problem and the value of a third unknown variable or function that you introduced into the problem when setting up the Lagrangian, Hamiltonian, or Bellman equation. Rewrite this three-equation system as one involving just two equations in the two variables with direct economic interpretations: c_t and x_t.

e) Use your answers from part (d) above to answer the following question: is the optimal consumption c_t of the exhaustible resource rising, falling, or staying constant over time?

5 Saving with a Random Return

Consider the following dynamic, stochastic optimization problem in discrete time. An infinitely-lived representative consumer enters each period $t = 0, 1, 2, ...$ with a stock of financial wealth denoted by A_t. During period t, the consumer divides this wealth up into an amount c_t to be consumed and an amount s_t to be saved, subject to the constraint

$$A_t = c_t + s_t,$$

which must hold for all $t = 0, 1, 2, ...$

The gross return R_{t+1} on savings between t and $t+1$ is random and, more specifically, follows the first-order autoregressive process

$$\ln(R_{t+1}) = \rho \ln(R_t) + \varepsilon_{t+1}$$

where $0 < \rho < 1$ and ε_{t+1} is an independently and identically distributed shock that satisfies $E_t\varepsilon_{t+1} = 0$ for all $t = 0, 1, 2, ...$ Thus, last period's return R_t is known when the consumer chooses s_t, but R_{t+1} is still viewed as random. Given the consumer's choice of s_t and the realized value of R_{t+1}, the consumer's financial wealth A_{t+1} is determined by the constraint

$$R_{t+1}s_t \geq A_{t+1},$$

which must hold for all $t = 0, 1, 2, ...$ and for all possible realizations of R_{t+1}.

The representative consumer seeks to maximize expected utility

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where the discount factor satisfies $0 < \beta < 1$ and where, as indicated, the single-period utility function takes the natural logarithmic form. As a first step in solving the consumer's problem, it is convenient to substitute the constraint $A_t = c_t + s_t$ into the utility function to obtain

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t).$$

The consumer's problem can now be stated formally as: choose contingency plans for s_t, $t = 0, 1, 2, ...$, and A_t, $t = 1, 2, 3, ...$, to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t)$$

subject to the constraints

$$A_0 \text{ given}$$

and

$$R_{t+1}s_t \geq A_{t+1}$$

for all $t = 0, 1, 2, ...$ and all possible realizations of R_{t+1}.

a) Write down the Bellman equation for the household's problem. In doing so, you might find it helpful to view A_t as the state variable, s_t as the control variable, and R_t as the random variable entering into the problem. You might also find it helpful to recognize that since the consumer's utility function is strictly increasing, the constraint $R_{t+1}s_t \geq A_{t+1}$ will always bind at the optimum.

b) Now guess that the value function for this problem takes the form

$$v(A_t, R_t) = F \ln(A_t) + G \ln(R_t) + H,$$

where F, G, and H are unknown constants. Substitute this guess into your Bellman equation from above, and then write down the first-order condition for s_t and the envelope condition for A_t. In deriving these optimality conditions, you might find it helpful to note that since s_t is known at time t,

$$E_t \ln(s_t) = \ln(s_t).$$

You might also find it helpful to note that since

$$\ln(R_{t+1}) = \rho \ln(R_t) + \varepsilon_{t+1},$$

with $E_t \varepsilon_{t+1} = 0$, it follows that

$$E_t \ln(R_{t+1}) = \rho \ln(R_t).$$

c) Use your first-order condition from above to derive an expression linking the consumer's optimal choice s_t to the state variable A_t, the discount factor β, and the unknown constant F.

d) Next, use your envelope condition from above to solve for the unknown constant F in terms of the discount factor β.

e) Combine your results from above to obtain a set of two equations that can be used to construct the optimal paths for s_t and A_t, given the initial value of A_0 and the realized path for R_{t+1}. NOTE: It should not be necessary to solve for the unknown constants G and H to derive these equations and, since solving for G and H requires some extra algebra, DON'T BOTHER TO DO THIS.

Solutions to Final Exam

Economics 720: Mathematics for Economists

Fall 2007

1 The Kuhn-Tucker Theorem

The consumer solves

$$\max_{a,b} U(a,b) \text{ subject to } I \geq p^a a + p^b b,\ a \geq 0, \text{ and } b \geq 0.$$

a) There are a number of ways to define the Lagrangian for this problem, but one way is to treat the nonnegativity constraints symmetrically with the budget constraint and let

$$L(a,b,\lambda,\mu,\phi) = U(a,b) + \lambda(I - p^a a - p^b b) + \mu a + \phi b.$$

b) With the Lagragian defined as above, the first order conditions are

$$L_1(a^*,b^*,\lambda^*,\mu^*,\phi^*) = U_a(a^*,b^*) - \lambda^* p^a + \mu^* = 0 \tag{1}$$

and

$$L_2(a^*,b^*,\lambda^*,\mu^*,\phi^*) = U_b(a^*,b^*) - \lambda^* p^b + \phi^* = 0 \tag{2}$$

c) The Kuhn-Tucker theorem implies that in addition to the two first-order conditions stated above, the values a^* and b^* that solve the consumer's problem and the associated values λ^*, μ^*, and ϕ^* of the multipliers must satisfy the constraints

$$I \geq p^a a^* + p^b b^*,$$

$$a^* \geq 0,$$

and

$$b^* \geq 0$$

and the nonnegativity conditions

$$\lambda^* \geq 0,$$

$$\mu^* \geq 0,$$

and

$$\phi^* \geq 0.$$

d) The Kuhn-Tucker theorem also implies that the a^*, b^*, λ^*, μ^*, and ϕ^* must satisfy the complementary slackness conditions

$$\lambda^*(I - p^a a^* - p^b b^*) = 0,$$

$$\mu^* a^* = 0, \tag{3}$$

and

$$\phi^* b^* = 0. \tag{4}$$

e) The first-order conditions (1) and (2) imply that

$$\frac{U_a(a^*, b^*)}{U_b(a^*, b^*)} = \frac{\lambda^* p^a - \mu^*}{\lambda^* p^b - \phi^*}.$$

The right-hand side of this expression fails to equal p^a/p^b if either $\mu^* > 0$ or $\phi^* > 0$; the complementary slackness conditions (3) and (4) imply that one of these multipliers will be strictly positive if either $a^* = 0$ or $b^* = 0$. Hence, if the consumer is at a corner solution, consuming either no apples or no bananas, then the marginal rate of substitution $U_a(a^*, b^*)/U_b(a^*, b^*)$ between apples and bananas will not equal the relative price p^a/p^b of apples and bananas.

2 The Maximum Principle in Continuous Time

The problem involves choosing continuously differentiable functions $z(t)$ for $t \in [0, T]$ and $y(t)$ for $t \in (0, T]$ to maximize

$$\int_0^T e^{-\rho t} F(y(t), z(t); t) dt$$

subject to the constraints

$$Q(y(t), z(t); t) \geq \dot{y}(t) \text{ for all } t \in [0, T],$$

$$c \geq G(y(t), z(t); t) \text{ for all } t \in [0, T],$$

$$y(0) \text{ given}$$

and

$$y(T) \geq y^*.$$

a) Again, there are a number of different ways to set up the Hamiltonian for this problem, but one definition is

$$H(y(t), \pi(t); t) = \max_{z(t)} e^{-\rho t} F(y(t), z(t); t) + \pi(t) Q(y(t), z(t); t) \text{ subject to } c \geq G(y(t), z(t); t).$$

b) Letting $\lambda(t)$ denote the multiplier on the constraint that appears in the static optimization problem that appears in the definition of the Hamiltonian, the first-order condition for the value of $z(t)$ that solves this problem can be written as

$$e^{-\rho t}F_z(y(t),z(t);t)+\pi(t)Q_z(y(t),z(t);t)-\lambda(t)G_z(y(t),z(t);t)=0. \tag{5}$$

The maximum principle implies that this same first-order condition must also be satisfied by the value of $z(t)$ that solves the original, dynamic, optimization problem.

c) The maximum principle also implies that the values of $z(t)$ and $y(t)$ that solve the dynamic optimization problem must satisfy the pair of differential equations

$$\begin{aligned}\dot{\pi}(t) &= -H_y(y(t),\pi(t);t) \qquad (6)\\ &= -[e^{-\rho t}F_y(y(t),z(t);t)+\pi(t)Q_y(y(t),z(t);t)-\lambda(t)G_y(y(t),z(t);t)]\end{aligned}$$

and

$$\dot{y}(t)=H_\pi(y(t),\pi(t);t)=Q(y(t),z(t);t), \tag{7}$$

where the second equality in each of these expressions follows from an application of the envelope theorem to the Hamiltonian.

d) Together with the complementary slackness condition

$$\lambda(t)[c-G(y(t),z(t);t)]=0,$$

(5)-(7) form a system that can be used to solve for the values of $z(t)$ and $y(t)$ that solve the dynamic optimization problem and the associated values of $\pi(t)$ and $\lambda(t)$.

e) According to the maximum principle, the solution to the dynamic optimization problem must also satisfy two boundary conditions: the initial condition

$$y(0) \text{ given}$$

and the terminal, or transversality, condition

$$\pi(T)[y(T)-y^*]=0.$$

3 Optimal Growth in Continuous Time

The problem involves choosing continuously differential functions $c(t)$ for all $t\in[0,\infty)$ and $k(t)$ for all $t\in(0,\infty)$ to maximize

$$\int_0^\infty e^{-\rho t}\ln(c(t))dt$$

subject to the constraints $k(0)$ given and

$$k(t)^\alpha-\delta k(t)-c(t)\geq\dot{k}(t)$$

for all $t\in[0,\infty)$.

a) The Hamiltonian for this problem can be defined as

$$H(k(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

b) The first-order condition for the value of $c(t)$ that solves the static optimization problem that appears in the definition of the Hamiltonian can be written as

$$\frac{e^{-\rho t}}{c(t)} - \pi(t) = 0. \tag{8}$$

The maximum principle implies that this same first-order condition must also be satisfied by the value of $c(t)$ that solves the original, dynamic, optimization problem.

c) The maximum principle also implies that the values of $c(t)$ and $k(t)$ that solve the dynamic optimization problem must satisfy the pair of differential equations

$$\dot{\pi}(t) = -H_k(k(t), \pi(t); t) = -\pi(t)[\alpha k(t)^{\alpha-1} - \delta] \tag{9}$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = k(t)^\alpha - \delta k(t) - c(t), \tag{10}$$

where the second equality in each of these expressions follows from an application of the envelope theorem to the Hamiltonian.

d) A problem with the three-equation system (8)-(10) is that it involves not just the variables $c(t)$ and $k(t)$ from the original social planner's problem but also the new variable $\pi(t)$ that lacks a straightforward economic interpretation. To eliminate $\pi(t)$ from the system, rewrite (8) as

$$e^{-\rho t} = \pi(t)c(t)$$

and differentiate both sides to obtain

$$-\rho e^{-\rho t} = \pi(t)\dot{c}(t) + \dot{\pi}(t)c(t).$$

Now use (8) and (9) again to rewrite this last expression as

$$-\rho\pi(t)c(t) = \pi(t)\dot{c}(t) - \pi(t)[\alpha k(t)^{\alpha-1} - \delta]c(t).$$

Divide both sides of this expression by $\pi(t)$ and rearrange to obtain

$$\dot{c}(t) = [\alpha k(t)^{\alpha-1} - \delta - \rho]c(t). \tag{11}$$

Together, (10) and (11) now form a two-equation system involving only $c(t)$ and $k(t)$.

e) In a steady-state, $\dot{c}(t) = 0$. Hence, (11) requires that

$$k(t) = k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}.$$

Since $\dot{k}(t) = 0$ also holds in steady state, (10) implies that

$$c(t) = c^* = (k^*)^\alpha - \delta k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{\alpha/(\alpha-1)} - \delta\left(\frac{\delta + \rho}{\alpha}\right)^{1/(\alpha-1)}.$$

4 Optimal Resource Depletion

The problem involves choosing sequences $\{c_t\}_{t=0}^{\infty}$ and $\{x_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints x_0 given and $x_t - c_t \geq x_{t+1}$ for all $t = 0, 1, 2, ...$

a) With the Lagrangian defined as

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \pi_{t+1}(x_t - c_t - x_{t+1}),$$

the first-order condition for c_t is

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0 \tag{12}$$

and the first-order condition for x_t is

$$\pi_{t+1} - \pi_t = 0. \tag{13}$$

Together with the binding constraint

$$x_{t+1} = x_t - c_t, \tag{14}$$

(12) and (13) form a system of three equations in the three unknowns: the values of c_t and x_t that solve the dynamic optimization problem and the associated value of π_t.

b) With the Hamiltonian defined as

$$H(x_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) - \pi_{t+1} c_t,$$

the first-order condition for the static problem,

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0, \tag{15}$$

and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_x(x_t, \pi_{t+1}; t) = 0 \tag{16}$$

and

$$x_{t+1} - x_t = H_\pi(x_t, \pi_{t+1}; t) = -c_t \tag{17}$$

form a system of three equations in c_t, x_t, and π_t that coincides, of course, with (12)-(14).

c) The Bellman equation for the problem is

$$v(x_t; t) = \max_{c_t} \ln(c_t) + \beta v(x_t - c_t; t+1).$$

The first-order condition for the value of c_t that solves the static and unconstrained optimization problem on the right-hand side of the Bellman equation is

$$\frac{1}{c_t} - \beta v'(x_t - c_t; t+1) = 0. \tag{18}$$

The envelope condition for x_t is

$$v'(x_t; t) = \beta v'(x_t - c_t; t+1). \tag{19}$$

And the binding constraint is

$$x_{t+1} = x_t - c_t. \tag{20}$$

d) We know from our general analysis that (12)-(14), (15)-(17), and (18)-(20) provide identical information about the solution to the original dynamic optimization problem, so that we can work with whichever system is most convenient. Consider (18)-(20). Using (20), (18) can be written more simply as

$$\frac{1}{c_t} = \beta v'(x_{t+1}; t+1).$$

Since this condition must hold for all $t = 0, 1, 2, ...$, it also implies that

$$\frac{1}{c_{t-1}} = \beta v'(x_t; t).$$

Substituting (20) into (19) yields

$$v'(x_t; t) = \beta v'(x_{t+1}; t+1)$$

and hence

$$\frac{1}{\beta c_{t-1}} = \frac{1}{c_t}$$

or more simply

$$c_t = \beta c_{t-1}. \tag{21}$$

Equations (20) and (21) form a system of two equations the two unknowns: the values of c_t and x_t that solve the dynamic optimization problem.

e) Since the discount factor satisfies $0 < \beta < 1$, (21) implies that the optimal consumption c_t of the exhaustible resource is falling over time.

5 Saving with a Random Return

The problem involves choosing contingency plans for s_t, $t = 0, 1, 2, ...$, and A_t, $t = 1, 2, 3, ...$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t)$$

subject to the constraints

$$A_0 \text{ given}$$

and

$$R_{t+1} s_t \geq A_{t+1}$$

for all $t = 0, 1, 2, ...$ and all possible realizations of R_{t+1}.

a) The Bellman equation for this problem is

$$v(A_t, R_t) = \max_{s_t} \ln(A_t - s_t) + \beta E_t v(R_{t+1} s_t, R_{t+1}).$$

b) Using the conjectured form of the value function, the Bellman equation becomes

$$F \ln(A_t) + G \ln(R_t) + H = \max_{s_t} \ln(A_t - s_t) + \beta F E_t \ln(R_{t+1} s_t) + \beta G E_t \ln(R_{t+1}) + \beta H$$

or, more simply,

$$F \ln(A_t) + G \ln(R_t) + H = \max_{s_t} \ln(A_t - s_t) + \beta F \ln(s_t) + \beta \rho (F + G) \ln(R_t) + \beta H.$$

The first-order condition for s_t is therefore

$$-\frac{1}{A_t - s_t} + \frac{\beta F}{s_t} = 0$$

and the envelope condition for A_t is

$$\frac{F}{A_t} = \frac{1}{A_t - s_t}.$$

c) The first-order condition for s_t implies that

$$s_t = \left(\frac{\beta F}{1 + \beta F} \right) A_t.$$

d) Substituting this last expression into the envelope condition yields a solution for F in terms of β:

$$F A_t - F \left(\frac{\beta F}{1 + \beta F} \right) A_t = A_t$$

$$F \left[1 - \left(\frac{\beta F}{1 + \beta F} \right) \right] = 1$$

$$F\left(\frac{1}{1+\beta F}\right) = 1$$
$$F = 1+\beta F$$
$$F = \frac{1}{1-\beta}. \tag{22}$$

e) Now substitute the solution (22) for F back into the first-order condition for s_t to obtain

$$s_t = \beta A_t. \tag{23}$$

Finally, substitute (23) into the binding constraint to obtain

$$A_{t+1} = \beta R_{t+1} A_t. \tag{24}$$

Equations (23) and (24) form a system that can be used to construct the optimal paths for s_t and A_t, given the initial value of A_0 and the realized path for R_{t+1}. Although the solution does not require these last steps, it is also possible to substitute (22) -(24) back into the Bellman equation to find

$$G = \frac{\beta\rho}{(1-\beta\rho)(1-\beta)}$$

and

$$H = \left(\frac{1}{1-\beta}\right)\left[\ln(1-\beta) + \left(\frac{\beta}{1-\beta}\right)\ln(\beta)\right]$$

and verify that the conjectured form of the value function does satisfy the Bellman equation.

Midterm Exam

Economics 720: Mathematics for Economists

Fall 2008

This exam has three questions on three pages; before you begin, please make sure that your copy has all three questions and all three pages. The three questions will receive equal weight in determining your overall exam score. Since this is a 75-minute exam, I suggest that you spend approximately (75 minutes/3 questions) = 25 minutes on each question.

1) A consumer derives utility $U(c_1, c_2)$ from consuming two goods, where c_1 is his or her consumption of good 1 and c_2 is his or her consumption of good 2. Let $p_1 > 0$ denote the price of good 1, let $p_2 > 0$ denote the price of good 2, and assume that the consumer can purchase as many or as few units of each good as he or she likes at these competitively determined prices. Assume that the utility function is strictly increasing in both of its arguments, so that $U_1(c_1, c_2) > 0$ and $U_2(c_1, c_2) > 0$ for all values of $c_1 > 0$ and $c_2 > 0$, where U_i denotes the derivative of U with respect to its ith argument. Finally, assume that the utility function satisfies $\lim_{c_1 \to 0} U_1(c_1, c_2) = \infty$ for all values of c_2 and $\lim_{c_2 \to 0} U_1(c_1, c_2) = \infty$ for all values of c_1; these last assumptions will allow you to avoid having to explicitly impose nonnegativity constraints on the consumer's choices of c_1 and c_2 in all of the analysis that follows, since they imply that the consumer will always wish to purchase at least a small, positive amount of each good.

   a) Suppose first that the consumer takes his or her income I as given, and seeks to maximize utility subject to a budget constraint that says that the consumer's total expenditures cannot exceed income I. Write down (define) the Lagrangian for this utility maximization problem; then, write down the set of conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve the consumer's problem together with the associated value of the Lagrange multiplier.

   b) Explain how you would use your results from part (a) to derive demand curves for the two consumption goods.

   c) Suppose next that the consumer seeks to minimize the total cost of achieving a level of utility that is greater than or equal to $\bar{U}$. Write down (define) the Lagrangian for this cost or expenditure minimization problem; then, write down the set of conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^{**} and c_2^{**} that solve the consumer's problem together with the associated value of the Lagrange multiplier.

d) Explain how you would use your results from part (c) to derive demand curves for the two consumption goods.

e) What is the difference between the demand curves that you derived in part (b) and the demand curves that you derived in part (d)? Under what conditions, if any, will the solutions c_1^* and c_2^* that you described in parts (a) and (b) coincide with the solutions c_1^{**} and c_2^{**} that you described in parts (c) and (d)?

2) An econometrician collects data on a firm's output $y > 0$, the competitively-determined wage rate $w > 0$ and the rental rate $r > 0$ at which it hired labor and capital to produce that output, as well as the firm's total costs $C > 0$. Then, by fitting the log-linear equation

$$\ln(C) = \alpha \ln(w) + \beta \ln(r) + \gamma \ln(y)$$

where ln denotes the natural logarithm of each variable, to those data, the econometrician obtains the parameter estimates $\alpha = 0.5$, $\beta = 0.5$, and $\gamma = 1$. Using these estimates, assuming that the firm has always acted to minimize the costs of producing output, and assuming that the firm produces output y with capital k and l according to the production function $y = f(k, l)$, what conclusions can you draw about the specific form of this production function f?

3) A consumer works for periods $t = 0, 1, ..., T$, receiving constant labor income w during each of those periods. Let k_t denote the consumer's stock of wealth, that is, his or her bank account balance at the beginning of period t. Suppose that the consumer starts period $t = 0$ with nothing in the bank: $k_0 = 0$. Assume, also, that the bank does not allow the consumer to borrow, so that $k_t \geq 0$ must hold for all $t = 1, 2, ..., T+1$ as well. Let r denote the constant interest rate, and let c_t denote the consumer's consumption during period t. Finally, suppose that the consumer's utility function is additively-time separable and discounted, and that the single-period utility function is logarithmic in form. Putting all of these building blocks together, the consumer's problem can be stated as: choose sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ to maximize utility

$$\sum_{t=0}^{T} \beta^t \ln(c_t),$$

where β, with $0 < \beta < 1$, denotes the discount factor, subject to the constraints

$$w + rk_t - c_t \geq k_{t+1} - k_t$$

governing the evolution of the bank account balance for all $t = 0, 1, ..., T$, the constraints

$$k_t \geq 0$$

ruling out borrowing for all $t = 1, 2, ..., T$, the initial condition

$$k_0 = 0 \text{ given,}$$

and the constraint

$$k_{T+1} \geq 0$$

on the terminal value of the bank account balance (assuming here for simplicity that the consumer receives public pension income when retired and does not need to draw on funds from his or her bank account to consume after he or she has retired).

a) Write down the Hamiltonian for the consumer's problem. Hint: the easiest way to do this, and to be ready to derive the additional results below, is to note that since the constraints governing the evolution of the bank account balance will always bind, the no-borrowing constraints from the consumer's dynamic problem can be expressed equivalently as

$$w + (1+r)k_t - c_t \geq 0$$

for all $t = 0, 1, ..., T$. Writing the no-borrowing constraint in this way is convenient, since it clarifies how that constraint restricts the feasible choice of c_t during each period $t = 0, 1, ...T$ given the beginning-of-period bank account balance k_t.

b) Now write down the first-order condition, the pair of difference equations, and the initial and transversality conditions that, according to the maximum principle, must be satisfied by the sequences for $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ that solve the consumer's dynamic optimization problem.

c) Suppose now that the discount factor and the interest rate satisfy $\beta(1+r) = 1$ and guess (correctly, it turns out) that with this additional restriction on the parameters, the no-borrowing constraint will never bind, in the sense that the Lagrange multiplier attached to the additional constraint

$$w + (1+r)k_t - c_t \geq 0$$

will equal zero for all $t = 0, 1, ..., T$. Use your results from part (b) to answer the follow question: under these conditions, will the consumer's optimal consumption be rising, falling, or staying constant over time?

d) Use your results from parts (b) and (c) to find the sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ that solve the consumer's dynamic optimization problem.

Solutions to Midterm Exam

Economics 720: Mathematics for Economists

Fall 2008

1)

a) With the Lagrangian for the utility maximization problem defined as

$$L^u(c_1, c_2, \lambda) = U(c_1, c_2) + \lambda(I - p_1 c_1 - p_2 c_2),$$

the Kuhn-Tucker theorem implies that the optimal values c_1^* and c_2^* and the associated value of the Lagrange multiplier λ^* must satisfy the first-order conditions

$$L_1^u(c_1^*, c_2^*, \lambda^*) = U_1(c_1^*, c_2^*) - \lambda^* p_1 = 0$$

and

$$L_2^u(c_1^*, c_2^*, \lambda^*) = U_2(c_1^*, c_2^*) - \lambda^* p_2 = 0,$$

the constraint

$$I - p_1 c_1^* - p_2 c_2^* \geq 0,$$

the nonnegativity condition for the multiplier

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0.$$

b) Since U is strictly increasing and p_1 and p_2 are strictly positive, the first-order conditions imply that λ^* is strictly positive as well. It then follows from the complementary slackness condition that the budget constraint binds at the optimum. Use the two first-order conditions, together with the binding constraint, as a system of three equations to solve for the three unknowns c_1^*, c_2^*, and λ^* in terms of the parameters p_1, p_2, and I. The solutions $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$ can be interpreted as the demand curves for the two consumption goods.

c) With the Lagrangian for the expenditure minimization problem defined as

$$L^e(c_1, c_2, \mu) = p_1 c_1 + p_2 c_2 - \mu[U(c_1, c_2) - \bar{U}],$$

the Kuhn-Tucker theorem implies that the optimal values c_1^{**} and c_2^{**} and the associated value of the Lagrange multiplier μ^* must satisfy the first-order conditions

$$L_1^e(c_1^{**}, c_2^{**}, \mu^*) = p_1 - \mu^* U_1(c_1^{**}, c_2^{**}) = 0$$

and

$$L_2^e(c_1^{**}, c_2^{**}, \mu^*) = p_2 - \mu^* U_2(c_1^{**}, c_2^{**}) = 0,$$

the constraint

$$U(c_1^{**}, c_2^{**}) - \bar{U} \geq 0,$$

the nonnegativity condition for the multiplier

$$\mu^* \geq 0,$$

and the complementary slackness condition

$$\mu^*[U(c_1^{**}, c_2^{**}) - \bar{U}] = 0.$$

d) Since U is strictly increasing and p_1 and p_2 are strictly positive, the first-order conditions imply that μ^* is strictly positive as well. It then follows from the complementary slackness condition that the constraint binds at the optimum. Use the two first-order conditions, together with the binding constraint, as a system of three equations to solve for the three unknowns c_1^{**}, c_2^{**}, and μ^* in terms of the parameters p_1, p_2, and $\bar{U}$. The solutions $c_1^{**}(p_1, p_2, \bar{U})$ and $c_2^{**}(p_1, p_2, \bar{U})$ can be interpreted as the demand curves for the two consumption goods.

e) The demand curves $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$ describe how the consumer's choices change as prices and income vary; these are Marshallian demand curves. The demand curves $c_1^{**}(p_1, p_2, \bar{U})$ and $c_2^{**}(p_1, p_2, \bar{U})$ describe how the consumer's choices change as prices and utility vary; these are Hicksian demand curves. Define the indirect utility function associated with the utility maximization problem as

$$V(p_1, p_2, I) = \max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2$$

and define the expenditure function associated with the expenditure minimization problem as as

$$E(p_1, p_2, \bar{U}) = \min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

The solutions to the two problems will coincide when I is such that

$$I = E(p_1, p_2, \bar{U})$$

or, equivalently, when $\bar{U}$ is such that

$$\bar{U} = V(p_1, p_2, I).$$

2) The econometrician's estimates, together with the assumptions made about the firm's behavior, imply that the firm's minimum cost function, defined by

$$C(w,r,y) = \min_{w,l} wl + rk \text{ subject to } f(k,l) \geq y$$

takes the specific form

$$C(w,r,y) = w^{1/2}r^{1/2}y.$$

Shephard's lemma, which is just a special case of the envelope theorem applied to the firm's minimum cost function, implies that the firm's conditional factor demand curves can be found by differentiating the minimum cost function with respect to the two factor prices. Hence, in particular,

$$l^*(w,r,y) = C_1(w,r,y) = (1/2)w^{-1/2}r^{1/2}y$$

and

$$k^*(w,r,y) = C_2(w,r,y) = (1/2)w^{1/2}r^{-1/2}y.$$

There are a number of ways to use these last two expressions to determine the specific form of the firm's production function, but the easiest is to just multiply the first by the second to obtain

$$l^*(w,r,y)k^*(w,r,y) = (1/4)y^2.$$

Multiply both sides of this last equation by 4 and take the square root of both sides to obtain

$$y = 2[k^*(w,r,y)]^{1/2}[l^*(w,r,y)]^{1/2},$$

which, when compared to the binding constraint $y = f(k,l)$, reveals that

$$f(k,l) = 2k^{1/2}l^{1/2}.$$

Evidently, the firm's production function takes the Cobb-Douglas form, with equal shares for capital and labor.

3)

a) The Hamiltonian for the consumer's problem is

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(w + rk_t - c_t) \text{ subject to } w + (1+r)k_t - c_t \geq 0.$$

b) The maximum principle implies that if λ_t denotes the Lagrange multiplier on the constraint from the static optimization problem on the right-hand-side of the equation defining the Hamiltonian, the sequences $\{c_t\}_{t=0}^{T}$ and $\{k_t\}_{t=1}^{T+1}$ that solve the original, dynamic optimization problem must satisfy the first-order condition

$$\frac{\beta^t}{c_t} - \pi_{t+1} - \lambda_t = 0,$$

the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}r - \lambda_t(1+r)$$

and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = w + rk_t - c_t,$$

the initial condition

$$k_0 = 0,$$

and the transversality condition

$$\pi_{T+1} k_{T+1} = 0.$$

c) After guessing that $\lambda_t = 0$ for all $t = 0, 1, ..., T$, the first-order condition simplifies to

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0$$

and the first of the two difference equations simplifies to

$$(1 + r)\pi_{t+1} = \pi_t.$$

Using the first-order condition to substitute out for π_{t+1} and π_t and imposing the restriction that $\beta(1 + r) = 1$, this last expression becomes

$$c_t = c_{t-1},$$

revealing that optimal consumption is constant over time, with

$$c_t = c$$

for some constant c for all $t = 0, 1, ...T$.

d) To pin down the constant level of consumption c and to determine the optimal time path for the bank account balance k_t, consider the second of the two difference equations, which just corresponds to the binding constraint

$$k_{t+1} = w + (1 + r)k_t - c_t.$$

Using the result that $c_t = c$ for all $t = 0, 1, ..., T$, this equation implies that for $t = 0$,

$$k_1 = (1 + r)k_0 + w - c.$$

For $t = 1$,

$$k_2 = (1 + r)k_1 + w - c = (1 + r)^2 k_0 + (1 + r)(w - c) + w - c.$$

For $t = 2$,

$$k_3 = (1 + r)k_2 + w - c = (1 + r)^3 k_0 + (1 + r)^2(w - c) + (1 + r)(w - c) + w - c.$$

And hence for $t = T$,

$$k_{T+1} = (1 + r)^{T+1} k_0 + \sum_{t=0}^{T} (1 + r)^t (w - c).$$

The initial condition $k_0 = 0$ has been given to us by the specification of the original problem. Meanwhile, since the first-order condition from part (b) implies that

$$\pi_{T+1} = \frac{\beta^T}{c_T} > 0,$$

the transversality condition requires that $k_{T+1} = 0$ as well. So it must be true that

$$0 = \sum_{t=0}^{T} (1+r)^t (w - c),$$

which can only hold if $c = w$. Putting all these results together, it follows that the optimal path for consumption has

$$c_t = w$$

for all $t = 0, 1, ..., T$ and the optimal path for the bank account balance has

$$k_t = 0$$

for all $t = 0, 1, ..., T + 1$.

Final Exam

Economics 720: Mathematics for Economists

Fall 2008

This exam has three questions on five pages; before you begin, please make sure that your copy has all three questions and all five pages. The three questions will receive equal weight in determining your overall exam score. Since this is a 120-minute exam, I suggest that you spend approximately (120 minutes/3 questions) = 40 minutes on each question.

1 A Taste for Variety

Consider a static model in which a continuum of goods indexed by $j \in [0, n]$ can be produced and consumed at a single date. Hence, in this model, n is a parameter that measures the number of distinct goods that are available. Let $c(j)$ denote the number of units of each good $j \in [0, n]$ that a representative household consumes, let

$$C = \left[\int_0^n c(j)^\alpha dj \right]^{1/\alpha}$$

define an index of total consumption enjoyed by the household, and suppose that the household's preferences are described by the utility function

$$\ln(C),$$

defined in terms of the index C. With this specification, the parameter α lies between zero and one, $0 < \alpha < 1$, and is related to the degree of substitutability between the various goods; in particular, this specification implies that the elasticity of substitution between any two goods is $1/(1-\alpha)$, so that a value of α closer to zero means that the household is less willing to substitute between goods as their relative price changes and a value of α closer to one means that the household is more willing to substitute between goods as their relative price changes.

Suppose that each unit of each good can be produced with one unit of labor and that the household has X units of labor available in total to produce all of the various goods. Hence, this household chooses $c(j)$ for all $j \in [0, n]$ to maximize its utility

$$\ln \left\{ \left[\int_0^n c(j)^\alpha dj \right]^{1/\alpha} \right\}$$

subject to the production possibilities constraint

$$X \geq \int_0^n c(j)dj.$$

a) Set up (define) the Lagrangian for this problem.

b) Next, write down the set of conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values $c^*(j)$, $j \in [0, n]$, that solve the household's problem together with the associated value of the Lagrange multiplier.

c) Guess (correctly, it turns out) that it is optimal for the household to produce and consume equal amounts of all goods, so that $c^*(j) = c^*$ for all $j \in [0, n]$. Use this conjecture, together with your results from above, to solve for the constant c^* in terms of the model's parameters: X, n, and α.

d) Finally, define the household's indirect utility function as

$$V(X, n, \alpha) = \max_{c(j),\, j \in [0,n]} \ln \left\{ \left[\int_0^n c(j)^\alpha dj \right]^{1/\alpha} \right\} \text{ subject to } X \geq \int_0^n c(j)dj.$$

Using this definition, together with your results from above, to derive an expression that shows how the household's maximized utility depends on the parameters X, n, and α.

e) Does the household's maximized utility increase, decrease, or stay the same when the number n of available goods increases?

2 Expanding Product Variety

The previous static problem suggests that in a dynamic model, households with preferences as described above might be willing to allocate scare labor resources not just to the production of consumption goods but also to a process of research and development that leads to the invention of new products over time. To explore this possibility, consider a dynamic model cast in continuous time with an infinite horizon, so that $t \in [0, \infty)$, and let the representative household have preferences over this infinite horizon as described by the utility function

$$\int_0^\infty e^{-\rho t} U(C(t))dt,$$

where $\rho > 0$ is the discount rate and U is a single-period utility function defined in terms of the index of total consumption $C(t)$ at each date t, given as before by

$$C(t) = \left[\int_0^{n(t)} c(j;t)^\alpha dj \right]^{1/\alpha},$$

where $c(j;t)$ denotes consumption of each good $j \in [0, n(t)]$ at time t and now the number of product varieties $n(t)$ is also allowed to depend on time.

In this dynamic model, the household allocates its time across two activities. First, as before, it allocates total time $X(t)$ during each period t to producing the various consumption goods in existence using technologies that yield one unit of output for each unit of labor input. Hence, production possibilities in this "manufacturing sector" are described by

$$X(t) \geq \int_0^{n(t)} c(j;t)dj$$

for all $t \in [0, \infty)$. Second, the household allocates total time $l(t)$ to the process of research and development and, by doing so, comes up with new inventions that expand the range of goods available for production and consumption in future periods according to

$$zl(t) \geq \dot{n}(t)$$

for all $t \in [0, \infty)$, where z is a parameter that measures the productivity of labor in inventive activity: a higher value of z means that more inventions are found using a given amount of time allocated to research and development.

To complete the specification of the model, let the household be endowed with a constant amount of time L to be allocated within each period to either manufacturing or research and development, as summarized by the constraint

$$L \geq X(t) + l(t)$$

for all $t \in [0, \infty)$. Finally, assume that the form of the household's utility function implies that the household will always find it optimal to produce at least some consumption goods by devoting at least some time to manufacturing, so that the nonnegativity constraint $X(t) \geq 0$, if imposed, will never bind. On the other hand, it is possible that the household will choose not to "invest" any time in research and development, so that the nonnegativity constraint $l(t) \geq 0$ might bind at the optimum. It turns out that it is most convenient to express this nonnegativity constraint for $l(t)$ by requiring that

$$L - X(t) \geq 0$$

hold for all $t \in [0, \infty)$.

The household's optimization problem is easiest to solve in two stages. First, during any period t, let's ask how the household will use the time $X(t)$ that it allocates to manufacturing to produce an optimal mixture of the $n(t)$ goods that are available during that period. This static problem involves choosing $c(j;t)$ for all $j \in [0, n(t)]$ to maximize

$$U\left\{\left[\int_0^{n(t)} c(j;t)^\alpha dj\right]^{1/\alpha}\right\}$$

subject to the production possibilities constraint for manufacturing

$$X(t) \geq \int_0^{n(t)} c(j;t)dj.$$

This problem is very similar to the static problem that you solved above, except that now $n(t)$ and $X(t)$ may vary from period to period, so rather than work through all of the details of this sort of problem all over again, just assume for the sake of convenience that the single-period utility function U is such that the indirect utility function for each period t, defined in a manner that is similar to before as

$$V[X(t), n(t), \alpha] = \max_{c(j;t),\ j\in[0,n]} U\left\{\left[\int_0^{n(t)} c(j;t)^\alpha dj\right]^{1/\alpha}\right\} \text{ subject to } X(t) \geq \int_0^{n(t)} c(j;t)dj,$$

takes the double-log form

$$V[X(t), n(t), \alpha] = \ln[X(t)] + \beta \ln[n(t)],$$

where $\beta > 0$ is a positive parameter that depends in a nonlinear way on the elasticity of substitution $1/(1-\alpha)$.

Now the second, dynamic, part of the household's problem can be stated as: choose continuously differentiable functions $X(t)$ for $t \in [0, \infty)$ and $n(t)$ for $t \in (0, \infty)$ to maximize utility over the infinite horizon,

$$\int_0^\infty e^{-\rho t}\{\ln[X(t)] + \beta \ln[n(t)]\}dt,$$

subject to the constraint

$$z[L - X(t)] \geq \dot{n}(t)$$

linking the invention of new goods $\dot{n}(t)$ to the amount of labor $l(t) = L - X(t)$ allocated to research and development in each period $t \in [0, \infty)$, and the constraint

$$L - X(t) \geq 0$$

that requires the amount of labor $l(t) = L - X(t)$ allocated to research and development to be nonnegative in each period $t \in [0, \infty)$, taking the initial condition $n(0) = n_0$ as given.

a) Set up (define) the Hamiltonian for the household's dynamic problem as just described.

b) Now write down the first-order condition, the pair of differential equations, and the complementary slackness condition that, according to the maximum principle, must be satisfied by the functions $X(t)$ and $n(t)$ that solve the household's problem.

c) The most interesting solutions to this problem are such that Lagrange multiplier on the nonnegativity constraint $L - X(t) \geq 0$ equals zero for all $t \in [0, \infty)$; solutions of this form exist, it turns out, so long as the initial number of product varieties $n(0) = n_0$ is not too large. So suppose that the multipliers on the constraints $L - X(t) \geq 0$ are zero for all $t \in [0, \infty)$ and assume, as well, that the economy has reached a steady state in which $X(t) = X$ and $n(t) = n$ and hence $\dot{X}(t) = \dot{n}(t) = 0$ as well. Then, using your results from above, solve for the steady-state values X and n in terms of the parameters ρ, β, L, and z.

d) Recalling that n measures the number of distinct goods that are available in the steady state that you just described, answer the following questions. Does the number n of distinct goods in the steady state increase, decrease, or stay the same when the household gets more patient, that is, when ρ falls? Does n increase, decrease, or stay the same when the "weight" β on product variety in the household's indirect utility function rises? Does n increase, decrease, or stay the same when the total amount of productive time L rises? Does n increase, decrease, or stay the same when productivity in research and development z rises?

3 Stochastic Growth

Consider a discrete-time version of the optimal growth (Ramsey) model in which the horizon is infinite, so that time periods are indexed by $t = 0, 1, 2, ...$, the single-period utility function is logarithmic and the aggregate production function is Cobb-Douglas, capital depreciates fully in use during each period, and there are random, but serially uncorrelated, shocks to productivity. In this model, the representative consumer maximizes the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints k_0 given and

$$z_t k_t^\alpha \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, ...$, where z_t represents the shock to the productivity of capital. The value of z_t is known when c_t and k_{t+1} are chosen during period t, but the value of z_{t+1} is random and satisfies $E_t \ln(z_{t+1}) = 0$ for all $t = 0, 1, 2, ...$

a) Write down the Bellman equation for this problem.

b) Now guess that the value function takes the form

$$v(k_t, z_t) = E + F \ln(k_t) + G \ln(z_t),$$

where E, F, and G are unknown constants. Using this guess, derive (write down) the first order condition for the control variable c_t and the envelope condition for the state variable k_t.

c) Use your results from above to derive two equations that can be used to recursively construct the optimal values for c_t and k_{t+1} for each period $t = 0, 1, 2, ...$, given values for the parameters α and β, the initial condition for k_0, and the productivity shocks z_t that are realized at each date $t = 0, 1, 2, ...$

d) For the sake of completeness, write down solutions that show how the unknown constants E, F, and G depend on the parameters α and β.

Solutions to Final Exam

Economics 720: Mathematics for Economists

Fall 2008

1 A Taste for Variety

In the static model, the representative household chooses $c(j)$ for all $j \in [0, n]$ to maximize its utility

$$\ln \left\{ \left[\int_0^n c(j)^\alpha dj \right]^{1/\alpha} \right\}$$

subject to the production possibilities constraint

$$X \geq \int_0^n c(j) dj.$$

a) The Lagrangian for this problem can be defined as

$$L[c(j), \lambda] = \ln \left\{ \left[\int_0^n c(j)^\alpha dj \right]^{1/\alpha} \right\} + \lambda \left[X - \int_0^n c(j) dj \right].$$

b) According to the Kuhn-Tucker theorem, if the values $c^*(j)$, $j \in [0, n]$, solve the household's problem, then there exists a value λ^* of the Lagrange multiplier such that, together, these values satisfy the first-order conditions

$$\frac{c^*(j)^{\alpha-1}}{\int_0^n c^*(j)^\alpha dj} - \lambda^* = 0$$

for all $j \in [0, n]$, the constraint

$$X \geq \int_0^n c^*(j) dj,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^* \left[X - \int_0^n c^*(j) dj \right] = 0.$$

c) Rearrange the first-order conditions so that they read

$$c^*(j)^{\alpha-1} = \lambda^* \left[\int_0^n c^*(j)^\alpha dj\right].$$

for all $j \in [0, n]$. Since the right-hand side of this expression does not depend on j, the condition implies that it is optimal for the household to produce and consume equal amounts of all goods, so that $c^*(j) = c^*$ for all $j \in [0, n]$, where c^* and λ^* must satisfy

$$(c^*)^{\alpha-1} = \lambda^* \left[\int_0^n (c^*)^\alpha dj\right] = \lambda^*(c^*)^\alpha n$$

or, more simply,

$$1 = \lambda^*(c^*)^\alpha n.$$

Since c^* and n are both strictly positive, this last expression implies that λ^* must be strictly positive as well. The complementary slackness condition then implies that the budget constraint must bind at the optimum, so that c^* must also satisfy

$$X = \int_0^n c^* dj = c^* n.$$

Rearranging this last result yields the desired solution for c^*:

$$c^* = X/n.$$

d) The indirect utility function, defined as

$$V(X, n, \alpha) = \max_{c(j),\ j\in[0,n]} \ln\left\{\left[\int_0^n c(j)^\alpha dj\right]^{1/\alpha}\right\} \text{ subject to } X \geq \int_0^n c(j)dj,$$

is a maximum value function. Hence, to evaluate this function, we need to use a two-step procedure. But we've already finished step one: given settings for the parameters X, n, and α, we know that the optimal choices for consumptions are

$$c^*(j) = X/n$$

for all $j \in [0, n]$. Step two now involves substituting this solution back into the objective function to obtain

$$\begin{aligned} V(X, n, \alpha) &= \ln\left\{\left[\int_0^n c^*(j)^\alpha dj\right]^{1/\alpha}\right\} \\ &= \ln\left\{\left[\int_0^n (X/n)^\alpha dj\right]^{1/\alpha}\right\} \\ &= \ln[(X/n)^\alpha n]^{1/\alpha} \\ &= \ln(X) + \left(\frac{1-\alpha}{\alpha}\right)\ln(n). \end{aligned}$$

e) Since $0 < \alpha < 1$, the last expression from part (d) above implies that the household's maximized utility increases when the number n of available goods increases.

2 Expanding Product Variety

In the dynamic model, the representative household chooses continuously differentiable functions $X(t)$ for $t \in [0, \infty)$ and $n(t)$ for $t \in (0, \infty)$ to maximize utility over the infinite horizon,

$$\int_0^\infty e^{-\rho t}\{\ln[X(t)] + \beta \ln[n(t)]\}dt,$$

subject to the constraint

$$z[L - X(t)] \geq \dot{n}(t)$$

linking the invention of new goods $\dot{n}(t)$ to the amount of labor $l(t) = L - X(t)$ allocated to research and development in each period $t \in [0, \infty)$, and the constraint

$$L - X(t) \geq 0$$

that requires the amount of labor $l(t) = L - X(t)$ allocated to research and development to be nonnegative in each period $t \in [0, \infty)$, taking the initial condition $n(0) = n_0$ as given. Note that the form of the indirect utility function from part (d) of question 1 implies that the new parameter β is related to the elasticity parameter α according to $\beta = (1 - \alpha)/\alpha$.

a) The Hamiltonian for this problem can be defined as

$$H[n(t), \pi(t); t] = \max_{X(t)} e^{-\rho t}\{\ln[X(t)] + \beta \ln[n(t)]\} + \pi(t)z[L - X(t)] \text{ subject to } L - X(t) \geq 0.$$

b) According to the maximum principle, the functions $X(t)$ and $n(t)$ that solve the household's problem must satisfy the first-order condition for the static problem that is described on the right-hand side of the Hamiltonian. Letting $\lambda(t)$ denote the Lagrange multiplier on the constraint in that problem, this first-order condition can be written as

$$\frac{e^{-\rho t}}{X(t)} - \pi(t)z - \lambda(t) = 0.$$

According to the maximum principle, the solution to the household's problem must also satisfy the pair of differential equations

$$\dot{\pi}(t) = -H_n[n(t), \pi(t); t] = -\frac{e^{-\rho t}\beta}{n(t)}$$

and

$$\dot{n}(t) = H_\pi[n(t), \pi(t); t] = z[L - X(t)]$$

and the complementary slackness condition corresponding to the constraint from the static problem:

$$\lambda(t)[L - X(t)] = 0.$$

c) In a steady state with $X(t) = X$ and $n(t) = n$ and hence $\dot{X}(t) = \dot{n}(t) = 0$, the second differential equation from part (b), above, implies that the steady-state value of X is given by

$$X = L.$$

This part of the solution simply confirms that in a steady state where no new products are being invented, all of the household's time gets allocated to manufacturing. When $\lambda(t) = 0$ as well, the first-order condition from above implies that

$$\pi(t) = \frac{e^{-\rho t}}{zX(t)} = \frac{e^{-\rho t}}{zL}$$

so that

$$\dot{\pi}(t) = -\frac{\rho e^{-\rho t}}{zL}.$$

Substituting this last result into the first differential equation yields

$$-\frac{\rho e^{-\rho t}}{zL} = -\frac{e^{-\rho t}\beta}{n(t)},$$

which can be rearranged to obtain the desired solution for the steady-state value of n:

$$n = \frac{\beta z L}{\rho}.$$

d) The solution described in part (c), above, implies that the steady-state number of product varieties n increases when ρ falls. This makes sense: a more patient household, with a smaller discount rate ρ, will be more willing to forgo consumption in order to "invest" in research and development. The solution also implies that the number of product varieties will increase when β increases, so that the household attaches a higher weight to product variety in preferences, when L increases, so that the household has more productive time to allocate to research and development as well as manufacturing, and when z increases, so that productivity in research in development is higher.

3 Stochastic Growth

In this stochastic version of the Ramsey model, the representative consumer maximizes the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints k_0 given and

$$z_t k_t^{\alpha} \geq c_t + k_{t+1}$$

for all $t = 0, 1, 2, \ldots$, where z_t represents the shock to the productivity of capital. The value of z_t is known when c_t and k_{t+1} are chosen during period t, but the value of z_{t+1} is random and satisfies $E_t \ln(z_{t+1}) = 0$ for all $t = 0, 1, 2, \ldots$.

a) The Bellman equation for this problem can be written as

$$v(k_t, z_t) = \max_{c_t} \ln(c_t) + \beta E_t v(z_t k_t^\alpha - c_t, z_{t+1}).$$

b) Using the guess

$$v(k_t, z_t) = E + F \ln(k_t) + G \ln(z_t),$$

the Bellman equation becomes

$$\begin{aligned} E + F \ln(k_t) + G \ln(z_t) &= \max_{c_t} \ln(c_t) + \beta E + \beta F E_t \ln(z_t k_t^\alpha - c_t) + \beta G E_t \ln(z_{t+1}) \\ &= \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(z_t k_t^\alpha - c_t). \end{aligned}$$

The first-order condition for c_t is therefore

$$\frac{1}{c_t} - \frac{\beta F}{z_t k_t^\alpha - c_t} = 0$$

and the envelope condition for k_t is

$$\frac{F}{k_t} = \frac{\alpha \beta F z_t k_t^{\alpha-1}}{z_t k_t^\alpha - c_t}.$$

c) There are many ways to derive the desired set of equations, but one way starts by rearranging the first-order condition to obtain

$$c_t = \left(\frac{1}{1+\beta F}\right) z_t k_t^\alpha,$$

then substituting this expression into the envelope condition and rearranging to obtain

$$\frac{1}{1+\beta F} = 1 - \alpha\beta.$$

Substitute the second of these last two results into the first to obtain

$$c_t = (1 - \alpha\beta) z_t k_t^\alpha,$$

then substitute this result into the binding constraint to obtain

$$k_{t+1} = \alpha\beta z_t k_t^\alpha.$$

These two equations confirm that even in the presence of uncertainty, it is optimal for the household to consume the fixed fraction $1-\alpha\beta$ of output and to save and invest the rest. Given the values of the parameters α and β, the initial condition for k_0, and the productivity shocks z_t that are realized at each date $t = 0, 1, 2, ...$, these two equations can also be used to construct the optimal values of c_t and k_{t+1} for all $t = 0, 1, 2, ...$

d) The condition

$$\frac{1}{1+\beta F} = 1-\alpha\beta.$$

derived above can be rearranged to obtain the desired solution for F:

$$F = \frac{\alpha}{1-\alpha\beta}.$$

Substituting this solution for F, together with the solutions for c_t and k_{t+1} derived above, back into the Bellman equation yields

$$\begin{aligned} E + \left(\frac{\alpha}{1-\alpha\beta}\right)\ln(k_t) + G\ln(z_t) &= \max_{c_t} \ln(c_t) + \beta E + \beta F \ln(z_t k_t^\alpha - c_t) \\ &= \ln(1-\alpha\beta) + \ln(z_t) + \alpha\ln(k_t) \\ &\quad +\beta E + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta) \\ &\quad + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(z_t) + \alpha\left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(k_t). \end{aligned}$$

Since

$$\alpha + \alpha\left(\frac{\alpha\beta}{1-\alpha\beta}\right) = \frac{\alpha}{1-\alpha\beta},$$

the terms involving $\ln(k_t)$ cancel from both sides, leaving

$$E + G\ln(z_t) = \ln(1-\alpha\beta) + \ln(z_t) + \beta E + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(z_t).$$

And since this last expression must hold for all possible values of z_t, it requires that

$$G = 1 + \frac{\alpha\beta}{1-\alpha\beta} = \frac{1}{1-\alpha\beta}$$

and

$$E = \ln(1-\alpha\beta) + \beta E + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta)$$

or

$$E = \left(\frac{1}{1-\beta}\right)\left[\ln(1-\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta)\right].$$

Final Exam

EC720.01 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2009

Sunday, December 20, 2009

This exam has two questions on four pages; before you begin, please check to make sure your copy has all two questions and all four pages. The first question is worth 40 points and the second question is worth 20 points; as indicated at the beginning of the semester, this 60-point final will account for 60 percent of your course grade and the problem sets will account for the remaining 40 percent of your course grade.

1. The Maximum Principle in Continuous Time

This question asks you to use the maximum principle to solve a continuous-time, infinite-horizon model in which economic growth is driven by the accumulation of both physical and human capital.

During each period $t \in [0, \infty)$, a representative consumer divides up a total stock of physical capital $k(t)$ into a fraction $u(t)$ used to produce goods for consumption and investment in physical capital and a fraction $1 - u(t)$ used to produce additional human capital. Likewise, the consumer "divides up" a total stock of human capital $h(t)$ by spending a fraction $v(t)$ of his or her time producing goods for consumption and investment in physical capital and a fraction $1 - v(t)$ of his or her time accumulating additional human capital. Hence, $u(t)k(t)$ and $v(t)h(t)$ measure the total amounts of physical and human capital used to produce goods and $[1-u(t)]k(t)$ and $[1-v(t)]h(t)$ measure the total amounts of physical and human capital used to accumulate more human capital.

Suppose that goods and new human capital both get produced according to the same Cobb-Douglas production function in which the parameter α, satisfying $0 < \alpha < 1$, measures the share of physical versus human capital in production. Suppose also that both capital stocks depreciate at the same rate, measured by the parameter δ satisfying $0 < \delta < 1$.

Then in this economy, physical capital gets accumulated according to the constraint

$$[u(t)k(t)]^\alpha [v(t)h(t)]^{1-\alpha} - \delta k(t) - c(t) \geq \dot{k}(t), \tag{1}$$

for all $t \in [0, \infty)$. In (1), $c(t)$ denotes the amount of goods consumed during period t. Thus, when this constraint holds as an equality, as it will when the consumer chooses quantities optimally, it says that the total amount $[u(t)k(t)]^\alpha [v(t)h(t)]^{1-\alpha}$ of goods produced during each period t gets used for three purposes: (1) to replace physical capital that depreciates away, (2) for consumption, and (3) to add, on net, to the stock of physical capital.

Meanwhile, human capital gets accumulated according to the constraint

$$\{[1 - u(t)]k(t)\}^\alpha \{[1 - v(t)]h(t)\}^{1-\alpha} - \delta h(t) \geq \dot{h}(t), \tag{2}$$

for all $t \in [0, \infty)$. This constraint will also hold as an equality when the consumer chooses quantities optimally. Hence, it says that the total amount $\{[1-u(t)]k(t)\}^{\alpha}\{[1-v(t)]h(t)\}^{1-\alpha}$ of human capital produced during each period t gets used for two purposes: (1) to replace human capital that depreciates away and (2) to add, on net, to the stock of human capital.

Comparing (1) and (2) also reveals that even under the simplifying assumptions made above that the production functions and depreciation rates are the same for physical and human capital, there is a key economic distinction between goods and human capital, because goods can be eaten but human capital cannot.

Suppose, finally, that the representative consumer's utility from consuming $c(t)$ units of the good at each date $t \in [0, \infty)$ is given by

$$\int_0^{\infty} e^{-\rho t} \ln(c(t))\,\mathrm{d}t, \tag{3}$$

where the single-period utility function takes the natural log form and where the discount rate ρ satisfies $\rho > 0$.

Thus, the consumer's problem can be stated as one of choosing continuously differentiable functions $c(t)$, $u(t)$, $v(t)$, $k(t)$, and $h(t)$ for all $t \in [0, \infty)$ to maximize the utility function in (3) subject to the constraints in (1) and (2) for all $t \in [0, \infty)$, taking as given the initial stocks $k(0)$ and $h(0)$ of physical and human capital.

a. (10 points) Write down the Hamiltonian for the consumer's problem. In doing so, you will need to account for the fact that the underlying dynamic optimization problem has three flow variables, $c(t)$, $u(t)$, and $v(t)$, so that the static optimization problem on the right-hand side of your definition of the Hamiltonian should have three choice variables as well. You will also need to account for the fact that the underlying problem has two stock variables, $k(t)$ and $h(t)$, so that you will have to introduce two new variables into the problem, corresponding to multipliers on the capital accumulation constraints (1) and (2), and then add two extra terms to the period-t objective function when setting up the Hamiltonian. To keep things simple, however, you can assume at this point that nonnegativity constraints on $c(t)$, $u(t)$, $1-u(t)$, $v(t)$, $1-v(t)$, $k(t)$, and $h(t)$ will never bind at the optimum, so that these constraints can be left out when defining the Hamiltonian. And you can assume, as indicated above, that the constraints (1) and (2) will always hold with equality at the optimum.

b. (20 points) Now write down three first-order conditions and four differential equations that, according to the maximum principle, must be satisfied by the values of $c(t)$, $u(t)$, $v(t)$, $k(t)$, and $h(t)$ that solve the consumer's problem together with the associated values of the two new variables that you introduced into the problem when defining the Hamiltonian.

c. (10 points) The equations that you derived in part (b) above imply that this economy has a "balanced growth path," along which the fractions $u(t)$ and $v(t)$ are constant, with $u(t) = u$ and $v(t) = v$ for all $t \in [0, \infty)$ and along which consumption $c(t)$ and the two capital stocks $k(t)$ and $h(t)$ all grow at the same constant rate γ, so that

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{\dot{h}(t)}{h(t)} = \gamma$$

for all $t \in [0, \infty)$. Use your results from above to obtain solutions for these three constants, u, v, and γ, in terms of the model's parameters: α, δ, and ρ.

2. Stochastic Dynamic Programming

This question asks you to use dynamic programming to characterize the solution to a stochastic version of the Ramsey model in discrete time where labor supply as well as consumption gets chosen optimally during each period $t = 0, 1, 2, ...$ in an infinite horizon.

To start by fixing notation, let K_t denote the stock of physical capital at the beginning of each period $t = 0, 1, 2, ...$ Let C_t denote consumption and let H_t denote hours worked by a representative consumer during each period $t = 0, 1, 2, ...$ Suppose that during each period t, output gets produced according to the Cobb-Douglas production function

$$Z_t K_t^\alpha H_t^{1-\alpha}, \tag{4}$$

where the parameter α, measuring the share of capital versus labor in production, satisfies $0 < \alpha < 1$ and where Z_t is a stochastic (random) shock to productivity. As in our general analysis from class, assume that Z_t gets realized at the very beginning of period t. Hence, when the consumer chooses C_t and H_t during period t, he or she knows the true value of the period-t shock Z_t but still views the value of the period-$t+1$ shock Z_{t+1} as random. Assume that Z_t is serially correlated but follows a Markov process; that is, that the probability distribution of Z_{t+1}, though it can depend on Z_t, cannot depend also on Z_{t-1}, Z_{t-2}, and so on backwards in time.

During each period t, the representative consumer divides up the output measured by (4) into amounts to be consumed and invested. Thus, if physical capital depreciates at a rate measured by the parameter δ satisfying $0 < \delta < 1$, then the resource constraint

$$Z_t K_t^\alpha H_t^{1-\alpha} \geq C_t + K_{t+1} - (1 - \delta) K_t \tag{5}$$

must hold for all periods $t = 0, 1, 2, ...$ and all possible realizations of Z_t. Note that (5) is consistent with the fact that the capital stock K_{t+1} that is available at the beginning of period $t + 1$ equals the amount of capital K_t that is available at the beginning of period t, plus the new output $Z_t K_t^\alpha H_t^{1-\alpha}$ that gets produced during period t, minus the amount of output C_t consumed during period t, and minus the amount of capital δK_t that depreciates away during period t.

Finally, suppose that the representative consumer's expected utility over the infinite horizon is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln(C_t) + \theta \ln(1 - H_t)], \tag{6}$$

where the discount factor β satisfies $0 < \beta < 1$, the total amount of time available to the consumer during each period is normalized to equal one, so that $1 - H_t$ denotes leisure enjoyed by the consumer during each period $t = 0, 1, 2, ...$, and the positive parameter $\theta > 0$ measure the weight on leisure versus consumption in a single-period utility function where both consumption and leisure enter in the natural log form.

Although this problem can be solved in a variety of ways, it turns out to be easiest to start by defining a new variable

$$S_t = Z_t K_t^\alpha H_t^{1-\alpha} + (1-\delta)K_t - C_t \tag{7}$$

as a measure of saving during each period t and to use this definition to rewrite the expected utility function in (6) as

$$E\sum_{t=0}^{\infty} \beta^t \{\ln[Z_t K_t^\alpha H_t^{1-\alpha} + (1-\delta)K_t - S_t] + \theta \ln(1 - H_t)\} \tag{8}$$

and then to rewrite the capital accumulation constraint in (5) more simply as

$$S_t \geq K_{t+1} \tag{9}$$

for all $t = 0, 1, 2, ...$

Now the representative consumer's problem can be stated as one of choosing contingency plans for S_t and H_t for all $t = 0, 1, 2, ...$ and K_t for all $t = 1, 2, 3, ...$ to maximize the expected utility function in (8) subject to the capital accumulation constraint in (9), which again must hold for for all periods $t = 0, 1, 2, ...$, taking the initial conditions K_0 and Z_0 as given.

a. (10 points) Write down the Bellman equation for this stochastic problem. Then use the Bellman equation to derive the first-order and envelope conditions that characterize the values of S_t, H_t, and K_t that solve the consumer's problem.

b. (10 points) A difficulty that arises in trying to interpret the optimality conditions that you derived in part (a) above is that they make reference not just to variables like C_t, H_t, K_t, Z_t, and S_t and parameters like α, δ, β, and θ that have straightforward economic interpretations, but also to the derivative of the unknown value function that you introduced into this problem when setting up the Bellman equation. Accordingly, use one of the optimality conditions to eliminate the derivative of the value function from the system. Then, by invoking the constraints (5), (7), and (9) as needed, assuming correctly that all will hold as equalities at the optimum, rewrite your results from part (a) above as a system of three equations involving only the unknown variables C_t, H_t, and K_t that solve the consumer's problem, the model's parameters α, δ, β, and θ, and the exogenous shock Z_t. *Note:* You will not be able to solve this system of equations explicitly for optimal choices of C_t, H_t, and K_t, but you should be able to derive the system of three equations that, in principle at least, could be used to solve for these unknown values for given values of the parameters α, δ, β, and θ and the exogenous shock Z_t.

Solutions to Final Exam

EC720.01 - Math for Economists
Boston College, Department of Economics

Peter Ireland
Fall 2009

Sunday, December 20, 2009

1. The Maximum Principle in Continuous Time

The consumer's problem can be stated as one of choosing continuously differentiable functions $c(t)$, $u(t)$, $v(t)$, $k(t)$, and $h(t)$ for all $t \in [0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t, \tag{3}$$

subject to the constraints

$$[u(t)k(t)]^\alpha [v(t)h(t)]^{1-\alpha} - \delta k(t) - c(t) \geq \dot{k}(t) \tag{1}$$

and

$$\{[1-u(t)]k(t)\}^\alpha \{[1-v(t)]h(t)\}^{1-\alpha} - \delta h(t) \geq \dot{h}(t), \tag{2}$$

each of which must hold for all $t \in [0, \infty)$, taking $k(0)$ and $h(0)$ as given.

a. The Hamiltonian for this problem can be written in present-value terms as

$$\begin{aligned} H(k(t), h(t), \pi(t), \theta(t)) \quad = \quad & \max_{c(t),u(t),v(t)} e^{-\rho t} \ln(c(t)) \\ & +\pi(t)\{[u(t)k(t)]^\alpha [v(t)h(t)]^{1-\alpha} - \delta k(t) - c(t)\} \\ & +\theta(t)\{[(1-u(t))k(t)]^\alpha [(1-v(t))h(t)]^{1-\alpha} - \delta h(t)\}. \end{aligned}$$

b. According to the maximum principle, the values of $c(t)$, $u(t)$, $v(t)$, $k(t)$, and $h(t)$ that solve the consumer's problem together with the associated values of $\pi(t)$ and $\theta(t)$, must satisfy the first-order conditions

$$\frac{e^{-\rho t}}{c(t)} - \pi(t) = 0, \tag{10}$$

$$\begin{aligned} &\alpha\pi(t)u(t)^{\alpha-1}k(t)^\alpha v(t)^{1-\alpha}h(t)^{1-\alpha} \\ &- \alpha\theta(t)(1-u(t))^{\alpha-1}k(t)^\alpha(1-v(t))^{1-\alpha}h(t)^{1-\alpha} = 0, \end{aligned} \tag{11}$$

and

$$\begin{aligned} &(1-\alpha)\pi(t)u(t)^\alpha k(t)^\alpha v(t)^{-\alpha}h(t)^{1-\alpha} \\ &- (1-\alpha)\theta(t)(1-u(t))^\alpha k(t)^\alpha(1-v(t))^{-\alpha}h(t)^{1-\alpha} = 0, \end{aligned} \tag{12}$$

and the differential equations

$$\begin{aligned}\dot{\pi}(t) =& -H_k(k(t), h(t), \pi(t), \theta(t)) \\ =& -\pi(t)[\alpha u(t)^\alpha k(t)^{\alpha-1} v(t)^{1-\alpha} h(t)^{1-\alpha} - \delta] \\ & -\theta(t)[\alpha(1-u(t))^\alpha k(t)^{\alpha-1}(1-v(t))^{1-\alpha} h(t)^{1-\alpha}],\end{aligned} \tag{13}$$

$$\begin{aligned}\dot{\theta}(t) =& -H_h(k(t), h(t), \pi(t), \theta(t)) \\ =& -\pi(t)[(1-\alpha)u(t)^\alpha k(t)^\alpha v(t)^{1-\alpha} h(t)^{-\alpha}] \\ & -\theta(t)[(1-\alpha)(1-u(t))^\alpha k(t)^\alpha (1-v(t))^{1-\alpha} h(t)^{-\alpha} - \delta],\end{aligned} \tag{14}$$

$$\dot{k}(t) = H_\pi(k(t), h(t), \pi(t), \theta(t)) = [u(t)k(t)]^\alpha [v(t)h(t)]^{1-\alpha} - \delta k(t) - c(t), \tag{15}$$

and

$$\dot{h}(t) = H_\theta(k(t), h(t), \pi(t), \theta(t)) = [(1-u(t))k(t)]^\alpha [(1-v(t))h(t)]^{1-\alpha} - \delta h(t) \tag{16}$$

c. Along the balanced growth path, $u(t) = u$, $v(t) = v$, and

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{\dot{h}(t)}{h(t)} = \gamma$$

for all $t \in [0, \infty)$. Hence, (11) and (12) imply that along this balanced growth path

$$\pi(t) u^{\alpha-1} v^{1-\alpha} = \theta(t)(1-u)^{\alpha-1}(1-v)^{1-\alpha}$$

and

$$\pi(t) u^\alpha v^{-\alpha} = \theta(t)(1-u)^\alpha (1-v)^{-\alpha}.$$

Dividing the first of these equations by the second yields

$$\frac{v}{u} = \frac{1-v}{1-u},$$

which requires that

$$u = v. \tag{17}$$

Hence, (11) and (12) also require that

$$\pi(t) = \theta(t) \tag{18}$$

and hence

$$\dot{\pi}(t) = \dot{\theta}(t) \tag{19}$$

for all $t \in [0, \infty)$ along the balanced growth path as well. In light of (17)-(19), equations (13) and (14) require that

$$\dot{\pi}(t) = -\pi(t)[\alpha k(t)^{\alpha-1} h(t)^{1-\alpha} - \delta]$$

and

$$\dot{\pi}(t) = -\pi(t)[(1-\alpha)k(t)^\alpha h(t)^{-\alpha} - \delta].$$

These last two equations imply that the ratio of physical to human capital must be constant, with

$$\frac{k(t)}{h(t)} = \frac{\alpha}{1-\alpha} \tag{20}$$

for all $t \in [0, \infty)$, along the balanced growth path. Substituting (20) back into either (13) or (14) then reveals that

$$\frac{\dot{\pi}(t)}{\pi(t)} = \delta - \alpha^{\alpha}(1-\alpha)^{1-\alpha} \tag{21}$$

for all $t \in [0, \infty)$ as well. Rewriting (10) as

$$e^{-\rho t} = c(t)\pi(t)$$

and differentiating both sides with respect to t yields

$$-\rho e^{-\rho t} = \dot{c}(t)\pi(t) + c(t)\dot{\pi}(t)$$

or, using (10) again,

$$-\rho c(t)\pi(t) = \dot{c}(t)\pi(t) + c(t)\dot{\pi}(t).$$

Dividing this last expression through by $\pi(t)$ and using (21) yields

$$\frac{\dot{c}(t)}{c(t)} = \alpha^{\alpha}(1-\alpha)^{1-\alpha} - \delta - \rho$$

providing the solution for γ:

$$\gamma = \alpha^{\alpha}(1-\alpha)^{1-\alpha} - \delta - \rho. \tag{22}$$

To determine $u = v$, return to the binding constraint (16) for human capital accumulation, which along the balanced growth path requires

$$\dot{h}(t) = (1-u)k(t)^{\alpha}h(t)^{1-\alpha} - \delta h(t)$$

or

$$\frac{\dot{h}(t)}{h(t)} = (1-u)k(t)^{\alpha}h(t)^{-\alpha} - \delta.$$

In light of (20) and (22), this last expression requires that

$$\alpha^{\alpha}(1-\alpha)^{1-\alpha} - \delta - \rho = (1-u)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} - \delta$$

or more simply

$$u = v = \alpha + \rho\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}. \tag{23}$$

2. Stochastic Dynamic Programming

The consumer's problem can be stated as one of choosing contingency plans for S_t and H_t for all $t = 0, 1, 2, ...$ and K_t for all $t = 1, 2, 3, ...$ to maximize the expected utility function

$$E \sum_{t=0}^{\infty} \beta^t \{\ln[Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t - S_t] + \theta \ln(1 - H_t)\} \tag{8}$$

subject to the constraints

$$S_t \geq K_{t+1} \tag{9}$$

for all $t = 0, 1, 2, ...$, taking K_0 and Z_0 as given.

a. With the Bellman equation for this stochastic problem written as

$$v(K_t, Z_t) = \max_{S_t, Ht} \ln[Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t - S_t] + \theta \ln(1 - H_t) + \beta E_t v(S_t, Z_{t+1}),$$

the first-order conditions characterizing the optimal choices of S_t and H_t can be written as

$$-\frac{1}{Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t - S_t} + \beta E_t v_1(S_t, Z_{t+1}) = 0$$

and

$$\frac{(1-\alpha) Z_t K_t^{\alpha} H_t^{-\alpha}}{Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t - S_t} - \frac{\theta}{1 - H_t} = 0$$

and the envelope condition characterizing the optimal choice of K_t can be written as

$$v_1(K_t, Z_t) = \frac{\alpha Z_t K_t^{\alpha-1} H_t^{1-\alpha} + 1 - \delta}{Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t - S_t}.$$

b. To simplify the optimality conditions to facilitate their economic interpretation, start by using the definition of S_t and the binding constraint $S_t = K_{t+1}$ to rewrite the first-order and envelope conditions from above as

$$\frac{1}{C_t} = \beta E_t v_1(K_{t+1}, Z_{t+1}),$$

$$\frac{(1-\alpha) Z_t K_t^{\alpha} H_t^{-\alpha}}{C_t} = \frac{\theta}{1 - H_t},$$

and

$$v_1(K_t, Z_t) = \frac{\alpha Z_t K_t^{\alpha-1} H_t^{1-\alpha} + 1 - \delta}{C_t}.$$

Next, note that in characterizing the solution to the original dynamic problem, the envelope condition must hold for period $t+1$ as well as for period t. Hence, it implies that

$$v_1(K_{t+1}, Z_{t+1}) = \frac{\alpha Z_{t+1} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + 1 - \delta}{C_{t+1}}.$$

Use this last result to rewrite the first-order condition for C_t as

$$\frac{1}{C_t} = \beta E_t \left[\frac{\alpha Z_{t+1} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + 1 - \delta}{C_{t+1}} \right]. \tag{24}$$

In terms of the economics, (24) indicates that the optimizing consumer acts so that the intertemporal marginal rate of substitution

$$m_{t+1} = \frac{\beta C_t}{C_{t+1}}$$

and the stochastic return on capital

$$R_{t+1}^K = \alpha Z_{t+1} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + 1 - \delta$$

satisfy the familiar capital asset pricing relationship

$$1 = E_t(m_{t+1} R_{t+1}^K).$$

The first-order condition for H_t can be rearranged slightly to obtain

$$\frac{\theta C_t}{1 - H_t} = (1 - \alpha) Z_t K_t^{\alpha} H_t^{-\alpha}, \tag{25}$$

which states that the optimizing consumer acts so as to equate the marginal rate of substitution between consumption and leisure to the marginal product of labor. Hence (24) and (25) are just generalizations of basic results from consumer and producer theory to this dynamic, stochastic economic environment. Together with the binding resource constraint (5):

$$Z_t K_t^{\alpha} H_t^{1-\alpha} = C_t + K_{t+1} - (1 - \delta) K_t, \tag{26}$$

equations (24) and (25) form a system of three equations that could, in principle, be used to characterize the optimal choice of C_t, H_t, and K_t for given values of the parameters α, δ, β, and θ and the exogenous shock Z_t.

Final Exam

EC720.01 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2010

Due Tuesday, December 14, 2010 at 11:00am

This exam has three questions on five pages; before you begin, please check to make sure your copy has all three questions and all five pages. Each of the three questions is worth 20 points; as indicated at the beginning of the semester, this 60-point final will account for 60 percent of your course grade and the problem sets will account for the remaining 40 percent of your course grade.

This is an open-book exam, meaning that it is fine for you to consult your notes, textbooks, and other references when working on your answers to the questions. I expect you to work independently on the exam, however, so that the answers you submit are yours and yours alone.

1. Growth and Pollution, Part I

This question asks you to use the Kuhn-Tucker theorem to solve a *static* constrained optimization problem that can be used to think about patterns of pollution and environmental quality seen when looking across a sample of countries at different stages of economic development.

Suppose, in particular, that a country has $k > 0$ units of physical capital that can be used to produce output of a single consumption good using any one from a continuum of technologies indexed by $z \in [0, 1]$. Lower values of z represent "cleaner" technologies that generate less pollution and higher values of z represent "dirtier" technologies that generate more pollution. People dislike pollution, but they also like consumption; and because dirtier technologies can be used to produce output at a lower cost, a trade-off arises between economic growth and environmental quality.

To make these ideas more specific, suppose that the country's k units of capital, when used with technology z, produce $c = Akz$ units of output, where $A > 0$ is a parameter governing the overall level of productivity across all available technologies. Hence, according to this specification, more capital used with dirtier technologies yields more output. Suppose also, however, that the same k units of capital, when used with the same technology z, generate $p = Akz^\beta$ units of pollution, where $\beta > 1$ is a parameter governing how the use of dirtier technologies causes environmental quality to deteriorate.

Suppose, finally, that a representative consumer in this country gets utility

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

from consuming c units of output but suffers disutility

$$v(p) = \left(\frac{B}{\gamma}\right) p^\gamma$$

when the level of pollution is given by p, where the preference parameters satisfy $\sigma > 0$, $B > 0$, and $\gamma > 1$.

A social planner for this economy takes the stock of physical capital k as given, and chooses the levels of consumption c and pollution p to maximize the representative consumer's total utility subject to the constraints imposed by the description of the technologies from above. Although there are a number of ways to formalize the social planner's problem, perhaps the easiest is to start by observing that because preferences over consumption are such that $\lim_{c\to 0} u'(c) = \infty$ and because the cleanest technology $z = 0$ yields no output, the constraints $c \geq 0$ and $z \geq 0$ will never bind at the optimum and can therefore be dropped from further consideration. On the other hand, for certain parameter configurations at least, it may be optimal to chose the dirtiest technology $z = 1$; hence the constraint $1 \geq z$ must still be accounted for.

Substituting the technological relations $c = Akz$ and $p = Akz^\beta$ into the utility and disutility functions $u(c)$ and $v(p)$ then allows the social planner's problem to be written as one with a single choice variable and a single constraint:

$$\max_z \frac{(Akz)^{1-\sigma} - 1}{1-\sigma} - \left(\frac{B}{\gamma}\right) (Akz^\beta)^\gamma \text{ subject to } 1 \geq z.$$

a. Define (write down) the Langragian for this problem.

b. Next, write down the full set of first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, characterize the value z^* that solves the social planner's problem together with the associated value of the Lagrange multiplier.

c. Now use your results from part (b), above, to solve for the optimal choice of z^* as a function of the model's parameters: k, A, β, σ, B, and γ.

d. Next, use your result from part (c), above, to obtain an expression that shows how total pollution depends on the model's parameters k, A, β, σ, B, and γ when z^* is chosen optimally.

e. Finally, in order to obtain a sharper economic characterization of your result from part (d), above, suppose that the preference parameter σ is larger than one: $\sigma > 1$. Under this extra assumption, what does your result imply for the relationship between the economy's level of economic development, as measured by its stock of physical capital k, and its optimal choice of pollution? Looking at a cross-section of countries, under what circumstances will more developed countries have better environmental quality? Under what circumstances will more developed countries have worse environmental quality?

2. Growth and Pollution, Part II

This question asks you to use the maximum principle to characterize the solution to a dynamic version of the problem that you just considered in a static context. The dynamics are added to the model by allowing the social planner to increase the capital stock k over time by allocating output to investment instead of consumption.

Suppose that time is continuous and the horizon is infinite, so that periods can be indexed by $t \in [0, \infty)$. Let $k(t)$ denote the economy's stock of physical capital at time t and, extending the specification from above, suppose that by using a technology indexed by $z(t)$ during time t, the economy produces $Ak(t)z(t)$ units of output but also generates $p(t) = Ak(t)z(t)^{\beta}$ units of pollution where, as before, $A > 0$ and $\beta > 1$ are parameters.

Now, however, suppose that in addition to choosing an optimal technology $z(t)$ at each date $t \in [0, \infty)$, the social planner must also choose how to optimally divide up output $Ak(t)z(t)$ into an amount $c(t)$ to be consumed and an amount $Ak(t)z(t) - c(t)$ to be invested. If, in addition, the parameter δ, with $0 < \delta < 1$, measures the depreciation rate of physical capital, then the stock of capital evolves according to

$$Ak(t)z(t) - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$.

With the additional dynamics introduced through physical capital accumulation complicating the problem, assume now that the constraints $1 > z(t)$ and $z(t) > 0$ can be ignored, either because there are ever "dirtier" technologies than the dirtiest $z = 1$ considered before or because the configuration of parameters insures that these constraints will not bind at the optimum. Then, with the specification for the utility over consumption

$$u[c(t)] = \frac{c(t)^{1-\sigma} - 1}{1 - \sigma}$$

with $\sigma > 0$ and the disutility from pollution

$$v[p(t)] = \left(\frac{B}{\gamma}\right) p(t)^{\gamma}$$

with $B > 0$ and $\gamma > 1$ the same as before, the social planner's problem can be stated as follows: given the initial capital stock $k(0)$, choose continuously differentiable functions $c(t)$, $z(t)$, and $k(t)$ for all $t \in [0, \infty)$ to maximize

$$\int_0^{\infty} e^{-\rho t} \left\{ \frac{c(t)^{1-\sigma} - 1}{1 - \sigma} - \left(\frac{B}{\gamma}\right) [Ak(t)z(t)^{\beta}]^{\gamma} \right\} \mathrm{d}t$$

subject to

$$Ak(t)z(t) - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$, where $\rho > 0$ denotes the discount rate.

a. Write down the Hamiltonian for the social planner's problem.

b. Next, write down the first-order conditions and the pair of differential equations that, according to the maximum principle, characterize the solution to the social planner's problem.

c. The conditions you derived for part (b), above, should form a system of equations involving the not only optimal choices of $c(t)$, $z(t)$, and $k(t)$ but also the associated values for a new variable, corresponding to the Lagrange multiplier on the capital accumulation constraint that you would have introduced into the problem if you had decide to solve it using the Kuhn-Tucker theorem instead. Use one of your optimality conditions to eliminate this new variable from the system, so as to obtain a smaller set of three equations involving only the model's parameters and the three variables $c(t)$, $z(t)$, and $k(t)$ that enter into the original statement of the planner's problem.

d. The smaller system you derived for part (c), above, implies that starting from any initial value of $k(0)$, there are unique values of $c(0)$ and $z(0)$ that place the economy on a saddle path that converges to a steady state, in which $c(t) = c^*$, $z(t) = z^*$, and $k(t) = k^*$ for all t. Use your answers from part (c), above, to derive a system of three equations that can be used to solve for the steady-state values c^*, z^*, and k^* in terms of the model's parameters: A β, δ, σ, B, and γ. *Note:* you don't have to actually solve for the steady-state values; just write down the three-equation system that would, in principle, allow you to find these values.

3. Stochastic Growth

This problem asks you to use the guess-and-verify method to characterize the solution to a version of the stochastic growth model.

Suppose that time is discrete and the horizon is infinite, so that periods can be indexed by $t = 0, 1, 2, ...$ Let output produced during each period $t = 0, 1, 2, ...$ be given by $y_t = A_t k_t$, where k_t is the beginning-of-period capital stock and A_t is a random shock to productivity.

By consuming c_t units of output at each date $t = 0, 1, 2, ...$, the representative consumer gets expected utility

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where the discount factor satisfies $0 < \beta < 1$. Letting δ, with $0 < \delta < 1$, denote the depreciation rate for physical capital, the evolution of the capital stock is described by

$$A_t k_t + (1 - \delta) k_t - c_t \geq k_{t+1},$$

a constraint that must hold for all $t = 0, 1, 2, ...$ and all possible realizations of the productivity shock A_t at each date $t = 0, 1, 2, ...$

Assume, finally, that the productivity shock A_t gets realized at the very beginning of period t, before c_t and k_{t+1} are chosen, and that the total return on capital, accounting for depreciation,

$$R_t^K = A_t + 1 - \delta$$

follows the first-order autoregressive process

$$\ln(R_{t+1}^K) = (1-\rho)\ln(R) + \rho\ln(R_t^K) + \varepsilon_{t+1},$$

where the parameter ρ, satisfying $0 < \rho < 1$, measures the degree of serial correlation in the productivity shock, the parameter R, satisfying $R > 0$, determines the average value of R_t^K, and where ε_{t+1} is a serially uncorrelated shock, satisfying $E_t\varepsilon_{t+1} = 0$ for all $t = 0, 1, 2, ...$, that gets realized at the beginning of period $t+1$. Hence, when c_t and k_{t+1} are chosen, R_t^K is known but R_{t+1}^K is still viewed by the representative consumer as random.

The representative consumer's problem can now be stated as: given the initial capital stock k_0, choose contingency plans for c_t, $t = 0, 1, 2, ...$, and k_t, $t = 1, 2, 3, ...$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

subject to the constraints (which are more conveniently rewritten in terms of R_t^K)

$$R_t^K k_t - c_t \geq k_{t+1}$$

for all $t = 0, 1, 2, ...$ and for all possible realizations of R_t^K at each date $t = 0, 1, 2, ...$

a. Write down the Bellman equation for the representative consumer's problem.

b. Next, guess that the value function takes the time-invariant form

$$v(k_t, R_t^K) = E + F\ln(k_t) + G\ln(R_t^K)$$

where E, F, and G are unknown constants to be determined, and, using this guess, write down the first-order and envelope conditions that help characterize the consumer's optimal decisions.

c. Use your results from part (b), above, to solve for the constant F in terms of the model's parameters β, R, and ρ.

d. Use your results from parts (b) and (c) to derive a pair of equations that can be used to construct the optimal choices for c_t, $t = 0, 1, 2, ...$, and k_t, $t = 1, 2, 3, ...$, for a given value of the initial capital stock k_0, a given set of realizations for R_t^K at each date $t = 0, 1, 2, ...$, and for given values of the model's parameters β, R, and ρ.

Solutions to Final Exam

EC720.01 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2010

Due Tuesday, December 14, 2010 at 11:00am

1. Growth and Pollution, Part I

The social planner's problem is

$$\max_z \frac{(Akz)^{1-\sigma}-1}{1-\sigma} - \left(\frac{B}{\gamma}\right)(Akz^\beta)^\gamma \text{ subject to } 1 \geq z.$$

a. The Lagrangian for the social planner's problem can be written as

$$L(z,\lambda) = \frac{(Akz)^{1-\sigma}-1}{1-\sigma} - \left(\frac{B}{\gamma}\right)(Akz^\beta)^\gamma + \lambda(1-z).$$

b. According to the Kuhn-Tucker theorem, if z^* is the value of z that solves the social planner's problem, then there exists an associated value λ^* of λ such that, together, z^* and λ^* satisfy the first-order condition

$$L_1(z^*,\lambda^*) = (Ak)^{1-\sigma}(z^*)^{-\sigma} - \beta B(Ak)^\gamma (z^*)^{\beta\gamma-1} - \lambda^* = 0,$$

the constraint

$$L_2(z^*,\lambda^*) = 1 - z^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*(1-z^*) = 0.$$

c. Let's look first for a solution with a nonbinding constraint, that is, with $1 > z^*$. By the complementary slackness condition, this solution must have $\lambda^* = 0$. Hence, the first-order condition implies that

$$z^* = \left(\frac{1}{\beta B}\right)^{\frac{1}{\beta\gamma+\sigma-1}} \left(\frac{1}{Ak}\right)^{\frac{\gamma+\sigma-1}{\beta\gamma+\sigma-1}}$$

For this solution to apply, it must be that $z^* < 1$ or, in light of the expression just derived, that

$$Ak > \left(\frac{1}{\beta B}\right)^{\frac{1}{\gamma+\sigma-1}}.$$

If, on the other hand, the constraint is binding, then $z^* = 1$ and the first-order condition implies that

$$\lambda^* = (Ak)^{1-\sigma} - \beta B(Ak)^\gamma.$$

For this solution to apply, it must be that $\lambda^* \geq 0$ or, in light of the expression just derived, that

$$\left(\frac{1}{\beta B}\right)^{\frac{1}{\gamma+\sigma-1}} \geq Ak.$$

Putting these results together, the optimal choice for z^* is given by

$$z^* = \begin{cases} 1 & \text{if } \left(\frac{1}{\beta B}\right)^{\frac{1}{\gamma+\sigma-1}} \geq Ak \\ \left(\frac{1}{\beta B}\right)^{\frac{1}{\beta\gamma+\sigma-1}} \left(\frac{1}{Ak}\right)^{\frac{\gamma+\sigma-1}{\beta\gamma+\sigma-1}} & \text{if } Ak > \left(\frac{1}{\beta B}\right)^{\frac{1}{\gamma+\sigma-1}} \end{cases}.$$

Evidently, if we take k as an index of the level of economic development, then the optimal technology becomes progressively cleaner as the level of economic development increases.

d. Since total pollution is linked to the choice of technology via $p = Akz^\beta$, the solution for z^* implies that

$$p^* = \begin{cases} Ak & \text{if } \left(\frac{1}{\beta B}\right)^{\frac{1}{\gamma+\sigma-1}} \geq Ak \\ \left(\frac{1}{\beta B}\right)^{\frac{\beta}{\beta\gamma+\sigma-1}} \left(\frac{1}{Ak}\right)^{\frac{(\beta-1)(\sigma-1)}{\beta\gamma+\sigma-1}} & \text{if } Ak > \left(\frac{1}{\beta B}\right)^{\frac{1}{\gamma+\sigma-1}} \end{cases}.$$

e. The solution for p^* just derived implies that in general, total pollution increases with the index of economic development for low levels of k. This is because at low levels of development, the dirtiest technology is always chosen, so that pollution always increases with output. The same solution implies, however, that at higher levels of development the relationship between development and environmental quality is ambiguous. This is because as economies develop, they choose cleaner technologies, reducing pollution, but also produce more, increasing pollution; the total effect is ambiguous. However, with the additional condition $\sigma > 1$ imposed, the effects of cleaner technologies outweigh the effects of additional production, so that environmental quality improves with economic development. Hence, when $\sigma > 1$, the model predicts that in a cross-section of countries, environmental quality should at first first deteriorate but then improve with economic development. In fact, patterns of exactly this sort have been found in numerous studies: see, for example, Gene M. Grossman and Alan B. Krueger, "Economic Growth and the Environment," *Quarterly Journal of Economics*, vol.110 (May 1995), pp.353-377.

2. Growth and Pollution, Part II

The social planner's problem is to choose continuously differentiable functions $c(t)$, $z(t)$, and $k(t)$ for all $t \in [0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \left\{ \frac{c(t)^{1-\sigma} - 1}{1-\sigma} - \left(\frac{B}{\gamma}\right) [Ak(t)z(t)^\beta]^\gamma \right\} \mathrm{d}t$$

subject to

$$Ak(t)z(t) - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$, taking the initial capital stock $k(0)$ as given.

a. The Hamiltonian for the social planner's problem can be written as

$$\begin{aligned} H(k(t), \pi(t); t) \quad = \quad & \max_{c(t), z(t)} e^{-\rho t} \left\{ \frac{c(t)^{1-\sigma} - 1}{1-\sigma} - \left(\frac{B}{\gamma}\right) [Ak(t)z(t)^\beta]^\gamma \right\} \\ & + \pi(t)[Ak(t)z(t) - \delta k(t) - c(t)]. \end{aligned}$$

b. According to the maximum principle, the values of $c(t)$, $z(t)$, and $k(t)$ that solve the social planner's problem must satisfy the first-order conditions

$$e^{-\rho t} c(t)^{-\sigma} - \pi(t) = 0$$

and

$$-e^{-\rho t} \beta B [Ak(t)]^\gamma z(t)^{\beta\gamma - 1} + \pi(t) A k(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_k(k(t), \pi(t); t) = e^{-\rho t} B A^\gamma k(t)^{\gamma - 1} z(t)^{\beta\gamma} - \pi(t)[Az(t) - \delta]$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = Ak(t)z(t) - \delta k(t) - c(t).$$

c. The first-order condition for $c(t)$ implies that

$$\pi(t) = e^{-\rho t} c(t)^{-\sigma}$$

and hence

$$\dot{\pi}(t) = -\rho e^{-\rho t} c(t)^{-\sigma} - \sigma e^{-\rho t} c(t)^{-\sigma - 1} \dot{c}(t).$$

Using these expressions to eliminate all references to $\pi(t)$ and $\dot{\pi}(t)$, the first-order condition for $z(t)$ can be rewritten as

$$\beta B [Ak(t)]^\gamma z(t)^{\beta\gamma - 1} = c(t)^{-\sigma} A k(t)$$

and the two differential equations can be rewritten as

$$\dot{c}(t) = (1/\sigma)[Az(t) - \delta - \rho] c(t) - (1/\sigma) B A^\gamma k(t)^{\gamma - 1} z(t)^{\beta\gamma} c(t)^{\sigma + 1}$$

and

$$\dot{k}(t) = Ak(t)z(t) - \delta k(t) - c(t).$$

This three-equation system involves only the three variables, $c(t)$, $z(t)$, and $k(t)$, from the original economic problem.

d. In the steady state, $c(t) = c^*$, $z(t) = z^*$, $k(t) = k^*$, $\dot{c}(t) = 0$, and $\dot{k}(t) = 0$, where the three-equation system from above implies that

$$\beta B(Ak^*)^{\gamma}(z^*)^{\beta\gamma-1} = (c^*)^{-\sigma}Ak^*,$$

$$Az^* - \delta - \rho = BA^{\gamma}(k^*)^{\gamma-1}(z^*)^{\beta\gamma}(c^*)^{\sigma},$$

and

$$c^* = Ak^*z^* - \delta k^*.$$

The existence of a steady state with constant consumption, capital, and output is an interesting implication of this model. The simpler "Ak" model of economic growth, in which output $y(t)$ is produced with capital $k(t)$ according to the constant-returns-to-scale specification $y(t) = Ak(t)$, features sustained economic growth in which consumption, capital, and output all grow forever at the same constant, positive rate. Here, with pollution added to the specification, the added environmental effect makes it optimal to set "limits on growth." For a detailed analysis of this model and some further extensions, see Nancy L. Stokey. "Are There Limits to Growth?" *International Economic Review*, vol.39 (February 1998), pp.1-31.

3. Stochastic Growth

The representative consumer takes the initial capital stock k_0 as given and chooses contingency plans for c_t, $t = 0, 1, 2, ...$, and k_t, $t = 1, 2, 3, ...$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

subject to the constraints

$$R_t^K k_t - c_t \geq k_{t+1}$$

for all $t = 0, 1, 2, ...$ and for all possible realizations of R_t^K at each date $t = 0, 1, 2, ...$, where the total return on capital,

$$R_t^K = A_t + 1 - \delta,$$

follows the first-order autoregressive process

$$\ln(R_{t+1}^K) = (1-\rho)\ln(R) + \rho\ln(R_t^K) + \varepsilon_{t+1}.$$

a. The Bellman equation for the representative consumer's problem can be written as

$$v(k_t, R_t^K) = \max_{c_t} \ln(c_t) + \beta E_t v(R_t^K k_t - c_t, R_{t+1}^K).$$

b. Using the guess that the value function takes the time-invariant form

$$v(k_t, R_t^K) = E + F\ln(k_t) + G\ln(R_t^K)$$

where E, F, and G are unknown constants to be determined, as well as the assumptions made about the stochastic behavior of R^K_{t+1}, the Bellman equation becomes

$$\begin{aligned} & E + F\ln(k_t) + G\ln(R^K_t) \\ = \ & \max_{c_t} \ \ln(c_t) + \beta E + \beta F \ln(R^K_t k_t - c_t) + \beta(1-\rho)G\ln(R) + \beta\rho G \ln(R^K_t). \end{aligned}$$

The first-order condition for c_t is then

$$\frac{1}{c_t} - \frac{\beta F}{R^K_t k_t - c_t} = 0$$

and the envelope condition for k_t is

$$\frac{F}{k_t} = \frac{\beta F R^K_t}{R^K_t k_t - c_t}.$$

c. Rewrite the first-order condition as

$$c_t = \left(\frac{1}{1+\beta F}\right) R^K_t k_t,$$

and substitute this expression into the envelope condition to solve for

$$F = \frac{1}{1-\beta}.$$

d. Substituting the solution for F back into the first-order condition yields

$$c_t = (1-\beta)R^K_t k_t.$$

This equation, together with the constraint

$$k_{t+1} = R^K_t k_t - c_t = \beta R^K_t k_t$$

can be used to construct the optimal choices for c_t, $t = 0, 1, 2, ...$, and k_t, $t = 1, 2, 3, ...$ for any given value of k_0 and any given set of realizations for R^K_t at each date $t = 0, 1, 2, ...$

Midterm Exam

EC720.01 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2012

Thursday, October 25, 1:30 - 2:45pm

This exam has four questions on four pages; before you begin, please check to make sure that your copy has all four questions and all four pages. Each part of each question is worth five points. Hence, question 1 is worth 15 points, question 2 is worth 5 points, question 3 is worth 15 points, and question 4 is worth 10 points, for a total of 45 points overall.

1. Utility Maximization (15 points)

Consider a consumer who uses his or her income I to purchase c_1 units of good one at the price of p_1 per unit and c_2 units of good 2 at the price of p_2 per unit, subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2. \tag{1.1}$$

In addition, the consumer's choices of c_1 and c_2 must satisfy the nonnegativity constraints

$$c_1 \geq 0 \tag{1.2}$$

and

$$c_2 \geq 0. \tag{1.3}$$

Finally, suppose that the consumer has preferences over the two goods described by the utility function

$$U(c_1, c_2) = \ln(c_1) + \ln(c_2 + \alpha), \tag{1.4}$$

where ln denotes the natural logarithm and $\alpha > 0$ is a positive parameter.

a. Set up the Lagrangian for the consumer's problem: choose c_1 and c_2 to maximize the utility function (1.4) subject to the constraints (1.1)-(1.3). Recall from our conversations in class that this can be done in at least two ways: feel free to use whichever formulation of the Lagrangian you find most convenient. Then write down the first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values of c_1^* and c_2^* that solve the consumer's problem, together with the associated values of the Lagrange multiplier (or multipliers, if you use more than one). Again, as we discussed in class, there are a number of ways to do this, but make sure that your statement of the Kuhn-Tucker conditions is consistent with your chosen definition of the Lagrangian.

b. As a first step in using your answers from part (a), above, to find the solution to the consumer's problem, observe that since the utility function (1.4) implies that the marginal utility of consuming good 1 goes to infinity as c_1 approaches zero, it will always be optimal for the consumer to choose $c_1^* > 0$. With this in mind, use your results from above to find the condition on the parameters I, p_1, p_2, and α under which the consumer will find it optimal to consume *only* good 1, so that the constraint $c_2^* \geq 0$ binds at the optimum. In this case where $c_2^* = 0$, find the solution for c_1^* in terms of the parameters I, p_1, p_2, and α.

c. Still keeping in mind that $c_1^* > 0$ will always hold, find the condition on the parameters I, p_1, p_2, and α under which the consumer will find it optimal to consume *both* goods, so that $c_2^* > 0$ holds as well. In this case, find the solutions for c_1^* and c_2^* in terms of the parameters I, p_1, p_2, and α.

2. Cost Minimization (5 points)

Consider a firm that uses k units of capital and l units of labor to produce

$$(k^{\alpha} l^{1-\alpha})^{1/\beta}$$

units of output, where α and β, satisfying $0 < \alpha < 1$ and $\beta > 1$, are parameters. If the firm obtains these inputs at the competitive rental rate r for capital and the competitive wage rate w for labor, and seeks to minimize the cost of producing at least y units of output, it solves the problem

$$\min_{k,l} rk + wl \text{ subject to } (k^{\alpha} l^{1-\alpha})^{1/\beta} \geq y, \tag{2.1}$$

where, in this cost-minimization problem, the output requirement y is another parameter. It turns out that, for this firm, the minimum cost function given by

$$C(r, w, y) = \left[\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \right] r^{\alpha} w^{1-\alpha} y^{\beta}$$

measures the minimized value of costs in (2.1), when k and l are chosen optimally subject to the constraint.

Recall from our discussions in class that the conditional factor demand curves

$$k^* = k^*(r, w, y)$$

for capital and

$$l^* = l^*(r, w, y)$$

for labor express the values of k and l that solve the problem in (2.1) as functions of the three parameters r, w, and y.

Given this information, please find the explicit expressions for the two conditional factor demand curves $k^*(r, w, y)$ and $l^*(r, w, y)$.

3. Cigarette Smoking (15 points)

Consider a dynamic model with two periods, $t = 0$ and $t = 1$. During each period, a consumer divides his or her income up into an amount to be spent on cigarettes and an amount to be spent on all other goods. Let c_0 and c_1 denote his or her consumption of cigarettes during periods $t = 0$ and $t = 1$, and let x_0 and x_1 denote his or her consumption of all other goods during the same two periods. The consumer's preferences are described by the utility function

$$U(c_0, x_0, c_1, x_1) = \ln(c_0) + x_0 + \beta[\ln(c_1 - \gamma c_0) + x_1], \tag{3.1}$$

where ln denotes the natural logarithm and the parameters β and γ both lie between zero and one: $0 < \beta < 1$, and $0 < \gamma < 1$. Therefore, the parameter β is the consumer's discount factor – a higher value of β implies that the consumer is more patient – and the parameter γ measures the addictiveness of cigarettes – the more the consumer smokes during period $t = 0$, the higher will be his or her marginal utility of smoking during period $t = 1$.

For simplicity, suppose that the price of cigarettes p and the consumer's income I both remain constant across the two periods. Also for simplicity, suppose that the price of all other goods is constant and equal to one in both periods. Then the consumer will choose c_0, x_0, c_1, and x_1 to maximize the utility function in (3.1) subject to the budget constraints

$$I \geq pc_0 + x_0 \tag{3.2}$$

during period $t = 0$ and

$$I \geq pc_1 + x_1 \tag{3.3}$$

during period $t = 1$.

a. Set up the Lagrangian for the consumer's problem: choose c_0, x_0, c_1, and x_1 to maximize the utility function in (3.1) subject to the constraints in (3.2) and (3.3). In doing this, you can assume that the consumer's constant income level I is large enough that he or she will always consume positive amounts of cigarettes and all other goods in both periods, so that there is no need to explicitly impose nonnegativity constraints on c_0, x_0, c_1, and x_1. Then write down the first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values of c_0^*, x_0^*, c_1^*, and x_1^*, that solve the consumer's problem, together with the associated values of the Lagrange multipliers.

b. Use your results from part (a), above, to solve for the optimal choices of c_0^* and c_1^* as functions of the parameters β, γ, I, and p.

c. Use your results from part (b), above, to answer the following questions:

(i) Does the optimal choice of c_0^* increase or decrease when the parameter β rises?

(ii) Does the optimal choice of c_1^* increase or decrease when the parameter β rises?

(iii) Who smokes more: patient or impatient consumers?

4. Life Cycle Saving (10 points)

Consider a consumer who is employed for $T+1$ periods: $t = 0, 1, \ldots, T$. During each period of employment, the consumer receives labor income w; since w is not indexed by t, labor income remains constant over time. Let k_t denote the consumer's stock of financial assets at the beginning of period t, and assume that $k_0 = 0$, so that the consumer begins his or her career with no assets. For all $t = 1, 2, \ldots, T$, k_t can be negative; that is, the consumer is allowed to borrow. However, the consumer must eventually save for retirement, a requirement that is captured by imposing the constraint

$$k_{T+1} \geq k^* > 0 \tag{4.1}$$

on the terminal stock of wealth.

Let r denote the interest rate earned on savings, or paid on debt, during each period $t = 0, 1, \ldots, T$; since r is not indexed by t, this interest rate remains constant over time. Then the consumer's stock of assets evolves over time according to

$$k_{t+1} = k_t + w + rk_t - p_{1t}c_{1t} - p_{2t}c_{2t},$$

where c_{1t} and c_{2t} denote the amounts of two goods purchased by the consumer during each period $t = 0, 1, \ldots, T$ and p_{1t} and p_{2t} denote the prices of these two goods. Allowing for the possibility of free disposal of wealth, which will never be optimal under the preference specification to be introduced next, these constraints can be written as

$$w + rk_t - p_{1t}c_{1t} - p_{2t}c_{2t} \geq k_{t+1} - k_t, \tag{4.2}$$

for all $t = 0, 1, \ldots, T$.

Finally, assume that the consumer's preferences over consumption of the two goods during periods $t = 0, 1, \ldots, T$ are described by the utility function

$$\sum_{t=0}^{T} \beta^t [\alpha \ln(c_{1t}) + (1-\alpha)\ln(c_{2t})], \tag{4.3}$$

where ln denotes the natural logarithm and the discount factor β and the parameter α both lie between zero and one: $0 < \beta < 1$ and $0 < \alpha < 1$. The consumer's problem can now be stated as: choose sequences $\{c_{1t}\}_{t=0}^{T}$, $\{c_{2t}\}_{t=0}^{T}$, and $\{k_t\}_{t=1}^{T+1}$ to maximize the utility function in (4.3) subject to the constraints $k_0 = 0$ given, (4.1), and (4.2) for for all $t = 0, 1, \ldots, T$.

a. To derive the optimality conditions describing the solution to the consumer's problem using the maximum principle, begin by setting up the Hamilton.

b. Then write down the first-order conditions, the difference equations, and the initial and terminal (or transversality) conditions that, according to the maximum principle, characterize the solution to the consumer's dynamic optimization problem.

Solutions to Midterm Exam

EC720.01 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2012

Thursday, October 25, 1:30 - 2:45pm

1. Utility Maximization (15 points)

The consumer chooses c_1 and c_2 to maximize the utility function

$$U(c_1, c_2) = \ln(c_1) + \ln(c_2 + \alpha), \tag{1.4}$$

subject to the budget constrant

$$I \geq p_1 c_1 + p_2 c_2 \tag{1.1}$$

and the nonnegativity constraints

$$c_1 \geq 0 \tag{1.2}$$

and

$$c_2 \geq 0. \tag{1.3}$$

a. There are a number of ways in which the Lagrangian for this problem can be defined, but one simply treats the nonnegativity constraints symmetrically to the budget constraint and assigns a Lagrange multiplier to each. Taking this approach, and letting λ denote the multiplier of the budget constraint and μ_1 and μ_2 denote the multipliers on the nonnegativity constraints, the Lagrangian becomes

$$L(c_1, c_2, \lambda, \mu_1, \mu_2) = \ln(c_1) + \ln(c_2 + \alpha) + \lambda(I - p_1 c_1 - p_2 c_2) + \mu_1 c_1 + \mu_2 c_2.$$

With this definition of the Lagrangian, Kuhn-Tucker theorem implies that the values c_1^* and c_2^* that solve the consumer's problem and the associated values λ^*, μ_1^*, and μ_2^* of the multipliers must satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = \frac{1}{c_1^*} - \lambda^* p_1 + \mu_1^* = 0$$

and

$$L_2(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = \frac{1}{c_2^* + \alpha} - \lambda^* p_2 + \mu_2^* = 0,$$

the constraints

$$L_3(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = I - p_1 c_1^* - p_2^* \geq 0,$$
$$L_4(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = c_1^* \geq 0,$$

and

$$L_5(c_1^*, c_2^*, \lambda^*, \mu_1^*, \mu_2^*) = c_2^* \geq 0,$$

the nonnegativity conditions

$$\lambda^* \geq 0,$$

$$\mu_1^* \geq 0,$$

and

$$\mu_2^* \geq 0,$$

and the complementary slackness conditions

$$\lambda^*(I - p_1c_1^* - p_2^*) = 0,$$

$$\mu_1^* c_1^* = 0,$$

and

$$\mu_2^* c_2^* = 0.$$

b. Since the marginal utility of consuming good 1 goes to infinity as c_1 approaches zero, the nonnegativity constraint $c_1 \geq 0$ will never bind; the associated complementary slackness condition then implies that $\mu_1^* = 0$ will always hold. Suppose, on the other hand, that the constraint $c_2 \geq 0$ does bind at the optimum. With $c_2^* = 0$, the complementary slackness condition for that second nonnegativity constraint is satisfied for any value of $\mu_2^* \geq 0$. Under these circumstances, the first-order condition for c_1 implies that

$$p_1 c_1^* = \lambda^*.$$

Substituting this condition and $c_2^* = 0$ into the budget constraint, which will always bind at the optimum, yields

$$I = p_1c_1^* + p_2c_2^* = \frac{1}{\lambda^*}$$

and hence

$$\lambda^* = \frac{1}{I}.$$

Using this solution for λ^*, together with the first-order condition for c_1 again, provides the solution

$$c_1^* = \frac{I}{p_1}.$$

This makes sense: if it is optimal to consume only good 1, then it follows that the consumer will spend all of his or her income on that good. It only remains to determine the circumstances under which $c_2^* = 0$ is, in fact, the optimal choice. To do this, substitute $c_2^* = 0$ and $\lambda^* = 1/I$ into the first-order condition for c_2 to obtain

$$\frac{1}{\alpha} - \frac{p_2}{I} + \mu_2^* = 0$$

or

$$\mu_2^* = \frac{p_2}{I} - \frac{1}{\alpha}.$$

The nonnegativity condition $\mu_2^* \geq 0$ therefore requires that the parameters satisfy

$$p_2\alpha \geq I.$$

Evidently, when α is large, income is low, or the price p_2 is high, it will be optimal for the consumer to only purchase a positive amount of good 1.

c. In this case where it is optimal for the consumer to choose positive amounts of both goods, the complementary slackness conditions for both nonnegativity constraints imply that $\mu_1^* = \mu_2^* = 0$. Substituting these results into the two first-order conditions yields

$$p_1 c_1^* = \frac{1}{\lambda^*}$$

and

$$p_2 c_2^* = \frac{1}{\lambda^*} - p_2\alpha,$$

which can then be substituted into the binding budget constraint to obtain

$$I = \frac{1}{\lambda^*} + \frac{1}{\lambda^*} - p_2\alpha$$

or

$$\lambda^* = \frac{2}{I + p_2\alpha}.$$

Substituting this solution for λ^* back into the first-order conditions then yields the solutions

$$c_1^* = \frac{I + p_2\alpha}{2p_1}.$$

and

$$c_2^* = \frac{I + p_2\alpha}{2p_2} - \alpha.$$

The last expression reveals that c_2^* will only be positive, as conjectured, if the parameters satisfy

$$I > p_2\alpha.$$

Evidently, when α is small, income is high, or the price p_2 is low, it will be optimal for the consumer to purchase positive amounts of both goods.

2. Cost Minimization (5 points)

The firm solves solves

$$\min_{k,l} rk + wl \text{ subject to } (k^\alpha l^{1-\alpha})^{1/\beta} \geq y, \tag{2.1}$$

which leads to the minimum cost function

$$C(r, w, y) = \left[\frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}}\right] r^\alpha w^{1-\alpha} y^\beta,$$

measuring the minimized value of costs in (2.1), when k and l are chosen optimally subject to the constraint. Applying Shephard's lemma or, equivalently, the envelope theorem to this problem leads directly to solutions for the two conditional factor demand curves:

$$k^*(r,w,y) = C_r(r,w,y) = \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\frac{w}{r}\right)^{1-\alpha} y^{\beta}$$

and

$$l^*(r,w,y) = C_w(r,w,y) = \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\left(\frac{r}{w}\right)^{\alpha} y^{\beta}$$

3. Cigarette Smoking (15 points)

The consumer chooses c_0, x_0, c_1, and x_1 to maximize the utility function

$$U(c_0,x_0,c_1,x_1) = \ln(c_0) + x_0 + \beta[\ln(c_1-\gamma c_0) + x_1], \tag{3.1}$$

subject to the budget constraints

$$I \geq pc_0 + x_0 \tag{3.2}$$

during period $t = 0$ and

$$I \geq pc_1 + x_1 \tag{3.3}$$

during period $t = 1$.

a. Let λ_0 and λ_1 denote the multipliers on the two budget constraints, and define the Lagrangian as

$$L(c_0,x_0,c_1,x_1,\lambda_0,\lambda_1) = \ln(c_0)+x_0+\beta[\ln(c_1-\gamma c_0)+x_1]+\lambda_0(I-pc_0-x_0)+\lambda_1(I-pc_1-x_1).$$

The Kuhn-Tucker theorem then implies that the values c_0^*, x_0^*, c_1^*, and x_1^* that solve the consumer's problem, together with the associated values λ_0^* and λ_1^* of the Lagrange multipliers, must satisfy the first-order conditions

$$L_1(c_0^*,x_0^*,c_1^*,x_1^*,\lambda_0^*,\lambda_1^*) = \frac{1}{c_0^*} - \frac{\beta\gamma}{c_1^*-\gamma c_0^*} - \lambda_0^* p = 0,$$

$$L_2(c_0^*,x_0^*,c_1^*,x_1^*,\lambda_0^*,\lambda_1^*) = 1 - \lambda_0^* = 0,$$

$$L_3(c_0^*,x_0^*,c_1^*,x_1^*,\lambda_0^*,\lambda_1^*) = \frac{\beta}{c_1^*-\gamma c_0^*} - \lambda_1^* p = 0,$$

and

$$L_4(c_0^*,x_0^*,c_1^*,x_1^*,\lambda_0^*,\lambda_1^*) = \beta - \lambda_1^* = 0,$$

the constraints

$$L_5(c_0^*,x_0^*,c_1^*,x_1^*,\lambda_0^*,\lambda_1^*) = I - pc_0^* - x_0^* \geq 0$$

and

$$L_6(c_0^*,x_0^*,c_1^*,x_1^*,\lambda_0^*,\lambda_1^*) = I - pc_1^* - x_1^* \geq 0,$$

nonnegativity conditions

$$\lambda_0^* \geq 0$$

and

$$\lambda_1^* \geq 0,$$

and complementary slackness conditions

$$\lambda_0^*(I - pc_0^* - x_0^*) = 0$$

and

$$\lambda_1^*(I - pc_1^* - x_1^*) = 0.$$

b. The first-order conditions for x_0 and x_1 lead immediately to the solutions

$$\lambda_0^* = 1$$

and

$$\lambda_1^* = \beta,$$

allowing the first-order conditions for c_0 and c_1 to be written more simply as

$$\frac{1}{c_0^*} - \frac{\beta\gamma}{c_1^* - \gamma c_0^*} = p$$

and

$$\frac{1}{c_1^* - \gamma c_0^*} = p.$$

Substituting the second in this last pair of relationships into the first yields

$$\frac{1}{c_0^*} - \beta\gamma p = p,$$

which leads to the solution

$$c_0^* = \frac{1}{(1+\beta\gamma)p}.$$

Finally, substituting this solution back into the first-order condition for c_1 yields

$$1 = pc_1^* - \frac{\gamma}{1+\beta\gamma},$$

which leads to the solution

$$c_1^* = \frac{1+\beta\gamma+\gamma}{(1+\beta\gamma)p}.$$

c. Differentiating the solutions found above reveals that

$$\frac{\partial c_0^*}{\partial \beta} = -\frac{\gamma}{(1+\beta\gamma)^2 p} < 0$$

and

$$\frac{\partial c_1^*}{\partial \beta} = -\frac{\gamma^2}{(1+\beta\gamma)^2 p} < 0$$

It follows immediately from these expressions that

(i) The optimal choice of c_0^* decreases when the parameter β rises.

(ii) The optimal choice of c_1^* decreases when the parameter β rises.

(iii) Impatient consumers smoke more than patient consumers.

4. Life Cycle Saving (10 points)

The consumer chooses sequences $\{c_{1t}\}_{t=0}^{T}$, $\{c_{2t}\}_{t=0}^{T}$, and $\{k_t\}_{t=1}^{T+1}$ to maximize the utility function

$$\sum_{t=0}^{T} \beta^t[\alpha \ln(c_{1t}) + (1-\alpha)\ln(c_{2t})], \tag{4.3}$$

subject to the constraints

$$w + rk_t - p_{1t}c_{1t} - p_{2t}c_{2t} \geq k_{t+1} - k_t, \tag{4.2}$$

for all $t = 0, 1, \ldots, T$,

$$k_0 = 0,$$

and

$$k_{T+1} \geq k^* > 0 \tag{4.1}$$

a. The Hamiltonian for the consumer's problem can be defined as

$$H(k_t, \pi_{t+1}; t) = \max_{c_{1t}, c_{2t}} \beta^t[\alpha \ln(c_{1t}) + (1-\alpha)\ln(c_{2t})] + \pi_{t+1}(w + rk_t - p_{1t}c_{1t} - p_{2t}c_{2t}).$$

b. According to the maximum principle, the solution to the consumer's problem is characterized by the first-order conditions

$$\frac{\beta^t \alpha}{c_{1t}} - \pi_{t+1}p_{1t} = 0$$

and

$$\frac{\beta^t(1-\alpha)}{c_{2t}} - \pi_{t+1}p_{2t} = 0,$$

the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}r$$

and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = w + rk_t - p_{1t}c_{1t} - p_{2t}c_{2t},$$

the initial condition

$$k_0 = 0,$$

and the terminal, or transversality, condition

$$\pi_{T+1}(k_{t+1} - k^*) = 0.$$

Final Exam

EC720.01 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2012

Due Saturday, December 15, 2012 at 12 noon

This exam has three questions on seven pages; before you begin, please check to make sure that your copy has all three questions and all seven pages. Each question has three parts, and each part of each question is worth five points. Hence, each question is worth 15 points, for a total of 45 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, textbooks, and other written references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, inside or outside of the class: the answers you submit must be yours and yours alone.

1. Optimal Growth and the Relative Income Hypothesis

In his 1949 doctoral thesis, titled "Income, Saving and the Theory of Consumer Behavior," James Duesenberry developed his "relative income hypothesis," which builds on the assumption that people care less about the absolute level of their own consumption than about their own consumption level relative to everyone else's consumption in the economy as a whole. This problem asks you to apply the maximum principle to solve a version of the neoclassical (Ramsey) model of optimal growth in continuous time in which the representative consumer has a utility function that reflects Duesenberry's ideas.

In this version of the model, as in the standard case, output $y(t)$ gets produced with capital $k(t)$ during each period $t \in [0, \infty)$ according to the production function $y(t) = k(t)^\alpha$, where the parameter α lies between zero and one: $0 < \alpha < 1$. Also as in the standard case, capital depreciates at the constant rate $\delta > 0$, and $c(t)$ denotes the representative consumer's consumption during each period t. Hence, as usual, the capital stock evolves according to

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t), \tag{1.1}$$

for all $t \in [0, \infty)$, where $\dot{k}(t) = dk(t)/dt$. While this constraint as written allows for the free disposal of capital, the preference specification to be introduced next will imply that (1.1) will always hold with equality when the functions $c(t)$ and $k(t)$ are chosen optimally.

As noted above, what sets this version of the growth model apart from the standard case is the specification of the representative consumer's preferences. Suppose, in particular, that while $c(t)$ denotes the representative consumer's own consumption during each period t, $x(t)$ denotes the economy-wide average level of per-capita consumption, and that the representative consumer's utility at each date t depends on the level of $c(t)$ relative to $x(t)$, according to $\ln(c(t) - \gamma x(t))$, where the parameter γ also lies between zero and one: $0 < \gamma < 1$. Since utility at t depends on the natural logarithm of $c(t) - \gamma x(t)$, the larger the value of

γ the more the representative consumer cares about his or her own consumption relative to the average. Over the infinite horizon, the consumer's utility is then

$$\int_0^\infty e^{-\rho t} \ln(c(t) - \gamma x(t))\, \mathrm{d}t \tag{1.2}$$

where $\rho > 0$ is the discount rate.

Since the representative consumer is imagined to be just one of many identical consumers in the economy as a whole, he or she takes the behavior of average consumption $x(t)$ as given, and chooses continuously differentiable functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize the utility function in (1.2) subject to the constraint in (1.1) for all $t \in [0, \infty)$, also taking the initial capital stock $k(0)$ as given. Thus, for the representative consumer, $x(t)$ appears in (1.2) as a variable that changes the form of the single-period utility function over time, not as a variable that can be chosen.

a. Define (write down) the Hamiltonian for this problem. Then write down the first-order conditions and differential equations that, according to the maximum principle, characterize the solution to the representative consumer's problem. *Note:* as we discussed in class, there is more than one way of defining the Hamiltonian for a problem like this one; you can use whichever definition you want, but make sure your optimality conditions are consistent with your chosen form of the Hamiltonian. In addition, to derive these optimality conditions, you don't need to impose nonnegativity constraints on $c(t)$ and $k(t)$, as the specification of the utility and production functions implies that these constraints will never bind.

b. Recalling that the representative consumer is just one of many identical consumers in the economy as a whole, note that $c(t) = x(t)$ will hold for all $t \in [0, \infty)$ in the economy's equilibrium: while each individual does not take this into account when solving his or her optimization problem, in the end each individual's consumption must also equal the economy-wide average. If you use this equilibrium condition to substitute out for $x(t)$, your results from part (a) above should take the form of three equations in three unknowns: the values of $c(t)$ and $k(t)$ that solve the consumer's problem, plus the value of the additional variable that you introduced into the problem when setting up the Hamiltonian. Since that additional variable lacks an immediate economy interpretation, rewrite those three equations as a system of two equations that make reference only to the two functions $c(t)$ and $k(t)$ and the model's parameters α, δ, γ, and ρ.

c. If this economy starts in its nontrivial steady state, $c(t) = c^* > 0$ and $k(t) = k^* > 0$ for all $t \in [0, \infty)$, where c^* and k^* are positive constants. Thus, $\dot{c}(t) = \dot{k}(t) = 0$ for all $t \in [0, \infty)$ in this steady state as well. Use these steady-state conditions, together with the two equations you derived in part (b) above, to solve for the steady-state values c^* and k^* of consumption and capital in terms of the model's parameters α, δ, γ, and ρ.

2. Optimal Growth with Habit Formation

Duesenberry's model of preferences provides one explanation as to why people don't seem too much happier today that they were generations ago, even though their incomes and consumptions are much higher. Another model that explains the same phenomenon adapts to a growth framework the preference specifications used in microeconomics by Gary Becker and co-authors George Stigler ("De Gustibus Non Est Disputandum," *American Economic Review* 1977) and Kevin Murphy ("A Theory of Rational Addiction," *Journal of Political Economy* 1988) to explain the optimal consumption of addictive goods. In this alternative model of "habit formation," people care most about how their own consumption today compares to their own consumption in the recent past, and therefore feel badly when consumption grows slowly or not at all, even if the level of their own consumption remains quite high. This question asks you to use dynamic programming to solve a version of the neoclassical growth model in discrete time in which the representative consumer has a utility function that features habit formation.

In this version of the model, output y_t gets produced with capital k_t during each period $t = 0, 1, 2, \ldots$ according to the production function $y_t = k_t^\alpha$, with $0 < \alpha < 1$. Capital depreciates at the constant rate δ, and c_t denotes the representative consumer's consumption during period t. hence the capital stock evolves according to

$$k_t^\alpha + (1-\delta)k_t - c_t \geq k_{t+1} \tag{2.1}$$

for all $t = 0, 1, 2, \ldots$. Once more, this constraint as written allows for the free disposal of capital, but this will never be optimal with the preferences to be described next; (2.1) will always hold with equality when the sequences for c_t and k_t are chosen optimally.

Now what sets this version of the model apart from the standard case is that the representative consumer's utility at each date t depends on his or her consumption during period t relative to his or her own consumption during period $t-1$ according to $\ln(c_t - \gamma c_{t-1})$ where ln denotes the natural logarithm and the parameter γ, satisfying $0 < \gamma < 1$, governs the strength of "habits," or addictive effects, from consumption during the previous period. Over the infinite horizon, the consumer's utility is then

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t - \gamma c_{t-1}), \tag{2.2}$$

where the discount factor satisfies $0 < \beta < 1$. In this version of the model, therefore, the representative consumer takes the initial capital stock k_0 and the initial level of the habit in consumption c_{-1} as given, and chooses sequences $\{c_t\}_{t=0}^\infty$ and $\{k_t\}_{t=1}^\infty$ to maximize the utility function in (2.2) subject to the constraint in (2.1) for all $t = 0, 1, 2, \ldots$.

The utility function in (2.2) is not additively time-separable, since marginal utility at t depends not just on consumption during period t but also on consumption during period $t-1$. Still, the problem can be converted into one that can be solved using dynamic programming methods by defining a second state variable x_t that is equal to the previous period's consumption level c_{t-1} and therefore keeps track of the "habit stock" according to

$$x_t = c_{t-1} \tag{2.3}$$

for all $t = 0, 1, 2, \ldots$. In terms of this new state variable, the utility function in (2.2) can be rewritten equivalently as

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t - \gamma x_t). \tag{2.4}$$

The consumer's problem can now be restated as one of choosing a sequence $\{c_t\}_{t=0}^{\infty}$ for the control variable and sequences $\{x_t\}_{t=1}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ for the state variables to maximize the utility function in (2.4) subject to the constraints in (2.1) and (2.3) for all $t = 0, 1, 2, \ldots$, taking the initial state variables $x_0 = c_{-1}$ and k_0 as given. The extra constraint (2.3) then summarizes the key difference between this model and the previous model: here, the representative consumer explicitly recognizes that his or her choice of c_t at t will affect the level of the habit stock x_{t+1} at $t+1$; whereas in Duesenberry's model, each individual consumer ignores the effect that the choice of his or her own consumption has on the economy-wide average. Before moving on to solve this problem, note once again that the form of the utility and production functions imply that you don't need to explicitly impose nonnegativity constraints on the choices of c_t, x_t, and k_t, since these will never bind at the optimum.

a. In this model of habit formation, the stock variables x_t and k_t completely summarize the state of the economy as it appears to the representative consumer at the beginning of each period t. Hence, the value function that enters into the Bellman equation for the consumer's problem will depend on x_t and k_t, but not separately on t: that is, it can be written as $v(x_t, k_t)$. Using this insight, write down the Bellman equation for the consumer's problem. Then, write down the first-order condition describing the optimal choice of consumption c_t for period t and the two envelope conditions describing the optimal choices of x_t and k_t for period t.

b. Together with the two constraints, (2.1) with equality and (2.3), the first-order and envelope conditions you derived above form a system of five equations in five unknowns: the three unknown variables c_t, x_t, and k_t and the two partial derivatives $v_1(x_t, k_t)$ and $v_2(x_t, k_t)$ of the unknown value function that enters into the Bellman equation. Use the constraint (2.3) to replace x_t in this system with the lagged value of consumption c_{t-1}. Then use two of your three optimality conditions to eliminate reference to $v_1(x_t, k_t)$ and $v_2(x_t, k_t)$ in the two equations that remain: the result should be a system of two equations that make reference only to the variables c_t and k_t for different values of t (that is, two difference equations in the sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$) and the model's parameters α, δ, γ, and β.

c. If this economy starts in its nontrivial steady state, $c_t = c^* > 0$ and $k_t = k^* > 0$ for all $t = 0, 1, 2, \ldots$, where c^* and k^* are positive constants. Use the two equations involving c_t and k_t that you derived in part (b) above to solve for the steady-state values c^* and k^* of consumption and capital in terms of the model's parameters α, δ, γ, and β.

3. Real and Nominal Interest Rates

This problem asks you to use dynamic programming to solve a stochastic model that draws links between nominal interest rates, real interest rates, and the rate of inflation in a way that generalizes the relationships between the same three variables described, most famously, by Irving Fisher in his 1896 monograph, "Appreciation and Interest."

Let A_t denote the dollar value of a representative consumer's financial assets at the beginning of each period $t = 0, 1, 2, \ldots$. During each period, the consumer divides these assets up into an amount to be spent on consumption c_t and amounts to be invested in two types of bonds.

The first type of bond is called a "real bond" because its price at time t and its payoff at time $t+1$ are both expressed in "real terms," that is, in units of consumption. More specifically, each real bond costs the same as q_t units of consumption at t and pays off the cost of one unit of consumption at $t+1$. The second type of bond is called a "nominal bond" because its price at t and its payoff at $t+1$ are both expressed in "nominal terms." that is, in units of dollars. More specifically, each nominal bond costs Q_t dollars at time t and pays off one dollar at $t+1$.

Let P_t denote the dollar price of consumption at time t, let b_t denote the number of real bonds purchased by the consumer during period t, and let B_t denote the number of nominal bonds purchased by the consumer during period t. Then, during each period t, the consumer faces the constraint

$$A_t \geq P_t c_t + P_t q_t b_t + Q_t B_t, \tag{3.1}$$

where, on the right-hand side, the number of real bonds purchased gets multiplied by both P_t and q_t because each real bond costs the same as q_t units of consumption which, in turn, cost $P_t q_t$ dollars at time t.

Given that he or she purchases b_t real bonds and B_t nominal bonds during period t, the consumer then has assets of dollar value

$$A_{t+1} = P_{t+1} b_t + B_t \tag{3.2}$$

at the beginning of period $t+1$ where, on the right-hand side, the number of real bonds gets multiplied by P_{t+1} since each real bond pays off the cost one unit of consumption, which is P_{t+1} dollars at time $t+1$.

Suppose that in this economy, the prices of goods and bonds fluctuate stochastically, so that the vector $\varepsilon_t = (P_t, q_t, Q_t)$ appears as a random shock in the consumer's optimization problem. Suppose, as we did in class, that the shocks follow a Markov process, so that the distribution of ε_{t+1} depends on ε_t but not on previous values of $\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$. Consistent with timing assumptions we made in solving similar stochastic problems in class, assume that today's prices in ε_t are known, but next period's prices in ε_{t+1} are still viewed as random, when the consumer chooses c_t, b_t, and B_t during period t. Then, in this model, the consumer takes his or her initial value of assets A_0 and the initial set of prices in ε_0 as given and chooses contingency plans for c_t, b_t, and B_t for $t = 0, 1, 2, \ldots$ and A_{t+1} for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where the discount factor satisfies $0 < \beta < 1$, the single-period utility function u is strictly increasing, strictly concave, and satisfies $\lim_{c \to 0} u'(c) = \infty$, and E_0 denotes the consumer's expectation at $t = 0$, subject to the constraints (3.1) and (3.2), each of which must hold for all periods $t = 0, 1, 2, \ldots$ and for all possible realizations of the shocks in ε_{t+1}. The assumption on $u'(c)$ as c approaches zero implies that a nonnegativity constraint on the choice of c_t does not have to be imposed, as it will never bind at the optimum. Nonnegativity constraints on b_t, B_t, and A_t will not be imposed either; hence, in effect, the consumer is allowed to borrow as well as save by choosing negative values for b_t and/or B_t.

In order to solve this problem using dynamic programming, it is helpful to recognize, first, that because the single-period utility function is strictly increasing, the constraint in (3.1) will bind for all periods $t = 0, 1, 2, \ldots$. Substituting this binding constraint into the expected utility function yields

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(\frac{A_t}{P_t} - q_t b_t - \frac{Q_t B_t}{P_t}\right). \tag{3.3}$$

The consumer's problem now simplifies to one of choosing contingency plans for b_t and B_t for $t = 0, 1, 2, \ldots$ and A_{t+1} for $t = 1, 2, 3, \ldots$ to maximize (3.3) subject to the constraint in (3.2) for all $t = 0, 1, 2, \ldots$ and all possible realizations of ε_{t+1}, taking A_0 and ε_0 as given. In this problem, the bond purchases b_t and B_t are the control variables, the value of assets A_t is the state variable, and the value function $v(A_t, \varepsilon_t)$ will depend on both the state variable A_t and the vector of shocks ε_t.

a. Write down the Bellman equation for the consumer's problem. Then write down the first-order conditions for b_t and B_t and the envelope condition for A_t that help describe the solution of this problem.

b. Your three equations from part (a) above should make reference to the derivative $v_1(A_t, \varepsilon_t)$ of the unknown value function from the Bellman equation, which lacks an immediate economic interpretation. Use one of these optimality conditions, together with the constraints (3.1) with equality and (3.2), to rewrite the remaining two equations in a form that makes reference only to consumption c_t and the prices P_t, q_t, and Q_t at different periods t and the discount factor β.

c. Since in this model, the representative consumer can invest in real bonds to exchange q_t units of consumption at t for one unit of consumption at $t + 1$, the real interest rate r_t can be defined by

$$r_t = \frac{1}{q_t}.$$

Similarly, since the representative consumer can invest in nominal bonds to exchange Q_t dollars at t for one dollar at $t + 1$, the nominal interest rate R_t can be defined by

$$R_t = \frac{1}{Q_t}.$$

The gross rate of inflation π_{t+1} between t and $t+1$ is measured by

$$\pi_{t+1} = \frac{P_{t+1}}{P_t},$$

and the consumer's intertemporal marginal rate of substitution m_{t+1} between t and $t+1$ is

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$

Use these definitions to rewrite your two equations from part (b) above as two equations that link the real interest rate r_t and t and the nominal interest rate R_t at t to the inflation rate π_{t+1} between t and $t+1$ and the intertemporal marginal rate of substitution m_{t+1} between t and $t+1$.

Solutions to Final Exam

EC720.01 - Math for Economists
Boston College, Department of Economics

Peter Ireland
Fall 2012

Due Saturday, December 15, 2012 at 12 noon

1. Optimal Growth and the Relative Income Hypothesis

The representative consumer takes the initial capital stock $k(0)$ and the time path for the economy-wide average level of per-capita consumption $x(t)$ as given, and chooses continuously differentiable functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t) - \gamma x(t))\,\mathrm{d}t \tag{1.2}$$

subject to the constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t), \tag{1.1}$$

for all $t \in [0, \infty)$.

a. The Hamiltonian for this problem can be defined in present-value form as

$$H(k(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t) - \gamma x(t)) + \pi(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

The maximum principle then implies that the values of $c(t)$ and $k(t)$ that solve the representative consumer's problem must satisfy the first-order condition

$$\frac{e^{-\rho t}}{c(t) - \gamma x(t)} - \pi(t) = 0$$

for the value of $c(t)$ that solves the static, unconstrained problem on the right-hand side of the definition of the Hamilton, and the pair of differential equations

$$\dot{\pi}(t) = -H_k(k(t), \pi(t); t) = -\pi(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = k(t)^\alpha - \delta k(t) - c(t).$$

b. Use the equilibrium condition $c(t) = x(t)$ to rewrite the first-order condition for $c(t)$ as

$$e^{-\rho t} = (1 - \gamma) c(t) \pi(t),$$

then differentiate both sides of this expression to obtain

$$-\rho e^{-\rho t} = (1 - \gamma)\dot{c}(t)\pi(t) + (1 - \gamma)c(t)\dot{\pi}(t).$$

Use the first-order condition again, together with the differential equation for $\dot{\pi}(t)$, to rewrite this last equation as

$$-\rho(1-\gamma)c(t)\pi(t) = (1-\gamma)\dot{c}(t)\pi(t) - (1-\gamma)c(t)\pi(t)[\alpha k(t)^{\alpha-1} - \delta].$$

Dividing both sides of this expression by $(1-\gamma)\pi(t)$ and isolating $\dot{c}(t)$ on the left-hand side, the differential equation can be rewritten as

$$\dot{c}(t) = [\alpha k(t)^{\alpha-1} - \delta - \rho]c(t)$$

and combined with

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t)$$

to form a system of two equations that make reference only to the two functions $c(t)$ and $k(t)$ and the model's parameters α, δ, γ, and ρ.

c. If this economy starts in its nontrivial steady state, $c(t) = c^* > 0$ and $k(t) = k^* > 0$ for all $t \in [0, \infty)$, where c^* and k^* are positive constants. Since $\dot{c}(t) = \dot{k}(t) = 0$ for all $t \in [0, \infty)$ in this steady state as well, two differential equations from above require the steady-state values c^* and k^* to satisfy

$$0 = [\alpha(k^*)^{\alpha-1} - \delta - \rho]c^*$$

and

$$0 = (k^*)^{\alpha} - \delta k^* - c^*.$$

The first of these two equations can be used to solve for

$$k^* = \left(\frac{\delta+\rho}{\alpha}\right)^{1/(\alpha-1)},$$

and the second can be used to solve for

$$c^* = \left(\frac{\delta+\rho}{\alpha}\right)^{\alpha/(\alpha-1)} - \delta\left(\frac{\delta+\rho}{\alpha}\right)^{1/(\alpha-1)}.$$

Interestingly, the differential equations for $c(t)$ and $k(t)$ are the same in this version of the model as they are in the standard neoclassical growth model. The representative consumer in this model may not feel is happy as a consumer in the standard model, but the dynamic behavior of consumption and capital is unchanged.

2. Optimal Growth with Habit Formation

The representative consumer takes the initial capital stock k_0 and the initial habit stock $x_0 = c_{-1}$ as given, and chooses sequences $\{c_t\}_{t=0}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t - \gamma x_t), \tag{2.4}$$

subject to the constraints

$$k_t^\alpha + (1-\delta)k_t - c_t \geq k_{t+1} \tag{2.1}$$

and

$$x_t = c_{t-1} \tag{2.3}$$

for all $t = 0, 1, 2, \ldots$.

a. The Bellman equation for this problem is

$$v(x_t, k_t) = \max_{c_t} \ln(c_t - \gamma x_t) + \beta v[c_t, k_t^\alpha + (1-\delta)k_t - c_t].$$

The first-order condition for c_t is therefore

$$\frac{1}{c_t - \gamma x_t} + \beta v_1[c_t, k_t^\alpha + (1-\delta)k_t - c_t] - \beta v_2[c_t, k_t^\alpha + (1-\delta)k_t - c_t] = 0,$$

and the envelope conditions for x_t and k_t are

$$v_1(x_t, k_t) = -\frac{\gamma}{c_t - \gamma x_t}$$

and

$$v_2(x_t, k_t) = \beta(\alpha k_t^{\alpha-1} + 1 - \delta)v_2[c_t, k_t^\alpha + (1-\delta)k_t - c_t].$$

b. Use the constraints (2.1) and (2.3) to rewrite the first-order condition for c_t as

$$\frac{1}{c_t - \gamma c_{t-1}} + \beta v_1(x_{t+1}, k_{t+1}) - \beta v_2(x_{t+1}, k_{t+1}) = 0$$

and the envelope conditions for x_t and k_t as

$$v_1(x_t, k_t) = -\frac{\gamma}{c_t - \gamma c_{t-1}}$$

and

$$v_2(x_t, k_t) = \beta(\alpha k_t^{\alpha-1} + 1 - \delta)v_2(x_{t+1}, k_{t+1}).$$

Since the envelope condition for x_t must hold for all $t = 0, 1, 2, \ldots$, it implies that

$$v_1(x_{t+1}, k_{t+1}) = -\frac{\gamma}{c_{t+1} - \gamma c_t}.$$

Substitute this expression into the first-order condition for c_t to obtain

$$\beta v_2(x_{t+1}, k_{t+1}) = \frac{1}{c_t - \gamma c_{t-1}} - \frac{\beta\gamma}{c_{t+1} - \gamma c_t}.$$

Since this condition must also hold for all $t = 0, 1, 2, \ldots$, it implies that

$$v_2(x_t, k_t) = \frac{1}{\beta(c_{t-1} - \gamma c_{t-2})} - \frac{\gamma}{c_t - \gamma c_{t-1}}.$$

Substituting these last two expressions into the envelope condition for k_t yields

$$\frac{1}{\beta(c_{t-1} - \gamma c_{t-2})} - \frac{\gamma}{c_t - \gamma c_{t-1}} = (\alpha k_t^{\alpha-1} + 1 - \delta)\left(\frac{1}{c_t - \gamma c_{t-1}} - \frac{\beta\gamma}{c_{t+1} - \gamma c_t}\right),$$

which can be combined with the constraint

$$k_{t+1} = k_t^\alpha + (1-\delta)k_t - c_t$$

to form a system of two difference equations involving the variables c_t and k_t and the parameters α, δ, γ, and β.

c. If this economy starts in its nontrivial steady state, $c_t = c^* > 0$ and $k_t = k^* > 0$ for all $t = 0, 1, 2, \ldots$, where c^* and k^* are positive constants. The two difference equations derived above imply that the steady-state values c^* and k^* must satisfy

$$\frac{1}{\beta(1-\gamma)c^*} - \frac{\gamma}{(1-\gamma)c^*} = [\alpha(k^*)^{\alpha-1} + 1 - \delta]\left[\frac{1}{(1-\gamma)c^*} - \frac{\beta\gamma}{(1-\gamma)c^*}\right]$$

and

$$0 = (k^*)^\alpha - \delta k^* - c^*.$$

Since the first of these two conditions simplifies to

$$1 = \beta[\alpha(k^*)^{\alpha-1} + 1 - \delta],$$

it can be used to solve for

$$k^* = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{1/(\alpha-1)}.$$

The second condition can then be used to solve for

$$c^* = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{\alpha/(\alpha-1)} - \delta\left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{1/(\alpha-1)}.$$

Interestingly, although the difference equations for c_t and k_t in this version of the model with habit formation are quite different from those in the standard neoclassical growth model, the steady-state values c^* and k^* are the same. These observations imply that while habit formation affects the dynamic path towards the steady state, it does not affect the steady state to which the economy eventually converges.

3. Real and Nominal Interest Rates

The representative consumer chooses contingency plans for b_t, and B_t for $t = 0, 1, 2, \ldots$ and A_{t+1} for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(\frac{A_t}{P_t} - q_t b_t - \frac{Q_t B_t}{P_t}\right). \tag{3.3}$$

subject to the constraint

$$A_{t+1} = P_{t+1}b_t + B_t \tag{3.2}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of $\varepsilon_{t+1} = (P_{t+1}, q_{t+1}, Q_{t+1})$, taking A_0 and ε_0 as given.

a. The Bellman equation for this problem is

$$v(A_t, \varepsilon_t) = \max_{b_t, B_t} u\left(\frac{A_t}{P_t} - q_t b_t - \frac{Q_t B_t}{P_t}\right) + \beta E_t v(P_{t+1}b_t + B_t, \varepsilon_{t+1}).$$

The first-order conditions for b_t and B_t are therefore

$$-q_t u'\left(\frac{A_t}{P_t} - q_t b_t - \frac{Q_t B_t}{P_t}\right) + \beta E_t[P_{t+1}v_1(P_{t+1}b_t + B_t, \varepsilon_{t+1})] = 0$$

and

$$-\left(\frac{Q_t}{P_t}\right) u'\left(\frac{A_t}{P_t} - q_t b_t - \frac{Q_t B_t}{P_t}\right) + \beta E_t v_1(P_{t+1}b_t + B_t, \varepsilon_{t+1}) = 0,$$

and the envelope condition for A_t is

$$v_1(A_t, \varepsilon_t) = \left(\frac{1}{P_t}\right) u'\left(\frac{A_t}{P_t} - q_t b_t - \frac{Q_t B_t}{P_t}\right).$$

b. The constraints (3.2) and

$$A_t = P_t c_t + P_t q_t b_t + Q_t B_t \tag{3.1}$$

imply that the first-order and envelope conditions can be written more simply as

$$q_t u'(c_t) = \beta E_t[P_{t+1}v_1(A_{t+1}, \varepsilon_{t+1})],$$

$$\left(\frac{Q_t}{P_t}\right) u'(c_t) = \beta E_t v_1(A_{t+1}, \varepsilon_{t+1}),$$

and

$$v_1(A_t, \varepsilon_t) = \left(\frac{1}{P_t}\right) u'(c_t).$$

Moreover, since the envelope condition must hold for all $t = 0, 1, 2, \ldots$, it implies that

$$v_1(A_{t+1}, \varepsilon_{t+1}) = \left(\frac{1}{P_{t+1}}\right) u'(c_{t+1}).$$

Substituting this last expression into the two first-order conditions yields the simpler expressions

$$q_t u'(c_t) = \beta E_t u'(c_{t+1})$$

and

$$\left(\frac{Q_t}{P_t}\right) u'(c_t) = \beta E_t\left[\left(\frac{1}{P_{t+1}}\right) u'(c_{t+1})\right].$$

c. With the real interest rate defined as

$$r_t = \frac{1}{q_t}$$

and the intertemporal marginal rate of substitution m_{t+1} between t and $t+1$ denoted by

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)},$$

the first-order condition for b_t implies

$$\frac{1}{r_t} = E_t m_{t+1}.$$

And with the nominal interest rate defined by

$$R_t = \frac{1}{Q_t}$$

and the gross rate of inflation denoted by

$$\pi_{t+1} = \frac{P_{t+1}}{P_t},$$

the first-order condition for B_t implies

$$\frac{1}{R_t} = E_t\left(\frac{m_{t+1}}{\pi_{t+1}}\right).$$

These last two results can be used to generalize the relationship between the nominal interest rate, the real interest rate, and the inflation rate described by Irving Fisher. In particular, since

$$\frac{1}{R_t} = E_t\left(\frac{m_{t+1}}{\pi_{t+1}}\right) = \frac{1}{r_t}E_t\left(\frac{1}{\pi_{t+1}}\right) + cov_t\left(m_{t+1}, \frac{1}{\pi_{t+1}}\right),$$

the traditional "Fisher equation" linking the nominal interest rate to the real interest rate and the inflation rate must be modified in two ways to account for the effects of uncertainty. First, the relationship must be re-expressed in terms of the reciprocals of the nominal interest rate, the real interest rate, and the inflation rate; and, second, the relationship must account for the co-variance between the intertemporal marginal rate of substitution and the inflation rate. Suppose in particular that people begin to worry that the economy will fall into a deflationary recession. In this recession, the intertemporal marginal rate of substitution will be high, because the marginal utility of consumption will be high, and the reciprocal of the inflation rate will be high, because the inflation rate will be low, so that the covariance term in the last expression will be positive. The expression then indicates that the nominal interest rate will be R_t lower than what the real interest rate and the expected rate of inflation might otherwise suggest and, indeed, nominal interest rates in the United States have, in recent years, been quite low, perhaps reflecting fears of renewed deflation and recession.

Midterm Exam

EC720.01 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2013

Thursday, October 31, 1:30 - 2:45pm

This exam has three questions on three pages; before you begin, please check to make sure that your copy has all three questions and all three pages. The three questions will be will be weighted equally in determining your overall exam score.

1. The Kuhn-Tucker Theorem

Consider the problem of choosing values for two variables, x and y, to maximize the objective function

$$F(x, y) = xy,$$

subject to the constraints

$$8 \geq x^2 + y^2,$$

$$x \geq 0,$$

and

$$y \geq 0.$$

a. Set up the Lagrangian for this constrained optimization problem. Recall from our conversations in class that this can be done in a number of different ways: feel free to use whichever formulation of the Lagrangian you find most convenient.

b. Next, write down the first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values of x^* and y^* that solve this problem, together with the associated values of the Lagrange multiplier (or multipliers, if you use more than one). Again, as we discussed in class, there are a number of ways to do this, but make sure that your statement of the Kuhn-Tucker conditions is consistent with your chosen definition of the Lagrangian.

c. Finally, use your results from above to find the numerical values of x^* and y^* that solve the problem.

2. Expenditure Minimization

Suppose that a consumer has the utility function

$$U(c_1, c_2) = 2c_1^{1/2} + 2c_2^{1/2},$$

defined over consumption of two goods, where c_1 is consumption of the first good and c_2 is consumption of the second good. Let p_1 be the price of the first good and p_2 be the price of the second. Suppose that the consumer solves an expenditure minimization problem: choose c_1 and c_2 to minimize the cost of achieving a level of utility that is greater than or equal to $\bar{U}$. Assuming that the consumer operates in perfectly competitive markets that allow him or her to buy as many or as few units of each good as he or she likes at the given prices p_1 and p_2, this problem can be stated mathematically as

$$\min_{c_1,c_2} p_1c_1 + p_2c_2 \text{ subject to } 2c_1^{1/2} + 2c_2^{1/2} \geq \bar{U}.$$

a. Set up the Lagrangian for this constrained optimization problem. Recall from our conversations in class that this can be done in a number of different ways: feel free to use whichever formulation of the Lagrangian you find most convenient.

b. Next, write down the first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values of c_1^* and c_2^* that solve this problem, together with the associated value of the Lagrange multiplier. Again, as we discussed in class, there are a number of ways to do this, but make sure that your statement of the Kuhn-Tucker conditions is consistent with your chosen definition of the Lagrangian.

c. Finally, use your results from above to derive expressions for the optimal values of c_1^* and c_2^* as functions of the parameters p_1, p_2, and $\bar{U}$. In doing this, you can assume that the prices of the two goods are always strictly positive.

3. Production and Inventories

Consider a firm that produces y_t units of output during each period $t = 0, 1, \ldots, T$ in order to add to its stock of inventories x_t, which evolves according to

$$x_t + y_t \geq x_{t+1}$$

for all $t = 0, 1, \ldots, T$. The firm starts out with zero inventories, so that $x_0 = 0$, but must have at least $x^* > 0$ units in inventory by the end of the final period T, so that $x_{T+1} \geq x^*$. Suppose that the firm's total costs during each period $t = 0, 1, \ldots, T$ equal the sum of the cost of producing new output, given by $(a/2)y_t^2$, where $a > 0$ is a parameter, and the cost of storing its existing inventory, given by bx_t, where $b > 0$ is another parameter. Thus, if $r \geq 0$ denotes the constant interest rate, the present discounted value of the firm's costs over all periods $t = 0, 1, \ldots, T$ is

$$\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t \left[\left(\frac{a}{2}\right) y_t^2 + bx_t\right].$$

Assume that the firm acts to minimize these discounted costs while still producing at least x^* units of output over the $T + 1$ periods so as to satisfy the constraint $x_{T+1} \geq x^*$.

The easiest way to solve this problem using methods that we discussed in class is to convert it into a constrained maximization problem by multiplying the firm's costs by -1. Consider, therefore, the problem of choosing sequences $\{y_t\}_{t=0}^T$ for output and $\{x_t\}_{t=1}^{T+1}$ for the inventory stock in order to maximize

$$\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t \left[\left(-\frac{a}{2}\right) y_t^2 - bx_t\right].$$

subject to the constraints

$$x_t + y_t \geq x_{t+1}$$

for all $t = 0, 1, \ldots, T$,

$$x_0 = 0,$$

and

$$x_{T+1} \geq x^* > 0.$$

Strictly speaking, this problem only makes sense if the firm produces positive amounts of output y_t in each period $t = 0, 1, \ldots, T$. It is possible to show, however, that this will be true so long as the final inventory requirement x^* is large enough. For simplicity, you can assume that this condition holds, so that there is no need to explicitly impose nonnegativity constraints on y_t or x_t when solving the problem below.

a. Write down the Hamiltonian for this problem. Recall from our conversations in class that this can be done in a number of different ways ways: feel free to use whichever formulation of the Hamiltonian you find most convenient.

b. Next, write down the first-order condition and the pair of difference equations that, according to the maximum principle, must be satisfied by the sequences of values that solve the firm's problem. Again, as we discussed in class, there are a number of ways to do this, but make sure that your statement of the optimality conditions is consistent with your chosen definition of the Hamiltonian.

c. Finally, use your results to answer the following question. Assuming that, as indicated above, the firm's output y_t is always strictly positive, the cost parameters $a > 0$ and $b > 0$ are both strictly positive, and the interest rate $r \geq 0$ is nonnegative, will the firm optimally choose to have production y_t that is rising, falling, or constant over time?

Solutions to Midterm Exam

EC720.01 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2013

Thursday, October 31, 1:30 - 2:45pm

1. The Kuhn-Tucker Theorem

The problem is finding values for two variables, x and y, to maximize the objective function

$$F(x, y) = xy,$$

subject to the constraints

$$8 \geq x^2 + y^2,$$

$$x \geq 0,$$

and

$$y \geq 0.$$

a. One definition of the Lagrangian for this problem is

$$L(x, y, \lambda, \mu, \phi) = xy + \lambda(8 - x^2 - y^2) + \mu x + \phi y.$$

b. The Kuhn-Tucker theorem implies that there exist values λ^*, μ^*, and ϕ^* that, together with the values of x^* and y^* that solve the problem, satisfy the first-order conditions

$$L_1(x^*, y^*, \lambda^*, \mu^*, \phi^*) = y^* - 2\lambda^* x^* + \mu^* = 0,$$

and

$$L_2(x^*, y^*, \lambda^*, \mu^*, \phi^*) = x^* - 2\lambda^* y^* + \phi^* = 0,$$

the constraints

$$L_3(x^*, y^*, \lambda^*, \mu^*, \phi^*) = 8 - (x^*)^2 - (y^*)^2 \geq 0,$$

$$L_4(x^*, y^*, \lambda^*, \mu^*, \phi^*) = x^* \geq 0,$$

and

$$L_5(x^*, y^*, \lambda^*, \mu^*, \phi^*) = y^* \geq 0,$$

the nonnegativity conditions

$$\lambda^* \geq 0,$$

$$\mu^* \geq 0,$$

and

$$\phi^* \geq 0,$$

and the complementary slackness conditions

$$\lambda^*[8 - (x^*)^2 - (y^*)^2] = 0,$$

$$\mu^* x^* = 0,$$

and

$$\phi^* y^* = 0.$$

c. Suppose first that the solution to this problem has $x^* = 0$. Then the first-order condition for x requires that

$$y^* + \mu^* = 0.$$

But this last expression, coupled with the constraint that $y^* \geq 0$ and the nonnegativity condition that $\mu^* \geq 0$, can only hold if $y^* = \mu^* = 0$ as well. Now the first-order condition for y requires that $\phi^* = 0$ and the complementary slackness condition

$$\lambda^*[8 - (x^*)^2 - (y^*)^2] = 8\lambda^* = 0$$

requires that $\lambda^* = 0$ as well. Thus, the combination of values $(x^*, y^*, \lambda^*, \mu^*, \phi^*) = (0, 0, 0, 0, 0)$ satisfies all of the Kuhn-Tucker conditions and provides one candidate for a solution to the original problem.

Suppose next that the solution to the problem has $x^* > 0$. In this case, the first-order condition for y^*, rewritten as

$$x^* + \phi^* = 2\lambda^* y^*$$

implies that $y^* > 0$ must hold as well, since if instead $y^* = 0$, this equation would require that $\phi^* < 0$, which is ruled out by the nonnegativity condition for that Lagrange multiplier. Since $x^* > 0$ and $y^* > 0$ must both hold, complementary slackness requires that $\mu^* = 0$ and $\phi^* = 0$ must hold as well. Now the first-order condition for x implies that

$$\lambda^* = \frac{y^*}{2x^*}$$

while the first-order condition for y implies that

$$\lambda^* = \frac{x^*}{2y^*},$$

but, together, these conditions imply that $(x^*)^2 = (y^*)^2$. Moreover, since $\lambda^* = 1/2 > 0$, the complementary slackness condition

$$\lambda^*[8 - (x^*)^2 - (y^*)^2] = 0$$

requires that

$$8 = (x^*)^2 + (y^*)^2,$$

and hence that $(x^*)^2 = (y^*)^2 = 4$. Thus, the combination of values $(x^*, y^*, \lambda^*, \mu^*, \phi^*) = (2, 2, 1/2, 0, 0)$ satisfies all of the Kuhn-Tucker conditions and provides another candidate for a solution to the original problem. But $x^* = y^* = 2$ yields a higher value for the objective function than $x^* = y^* = 0$, implying that the solution to the problem sets $x^* = y^* = 2$.

2. Expenditure Minimization

The consumer solves

$$\min_{c_1,c_2} p_1c_1 + p_2c_2 \text{ subject to } 2c_1^{1/2} + 2c_2^{1/2} \geq \bar{U}.$$

a. The Lagrangian for this problem can be defined as

$$L(c_1, c_2, \lambda) = p_1c_1 + p_2c_2 - \lambda(2c_1^{1/2} + 2c_2^{1/2} - \bar{U}).$$

b. The Kuhn-Tucker theorem implies that there exists a value λ^* that, together with the values of c_1^* and c_2^* that solve the consumer's problem, satisfies the first-order conditions

$$L_1(c_1^*, c_2^*, \lambda^*) = p_1 - \lambda^*(c_1^*)^{-1/2} = 0$$

and

$$L_2(c_1^*, c_2^*, \lambda^*) = p_2 - \lambda^*(c_2^*)^{-1/2} = 0,$$

the constraint

$$2(c_1^*)^{1/2} + 2(c_2^*)^{1/2} \geq \bar{U}$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*[2(c_1^*)^{1/2} + 2(c_2^*)^{1/2} - \bar{U}] = 0.$$

c. The first-order conditions imply that

$$(c_1^*)^{1/2} = \lambda^*/p_1$$

and

$$(c_2^*)^{1/2} = \lambda^*/p_2.$$

Since the prices of the two goods are strictly positive and the utility function is strictly increasing, the constraint will always bind at the optimum. Hence, using the expressions for $(c_1^*)^{1/2}$ and $(c_2^*)^{1/2}$ just derived,

$$\bar{U} = 2(c_1^*)^{1/2} + 2(c_2^*)^{1/2} = 2\lambda^*/p_1 + 2\lambda^*/p_2,$$

or

$$\lambda^* = \frac{\bar{U}}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)^{-1} = \frac{\bar{U}}{2}\left(\frac{p_1p_2}{p_1 + p_2}\right).$$

Substituting this expression for λ^* back into the earlier ones for $(c_1^*)^{1/2}$ and $(c_2^*)^{1/2}$ yields

$$c_1 = \left[\frac{\bar{U}}{2}\left(\frac{p_2}{p_1 + p_2}\right)\right]^2$$

and

$$c_2 = \left[\frac{\bar{U}}{2}\left(\frac{p_1}{p_1 + p_2}\right)\right]^2$$

3. Production and Inventories

The firm chooses sequences $\{y_t\}_{t=0}^{T}$ for output and $\{x_t\}_{t=1}^{T+1}$ for the inventory stock in order to maximize

$$\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t\left[\left(-\frac{a}{2}\right)y_t^2 - bx_t\right].$$

subject to the constraints

$$x_t + y_t \geq x_{t+1}$$

for all $t = 0, 1, \ldots, T$,

$$x_0 = 0,$$

and

$$x_{T+1} \geq x^* > 0.$$

a. The Hamiltonian for this problem can be defined as

$$H(x_t, \pi_{t+1}; t) = \max_{y_t}\left(\frac{1}{1+r}\right)^t\left[\left(-\frac{a}{2}\right)y_t^2 - bx_t\right] + \pi_{t+1}y_t.$$

b. The maximum principle implies that the solution to the firm's dynamic optimization problem is characterized by the first-order condition for the value of y_t that solves the static and unconstrained optimization problem stated in the definition of the Hamiltonian,

$$-\left(\frac{1}{1+r}\right)^t ay_t + \pi_{t+1} = 0,$$

and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_x(x_t, \pi_{t+1}; t) = \left(\frac{1}{1+r}\right)^t b$$

and

$$x_{t+1} - x_t = H_\pi(x_t, \pi_{t+1}; t) = y_t.$$

c. The first-order condition for y_t implies that

$$\pi_{t+1} = \left(\frac{1}{1+r}\right)^t ay_t$$

and

$$\pi_t = \left(\frac{1}{1+r}\right)^{t-1} ay_{t-1}.$$

Substituting these expressions into the difference equation linking π_{t+1} to π_t yields

$$\left(\frac{1}{1+r}\right)^t ay_t - \left(\frac{1}{1+r}\right)^{t-1} ay_{t-1} = \left(\frac{1}{1+r}\right)^t b$$

or, more simply,

$$y_t - (1 + r)y_{t-1} = b/a.$$

Since $a > 0$, $b > 0$, $r \geq 0$, and output y_t is always strictly positive, this last question implies that

$$0 < b/a = y_t - (1 + r)y_{t-1} = y_t - y_{t-1} - ry_{t-1} \leq y_t - y_{t-1}.$$

Evidently, the firm finds it optimal to have output increasing over time, with $y_t > y_{t-1}$ for all $t = 0, 1, \ldots, T$. The cost of storing inventories counteracts the firm's desire to minimize the convex costs of production by smoothing output over time, so that the firm produces more output in later periods than in earlier periods.

Final Exam

EC720.01 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2013

Due Tuesday, December 17 at 12 noon

This exam has two questions on five pages; please check to make sure that your copy has both questions and all five pages. Question one has five parts and question two has four parts. Each part of each question will be weighted equally in determining your overall exam score, so that question one is worth 25 points in total and question two is worth 20 points in total.

This is an open-book exam, meaning that it is fine for you to consult your notes, textbooks, and other written references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, inside or outside of the class: the answers you submit must be yours and yours alone.

1. Dynamic Pricing by an Industry Leader

Consider a firm that is the "industry leader." This firm acts partly like a monopolist, in the sense that it perceives itself as facing a downward-sloping demand curve and sets the price for its output accordingly. A number of other firms produce the same good, however, and expand or contract the amount they produce and sell in a way, described in detail below, that the leading firm must take as given.

This industry leader produces output at constant marginal cost denoted by $c > 0$. The demand for its output is described by

$$d[p(t), x(t)] = a - bp(t) - x(t),$$

where $a > 0$ and $b > 0$ are positive parameters, $p(t)$ is the leader's price, and $x(t)$ measures the amount of output that the other firms produce and sell, thereby subtracting from the amount that the leader can sell. Hence, the leader's profits at each time $t \in [0, \infty)$ are

$$[p(t) - c][a - bp(t) - x(t)].$$

Assume that all other firms in the industry "follow the leader" by setting the same price for their output. This induces the followers to produce and sell more when the leader raises its price and to produce and sell less when the leader lowers its price. Suppose, in particular, that $x(t)$ evolves according to

$$\dot{x}(t) = k[p(t) - \bar{p}] \tag{1}$$

where, as usual, $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$ denotes the derivative of $x(t)$ with respect to t and where $k > 0$ and $\bar{p} > 0$ are also positive parameters.

The firm must therefore choose a continuously differentiable function $p(t)$ describing the time path for its price to maximize the present discounted value of its profits over the infinite horizon,

$$\int_0^\infty e^{-rt}[p(t) - c][a - bp(t) - x(t)]\mathrm{d}t$$

where $r > 0$ is the constant rate of interest, subject to the constraint shown in (1), which must hold for all $t \in [0, \infty)$, taking the initial value $x(0)$ as given.

This problem does not map directly into the form of the general, continuous-time dynamic optimization problem that we studied in class, because the constraint in (1) is depicted as an equality and because, if we allowed the industry leader to "freely dispose of market share," the version of (1) that would be written as an inequality would take the form

$$\dot{x}(t) \geq k[p(t) - \bar{p}],$$

indicating that $\dot{x}(t)$ could be larger that the value implied by the strict equality in (1) if the leader simply decided to allow the followers to take away additional demand at time t. Using the change of variables

$$y(t) = -x(t), \tag{2}$$

however, the industry leader's problem can be restated as one of choosing continuously differentiable functions $p(t)$ for $t \in [0, \infty)$ and $y(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-rt}[p(t) - c][a - bp(t) + y(t)]\mathrm{d}t$$

subject to the constraints

$$-k[p(t) - \bar{p}] \geq \dot{y}(t) \tag{3}$$

for all $t \in [0, \infty)$, taking $y(0)$ as given.

In all that follows, you can assume for simplicity that the leading firm will always choose a positive price for its output, so that there is no need to explicitly impose an nonnegativity constraint on $p(t)$, and you can assume in addition that the constraint in (3) will always bind when the leading firm behaves optimally.

a. Set up the Hamiltonian for the industry leader's problem. Recall from our conversations in class that this can be done in a number of different ways: feel free to use whichever formulation of the Hamiltonian you find most convenient.

b. Next, write down the first-order condition and the pair of differential equations that, according to the maximum principle, must be satisfied by the choices of $p(t)$ and $y(t)$ that solve the leader's problem. Again, as we discussed in class, there are a number of ways to do this, but make sure that your statement of the optimality conditions is consistent with your chosen definition of the Hamiltonian.

c. A problem in interpreting the optimality conditions as they are given by the maximum principle is that those conditions make reference to an additional function, which we called either $\pi(t)$ or $\theta(t)$ in our in-class discussions, that gets introduced in defining the Hamiltonian and that is closely related to the Lagrange multiplier on the constraint (3) that we would have introduced if we had decided to solve the problem using the Kuhn-Tucker theorem instead. By using the change of variables in (2) in reverse and by manipulating the optimality conditions you obtained in part (b), above, derive a pair of differential equations linking $\dot{p}(t)$ and $\dot{x}(t)$ to expressions involving only the functions $p(t)$ and $x(t)$ and the model's parameters a, b, c, k, $\bar{p}$, and r.

d. Use the pair of differential equations you just derived, in part (c), to find steady-state values of $p(t)$ and $x(t)$ that will prevail when the industry leader chooses these variables optimally and when $\dot{p}(t) = \dot{x}(t) = 0$.

e. Draw a phase diagram that shows that, for any given initial value of $x(0)$, there is a unique value of $p(0)$ such that, starting from $x(0)$ and $p(0)$, $x(t)$ and $p(t)$ both converge to their steady-state values. In doing this, you can assume that the model's parameters are such that the steady state values of $x(t)$ and $p(t)$ are both strictly positive.

2. Real Business Cycles

Consider a stochastic economy in which time is discrete, the horizon is infinite, and output Y_t gets produced using capital K_t and labor L_t according to the aggregate production function

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha},$$

with $0 < \alpha < 1$. In this model, Z_t represents a random shock to aggregate productivity that will give rise to "real," meaning non-monetary, business cycle fluctuations in output. This shock is assumed to follow a first-order autoregressive process in its natural logarithm, so that

$$\ln(Z_{t+1}) = \rho \ln(Z_t) + \varepsilon_{t+1},$$

where the parameter ρ, assumed to lie between zero and one, measures the degree of serial correlation in the shock and the innovation ε_{t+1} is independently and identically distributed (iid) over time, with $E_t \varepsilon_{t+1} = 0$.

During each period $t = 0, 1, 2, \ldots$, a social planner chooses the amount of labor L_t to be allocated to production, then divides the economy's output up into an amount C_t to be consumed and an amount I_t to be invested, subject to the aggregate resource constraint

$$Z_t K_t^{\alpha} L_t^{1-\alpha} \geq C_t + I_t.$$

The timing of the realization of the productivity shock is such that, when L_t, C_t, and I_t are chosen, the value of Z_t is known but the value of Z_{t+1} is still viewed as random.

As in the optimal growth example we studied in class, physical capital is assumed to depreciate fully in use during each period; as you are about to see, that makes it possible to find closed-form solutions for the model's key variables even in this stochastic version of the model and even with a variable labor supply. Thus, next period's capital equals this period's investment:

$$K_{t+1} = I_t$$

for all $t = 0, 1, 2, \ldots$.

The model's specification is completed by assuming that the economy has a representative consumer/worker, with preferences over consumption and leisure in each period described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln(C_t) + \theta \ln(1 - L_t)], \tag{4}$$

where the discount factor β lies between zero and one and where the total amount of time available within each period is normalized to equal one, so that $\theta > 0$ measures the weight on leisure $1 - L_t$ relative to consumption C_t in the single-period utility function.

The social planner's problem can now be stated as one of choose contingency plans for C_t, $t = 0, 1, 2, \ldots$, L_t, $t = 0, 1, 2, \ldots$, and K_t, $t = 1, 2, 3, \ldots$ to maximize the expected utility function in (4) subject to the constraints

$$Z_t K_t^{\alpha} L_t^{1-\alpha} \geq C_t + K_{t+1}, \tag{5}$$

for all $t = 0, 1, 2, \ldots$, taking the initial values of K_0 and Z_0 as given.

Once more, in solving this problem, you can assume that it is not necessary to explicitly impose nonnegativity constraints on any of the variables and that the aggregate resource constraint in (5) will always bind when quantities are chosen optimally.

a. Write down the Bellman equation for the social planner's problem.

b. As a first step in solving this problem, guess that the value function takes the time-invariant form

$$v(K_t, Z_t) = F + G \ln(K_t) + H \ln(Z_t),$$

where F, G, and H are constant coefficients that depend on the model's parameters. Then, using this guess, write down the first-order conditions and the envelope condition that help describe the social planner's optimal choices of C_t, L_t, and K_t.

c. Use your results from part (b), above, to find solutions that show how the optimal values of C_t, L_t, and $K_{t+1} = I_t$ chosen during each period $t = 0, 1, 2, \ldots$ depend on the realized value Z_t of the productivity shock, the beginning-of-period capital stock K_t, and the model's parameters α, ρ, β, and θ.

d. Use your results from part (c), above, to answer the following questions:

i. Does this real business cycle model account for the procyclicality of consumption, that is, the strong tendency seen in actual economies around the world for aggregate consumption C_t and aggregate output Y_t to move up and down together during booms and recessions?

ii. Does this model account for the procylicality of investment, that is, the strong tendency seen in actual economies around the world for aggregate investment I_t and aggregate output Y_t to move up and down together during booms and recessions?

iii. Does this model account for the procylicality of employment, that is, the strong tendency seen in actual economies around the world for aggregate time working L_t and aggregate output Y_t to move up and down together during booms and recessions?

Solutions to Final Exam

EC720.01 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2013

Due Tuesday, December 17 at 12 noon

1. Dynamic Pricing by an Industry Leader

The industry leader chooses continuously differentiable functions $p(t)$ for $t \in [0, \infty)$ and $y(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-rt}[p(t) - c][a - bp(t) + y(t)]\mathrm{d}t$$

subject to the constraints

$$-k[p(t) - \bar{p}] \geq \dot{y}(t)$$

for all $t \in [0, \infty)$, taking $y(0)$ as given.

a. The present-value Hamiltonian for the industry leader's problem can be defined as

$$H(y(t), \pi(t); t) = \max_{p(t)} e^{-rt}[p(t) - c][a - bp(t) + y(t)] - \pi(t)k[p(t) - \bar{p}].$$

b. With the Hamiltonian defined as above, the maximum principle implies that the choices of $p(t)$ and $y(t)$ that solve the leader's problem must satisfy the first-order condition

$$e^{-rt}[(a + bc) - 2bp(t) + y(t)] - \pi(t)k = 0$$

and the differential equations

$$\dot{\pi}(t) = -H_y(y(t), \pi(t); t) = -e^{-rt}[p(t) - c]$$

and

$$\dot{y}(t) = H_\pi(y(t), \pi(t); t) = -k[p(t) - \bar{p}].$$

c. To simplify the optimality conditions from part (b), above, replace $y(t)$ with $-x(t)$ and differentiate the first-order condition with respect to t to obtain

$$\dot{\pi}(t)k = -re^{-rt}[(a + bc) - 2bp(t) - x(t)] - e^{-rt}[2b\dot{p}(t) + \dot{x}(t)].$$

Substituting the differential equations $\dot{\pi}(t) = -e^{-rt}[p(t) - c]$ and $\dot{x}(t) = -\dot{y}(t) = k[p(t) - \bar{p}]$ into this last expression yields

$$-e^{-rt}[p(t) - c]k = -re^{-rt}[(a + bc) - 2bp(t) - x(t)] - e^{-rt}2b\dot{p}(t) - e^{-rt}k[p(t) - \bar{p}]$$

or, more simply,

$$\dot{p}(t) = \frac{k(\bar{p} - c) - r[(a + bc) - 2bp(t) - x(t)]}{2b},$$

which can be combined with

$$\dot{x}(t) = k[p(t) - \bar{p}]$$

to obtain a system of two differential equations in the two unknown functions $p(t)$ and $x(t)$.

d. When $\dot{p}(t) = \dot{x}(t) = 0$, the second of the two differential equations derived in part (c), above, requires that

$$p(t) = p^* = \bar{p}.$$

in any steady state. Hence, the first differential equation implies that

$$k(\bar{p} - c) = r[(a + bc) - 2b\bar{p} - x(t)]$$

or

$$x(t) = x^* = (k/r)(c - \bar{p}) + (a + bc) - 2b\bar{p}$$

in any steady-state as well. Assuming that the model's parameters are such that the expression on the right-hand side of this last expression is strictly positive, the model has a unique steady state in which the industry leader's price equals $p^* = \bar{p}$ and the remaining firms produce and sell x^* units of output.

e. The differential equation

$$\dot{x}(t) = k[p(t) - \bar{p}]$$

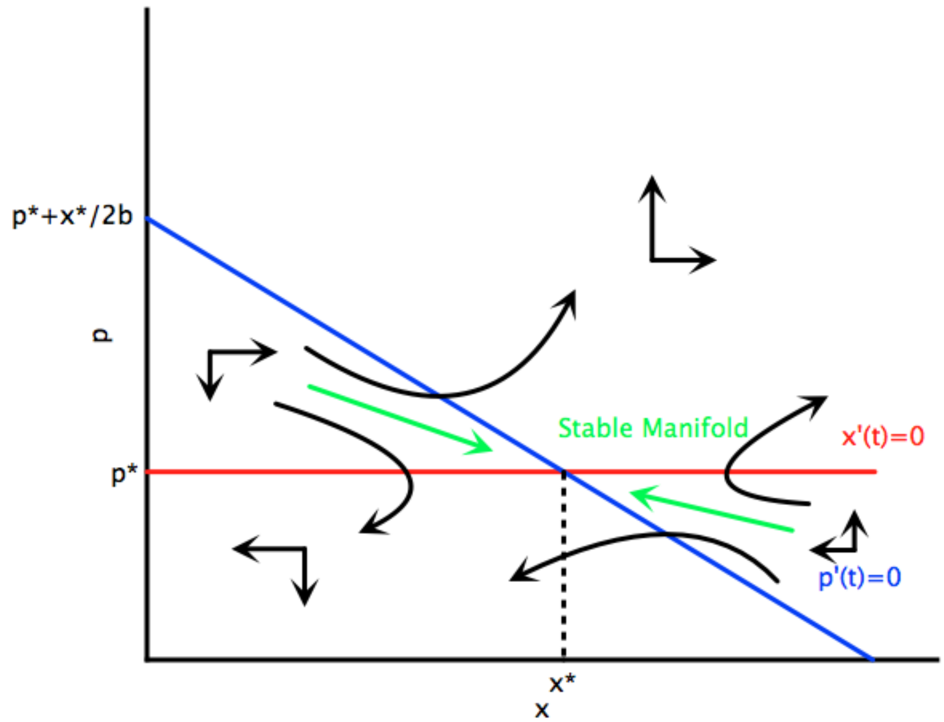
implies that, in a phase diagram with $x(t)$ on the horizontal axis and $p(t)$ on the vertical axis, the $\dot{x}(t) = 0$ locus will be a horizontal line at $p(t) = \bar{p}$. Above this line $x(t)$ will be increasing, and below this line, $x(t)$ will be decreasing. Meanwhile, the differential equation

$$\dot{p}(t) = \frac{k(\bar{p} - c) - r[(a + bc) - 2bp(t) - x(t)]}{2b}$$

implies that the $\dot{p}(t) = 0$ locus will be the straight, downward-sloping line described by the equation

$$p(t) = \frac{(k/r)(c - \bar{p}) + (a + bc)}{2b} - \frac{x(t)}{2b} = \frac{x^* + 2b\bar{p}}{2b} - \frac{x(t)}{2b}.$$

If, as assumed above, the model's parameters are such that the steady-state value x^* is strictly positive, then the y-intercept of this line lies above $\bar{p}$. Moreover, $p(t)$ will be increasing above the $\dot{p}(t) = 0$ locus and decreasing below the $\dot{p}(t) = 0$ locus. Putting all this information together yields a phase diagram that shows that, starting from any initial value of $x(0)$, there is a unique value of $p(0)$ such that, starting from $x(0)$ and $p(0)$, $x(t)$ and $p(t)$ both converge to their steady-state values.



For a more general and thorough analysis of this model, including an argument that confirms that trajectories along the stable manifold are, in fact, optimal, see the article by Darius W. Gaskins, "Dynamic Limit Pricing: Optimal Pricing under Threat of Entry," *Journal of Economic Theory* Vol.3 (1971): pp.306-322.

2. Real Business Cycles

The social planner chooses contingency plans for C_t, $t = 0, 1, 2, \ldots$, L_t, $t = 0, 1, 2, \ldots$, and K_t, $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln(C_t) + \theta \ln(1 - L_t)]$$

subject to the constraints

$$Z_t K_t^{\alpha} L_t^{1-\alpha} \geq C_t + K_{t+1}$$

for all $t = 0, 1, 2, \ldots$, taking as given the initial values of K_0 and Z_0 and the stochastic law of motion

$$\ln(Z_{t+1}) = \rho \ln(Z_t) + \varepsilon_{t+1},$$

for the aggregate productivity shock.

a. The Bellman equation for this problem can be written as

$$v(K_t, Z_t) = \max_{C_t, L_t} \ln(C_t) + \theta \ln(1 - L_t) + \beta E_t v(Z_t K_t^{\alpha} L_t^{1-\alpha} - C_t, Z_{t+1}).$$

b. Using the conjecture that the value function takes the time-invariant form

$$v(K_t, Z_t) = F + G\ln(K_t) + H\ln(Z_t),$$

the Bellman equation becomes

$$\begin{aligned} F + G\ln(K_t) + H\ln(Z_t) &= \max_{C_t, L_t} \ln(C_t) + \theta\ln(1-L_t) + \beta F \\ &\quad + \beta G\ln(Z_t K_t^{\alpha} L_t^{1-\alpha} - C_t) + \beta H\rho\ln(Z_t). \end{aligned}$$

The first-order conditions for C_t and L_t are

$$\frac{1}{C_t} = \frac{\beta G}{Z_t K_t^{\alpha} L_t^{1-\alpha} - C_t} \tag{1}$$

and

$$\frac{\theta}{1-L_t} = \frac{(1-\alpha)\beta G Z_t K_t^{\alpha} L_t^{-\alpha}}{Z_t K_t^{\alpha} L_t^{1-\alpha} - C_t}, \tag{2}$$

and the envelope condition for K_t is

$$\frac{G}{K_t} = \frac{\alpha\beta G Z_t K_t^{\alpha-1} L_t^{1-\alpha}}{Z_t K_t^{\alpha} L_t^{1-\alpha} - C_t}. \tag{3}$$

where F, G, and H are constant coefficients that depend on the model's parameters.

c. Although there are many other ways to derive these same solutions, start here by rearranging the terms in (1) to obtain

$$C_t = \left(\frac{1}{1+\beta G}\right) Z_t K_t^{\alpha} L_t^{1-\alpha}. \tag{4}$$

Next, rewrite (3) as

$$G Z_t K_t^{\alpha} L_t^{1-\alpha} - G C_t = \alpha\beta G Z_t K_t^{\alpha} L_t^{1-\alpha},$$

which, in light of (4), implies

$$\frac{1}{1+\beta G} = 1 - \alpha\beta, \tag{5}$$

and hence

$$G = \frac{\alpha}{1-\alpha\beta}. \tag{6}$$

Combining (4) and (5) yields

$$C_t = (1-\alpha\beta) Z_t K_t^{\alpha} L_t^{1-\alpha}. \tag{7}$$

Substitute this last expression into (2) to obtain

$$\frac{\theta}{1-L_t} = \frac{(1-\alpha)\beta G Z_t K_t^{\alpha} L_t^{-\alpha}}{\alpha\beta Z_t K_t^{\alpha} L_t^{1-\alpha}},$$

which simplifies to

$$\frac{\theta}{1-L_t} = \frac{(1-\alpha)G}{\alpha L_t},$$

and can then be used together with (6) to find

$$L_t = \frac{1-\alpha}{1-\alpha+\theta(1-\alpha\beta)}. \tag{8}$$

Equation (8) provides the desired solution for employment L_t. Substituting (8) into (7) yields the solution for C_t:

$$C_t = (1-\alpha\beta)\left[\frac{1-\alpha}{1-\alpha+\theta(1-\alpha\beta)}\right]^{1-\alpha} Z_t K_t^{\alpha}. \tag{9}$$

Finally, substituting (7) into the aggregate resource constraint and using (8) again yields the solution for

$$K_{t+1} = I_t = \alpha\beta\left[\frac{1-\alpha}{1-\alpha+\theta(1-\alpha\beta)}\right]^{1-\alpha} Z_t K_t^{\alpha}. \tag{10}$$

Equations (8)-(10) can be used to construct sequences for the optimal values of C_t, L_t, and K_t, starting from any initial value K_0 and given any realized sequences of aggregate productivity shocks Z_t. And while it is not necessary to find these values in order to obtain the results just derived, it is also possible to find the solutions

$$H = \left(\frac{1}{1-\beta\rho}\right)\left(\frac{1}{1-\alpha\beta}\right)$$

and

$$F = \left(\frac{1}{1-\beta}\right)\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \left(\frac{1-\alpha}{1-\alpha\beta}\right)\ln(L) + \theta\ln(1-L)\right],$$

where L is the constant value of L_t from (8).

d. To help interpret the results from above, note first that (8) implies that employment L_t is constant, while (9) and (10) can be written more simply as

$$C_t = (1-\alpha\beta)Y_t$$

and

$$I_t = \alpha\beta Y_t,$$

showing that it is optimal in this model to divide aggregate output Y_t up into the constant fraction $1-\alpha\beta$ to be consumed and the constant fraction $\alpha\beta$ is be invested. It then follows that:

i. The model accounts for the procyclicality of consumption, since C_t and Y_t move up and down together.

ii. The model also accounts for the procyclicality of investment, since I_t and Y_t move up and down together.

iii. But this simple model cannot account for the procyclicality of employment, since it implies that L_t is constant.

Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2014

Thursday, October 30, 1:30 - 2:45pm

This exam has three questions on three pages; before you begin, please check to make sure that your copy has all three questions and all three pages. The three questions will be will be weighted equally in determining your overall exam score.

1. Labor Supply

Consider a consumer who likes to consume two goods: goods 1 and 2. This consumer receives N in "non-labor income," which he or she can then augment by working h hours at the competitive wage rate of w per hour. Hence, if he or she purchases c_1 units of good 1 at the competitive price p_1 and c_2 units of good 2 at the competitive price p_2, his or her budget constraint is

$$N + wh \geq p_1 c_1 + p_2 c_2.$$

Suppose the consumer's utility function takes the form

$$\ln(c_1) + \alpha \ln(c_2) + \beta \ln(T - h),$$

where ln denotes the natural logarithm, α and β are positive parameters, and T measures the consumer's total "time endowment," so that $T - h$ measures time allocated to leisure activities instead of working.

a. Write down the Lagrangian for the consumer's problem: choose c_1, c_2, and h to maximize utility subject to the budget constraint. When doing this, make use of the fact that the form of the utility function implies that the consumer will always choose positive amounts of both consumption goods and leisure, so that nonnegativity conditions need not be explicitly imposed on any of these variables. Assume for simplicity, as well, that the parameters of this model are such that the consumer's optimal choice of hours worked h is always positive as well; thus, you don't have to explicitly impose on nonnegativity constraint on this variable, either. Then, write down the first-order conditions, constraint, nonnegativity condition, and complementary slackness condition that, according to the Kuhn-Tucker theorem, characterize the solution to this problem.

b. Use your results from part (a), above, to find solutions for the optimal choices of c_1, c_2, and h in terms of the model's parameters: N, T, α, β, p_1, p_2, and w.

c. According to your answers to part (b), above, what happens to hours optimally worked h^* when nonlabor income N rises: do they rise, fall, or stay the same? What happens to hours optimally worked h^* when the wage rate w rises: do they rise, fall, or stay the same?

2. Cost Minimization

Consider a firm that rents capital at the rental rate r and hires labor at the wage rate w in perfectly competitive factor markets. Suppose that the firm can produce y units of output with k units of capital and l units of labor according to the technology described by

$$k^{1/4}l^{1/4} \geq y.$$

For this problem, suppose that the firm's manager minimizes the total cost of producing y units of output.

a. Write down (define) the Lagrangian for the firm's problem: choose k and l to minimize the total cost $rk+wl$ of producing the y units of output, subject to the constraint $k^{1/4}l^{1/4} \geq y$ imposed by the technology. Then write down the first-order conditions, constraint, nonnegativity condition, and complementary slackness condition that, according to the Kuhn-Tucker theorem, characterize the solution to this problem.

b. Recall from our discussions in class that the firm's *conditional factor demand curves* describe how much capital and labor the firm optimally hires for any given combination of values for r, w, and y. Use your results from part (a), above, to solve for the conditional factor demand curves for capital and labor.

c. Recall also from our discussions in class that the firm's *minimum cost function* describes the firm's minimized costs as a function of r, w, and y. Use your results from part (b), above, to find expressions for the partial derivatives of that minimum cost function with respect to each of the factor prices r and w. *Note:* There are a number of ways to derive these expressions; it doesn't matter which approach you use, so long as you end up with the correct answer.

3. Learning-by-Doing and Long-Run Economic Growth

In the mid-1980s, Paul Romer developed an optimal growth model that, unlike the traditional Ramsey model, implies that there can be long-run growth in aggregate output and consumption. At first, the set up of Romer's model seems to resemble quite closely the Ramsey model. Output y_t is produced with capital k_t according to the aggregate production function

$$y_t = A_t k_t^\alpha,$$

where A_t is a time-varying parameter affecting the marginal product of capital and where the technological parameter α lies between zero and one: $0 < \alpha < 1$. A representative household consumes c_t units of output during each period $t = 0, 1, 2, \ldots$, and capital depreciates at the rate δ between periods, so that the aggregate capital stock evolves according to

$$A_t k_t^\alpha - \delta k_t - c_t \geq k_{t+1} - k_t. \tag{1}$$

The representative household's preferences are described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t), \tag{2}$$

where the discount factor β lies between zero and one, $0 < \beta < 1$, and ln denotes the natural logarithm. Thus, the representative household takes the initial capital stock k_0 as given, chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function in (2) subject to the constraint in (1), which must hold for all $t = 0, 1, 2, \ldots$.

a. Write down an equation that defines the maximized Hamiltonian for the representative household's problem. Then, write down the first-order condition the pair of difference equations that, according to the maximum principle, help describe the solution to this problem.

b. Borrowing an idea first suggested by Kenneth Arrow in the early 1960s, Romer then assumed that the aggregate productivity parameter A_t increases over time as part of a process of "learning-by-doing" according to which technical knowledge gets accumulated along with physical capital. An especially convenient special case that captures this idea assumes that

$$A_t = k_t^{1-\alpha}, \tag{3}$$

for all $t = 0, 1, 2, \ldots$, where α is the same technological parameter that appeared, originally, in the aggregate production function. Note that because Romer imposed the condition in (3) only after deriving the optimality conditions like you just did in part (a), above, he assumed that the creation of technical knowledge is like a positive externality that everyone benefits from but no one takes account of when making their own individual decisions. Following Romer, take (3) and substitute it into the three optimality conditions you derived in part (a), so as to eliminate the variable A_t from those expressions.

c. Romer then showed that when technological knowledge is generated according to (3), output, the capital stock, and consumption can grow in the long run at a constant positive rate, instead of converging to constant steady-state levels as in the Ramsey model. Let

$$\gamma = \frac{c_t}{c_{t-1}} = \frac{k_t}{k_{t-1}} = \frac{y_t}{y_{t-1}}$$

denote this long-run growth rate, and use your results from part (b), above, to find an expression for γ in terms of the parameters α, δ, and β.

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2014

Thursday, October 30, 1:30 - 2:45pm

1. Labor Supply

The consumer chooses c_1, c_2, and h to maximize the utility function

$$\ln(c_1) + \alpha \ln(c_2) + \beta \ln(T - h),$$

subject to the budget constraint

$$N + wh \geq p_1 c_1 + p_2 c_2.$$

a. With the Lagrangian for this problem defined as

$$L(c_1, c_2, h, \lambda) = \ln(c_1) + \alpha \ln(c_2) + \beta \ln(T - h) + \lambda(N + wh - p_1 c_1 - p_2 c_2),$$

the Kuhn-Tucker implies that there exists a value λ^* that, together with the values c_1^*, c_2^*, and h^* that solve the problem, satisfies the first-order conditions

$$L_1(c_1^*, c_2^*, h^*, \lambda^*) = \frac{1}{c_1^*} - \lambda^* p_1 = 0,$$

$$L_2(c_1^*, c_2^*, h^*, \lambda^*) = \frac{\alpha}{c_2^*} - \lambda^* p_2 = 0,$$

and

$$L_3(c_1^*, c_2^*, h^*, \lambda^*) = -\frac{\beta}{T - h^*} + \lambda^* w = 0,$$

the constraint

$$L_4(c_1^*, c_2^*, h^*, \lambda^*) = N + wh^* - p_1 c_1^* + p_2 c_2^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and complementary slackness condition

$$\lambda^*(N + wh^* - p_1 c_1^* - p_2 c_2^*) = 0.$$

b. The first-order conditions imply that

$$c_1^* = \frac{1}{\lambda^* p_1},$$

$$c_2^* = \frac{\alpha}{\lambda^* p_2},$$

and

$$h^* = T - \frac{\beta}{\lambda^* w}.$$

Substituting these expressions into the budget constraint, which will bind at the optimum, yields the solution for λ^* in terms of the model's parameters:

$$\lambda^* = \frac{1+\alpha+\beta}{N+wT}.$$

Substituting this expression back into the previous three yields the solutions for c_1^*, c_2^*, and h^*:

$$c_1^* = \left(\frac{1}{1+\alpha+\beta}\right)\left(\frac{N+wT}{p_1}\right),$$

$$c_2^* = \left(\frac{\alpha}{1+\alpha+\beta}\right)\left(\frac{N+wT}{p_2}\right),$$

and

$$h^* = T - \left(\frac{\beta}{1+\alpha+\beta}\right)\left(\frac{N+wT}{w}\right).$$

Notice that, consistent with our previous analyses using logarithmic utility functions, these expressions show that the consumer spends constant fractions of his or her "total income" $N + wT$, which includes the market value wT of the total time endowment, on each of the two consumption goods and on leisure, where the wage rate w also measures the "price" of leisure.

c. Since the solution for h^* implies that

$$\frac{\partial h^*}{\partial N} = -\left(\frac{\beta}{1+\alpha+\beta}\right)\left(\frac{1}{w}\right) < 0$$

and

$$\frac{\partial h^*}{\partial w} = \left(\frac{\beta}{1+\alpha+\beta}\right)\left(\frac{N}{w^2}\right) > 0,$$

hours worked go down, because of a wealth effect, when nonlabor income N rises but go up, because of a dominant substitution effect, when the wage rate w increases.

2. Cost Minimization

The firm chooses k and l to minimize its costs

$$rk + wl$$

subject to the technological constraint

$$k^{1/4} l^{1/4} \geq y.$$

a. With the Lagrangian for this problem defined as

$$L(k, l, \lambda) = rk + wl - \lambda(k^{1/4}l^{1/4} - y),$$

the Kuhn-Tucker theorem implies that there exists a value λ^* that, together with the values k^* and l^* the solve the problem, satisfies the first-order conditions

$$L_1(k^*, l^*, \lambda^*) = r - (1/4)\lambda^*(k^*)^{-3/4}(l^*)^{1/4} = 0$$

and

$$L_2(k^*, l^*, \lambda^*) = w - (1/4)\lambda^*(k^*)^{1/4}(l^*)^{-3/4} = 0,$$

the constraint

$$L_3(k^*, l^*, \lambda^*) = (k^*)^{1/4}(l^*)^{1/4} - y \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and complementary slackness condition

$$\lambda^*[(k^*)^{1/4}(l^*)^{1/4} - y] = 0.$$

b. Combine the first-order conditions to obtain

$$l^* = \left(\frac{r}{w}\right) k^*,$$

and substitute this expression into the binding constraint to obtain

$$k^*(r, w, y) = \left(\frac{w}{r}\right)^{1/2} y^2.$$

Substitute this last expression back into the one that preceded it to obtain

$$l^*(r, w, y) = \left(\frac{r}{w}\right)^{1/2} y^2.$$

These solutions $k^*(r, w, y)$ and $l^*(r, w, y)$ define the firm's conditional factor demand curves.

c. The minimum cost function is defined by

$$C(r, w, y) = \min_{k,l} rk + wl \text{ subject to } k^{1/4}l^{1/4} \geq y.$$

Although there are other ways of obtaining these expressions, probably the easiest is to apply the envelope theorem, which implies that

$$C_1(r, w, y) = k^*(r, w, y) = \left(\frac{w}{r}\right)^{1/2} y^2$$

and

$$C_2(r, w, y) = l^*(r, w, y) = \left(\frac{r}{w}\right)^{1/2} y^2.$$

These last results are a statement of Shephard's lemma: the derivatives of the minimum cost function coincide with the conditional factor demand curves.

3. Learning-by-Doing and Long-Run Economic Growth

In Romer's model, the representative household takes the initial capital stock k_0 as given, chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints

$$A_t k_t^{\alpha} - \delta k_t - c_t \geq k_{t+1} - k_t$$

for all $t = 0, 1, 2, \ldots$.

a. With the maximized Hamiltonian defined by

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1}(A_t k_t^{\alpha} - \delta k_t - c_t),$$

the maximum principle implies that the solution to the dynamic problem must satisfy the first-order condition for the value of c_t that solves the static problem from the right-hand side,

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0,$$

together with the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}(\alpha A_t k_t^{\alpha-1} - \delta)$$

and

$$k_{t+1} - k_t = H_\pi(k_t, \pi_{t+1}; t) = A_t k_t^{\alpha} - \delta k_t - c_t.$$

b. Using Romer's "learning-by-doing" equation

$$A_t = k_t^{1-\alpha}$$

to substitute out for A_t, the three optimality conditions from part (a), above, simplify to

$$\frac{\beta^t}{c_t} - \pi_{t+1} = 0,$$

$$\pi_{t+1} - \pi_t = -\pi_{t+1}(\alpha - \delta)$$

and

$$k_{t+1} - k_t = (1 - \delta)k_t - c_t.$$

c. Although there are a number of ways to solve for

$$\gamma = \frac{c_t}{c_{t-1}} = \frac{k_t}{k_{t-1}} = \frac{y_t}{y_{t-1}},$$

measuring the economy's long-run growth rate, perhaps the easiest its to use the first-order condition, which implies that

$$\pi_{t+1} = \frac{\beta^t}{c_t}$$

and

$$\pi_t = \frac{\beta^{t-1}}{c_{t-1}}$$

to rewrite the difference equation for π_{t+1} and π_t as

$$\frac{\beta^t}{c_t} - \frac{\beta^{t-1}}{c_{t-1}} = -\frac{\beta^t}{c_t}(\alpha - \delta),$$

which implies that

$$\gamma = \frac{c_t}{c_{t-1}} = \beta(\alpha + 1 - \delta).$$

So long as α is large enough, relative to β and δ, the expression on the right-hand side will be greater than one, indicating that Romer's model features long-run growth.

Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2014

Due Friday, December 19, 11:00am

This exam has three questions on five pages; please check to make sure that your copy has all three questions and all five pages. Each part of each equation has three parts. The three questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Exhaustible Resources, Technological Change, and Economic Growth

Consider an economy in which an exhaustible natural resource is used as an input to produce output of a consumption good. Technological progress allows more output to be produced with the same amount of the natural resource input. The question is: will consumption have to decline as the finite supply of the natural resource runs down, or will technological progress be fast enough to allow the economy to keep growing?

The continuous-time model has an infinite horizon, so that periods are indexed by $t \in [0, \infty)$. During each period, $n(t)$ units of the natural resource get allocated to producing $c(t)$ units of consumption, according to the technology described by

$$e^{zt} n(t)^{\alpha} \geq c(t), \tag{1}$$

where α is a parameter between zero and one, $0 < \alpha < 1$, and $z > 0$ measures the constant rate of technological progress. Let $s(t)$ denote the stock of the natural resource that remains at time t. Then the additional constraint

$$-n(t) \geq \dot{s}(t) \tag{2}$$

describes how this stock gets depleted as the resource is used up in production. Finally, assume that a representative consumer has preferences described by the utility function

$$\int_0^{\infty} e^{-\rho t} \ln(c(t)) \, \mathrm{d}t,$$

where ln denotes the natural logarithm and $\rho > 0$ is the constant discount rate.

Although this problem can be formulated and solved in a variety of ways, it is most convenient to observe that since the production and utility functions are strictly increasing, the constraint (1) will always bind at the optimum. Substitute this constraint into the utility function to obtain

$$\int_0^\infty e^{-\rho t} \ln(e^{zt} n(t)^\alpha)\, dt. \tag{3}$$

Now the representative consumer can be viewed as choosing continuously differentiable functions $n(t)$ for $t \in [0, \infty)$ and $s(t)$ for $t \in (0, \infty)$ in order to maximize the utility function as shown in (3), subject to the constraint (2), which must hold for all $t \in [0, \infty)$, taking the initial stock $s(0) = s_0 > 0$ of the natural resource as given.

a. Define (write down) the maximized Hamiltonian for this problem in present-value form, using $\pi(t)$ to denote the multiplier corresponding to the constraint (2) for period t. Then write down the first-order condition for $n(t)$ and the pair of differential equations for $\pi(t)$ and $s(t)$ that, according to the maximum principle, help characterize the solution to the consumer's problem.

b. The pair of differential equations that you just derived in part (a), above, have solutions of the general form

$$\pi(t) = \pi \tag{4}$$

and

$$s(t) = \left(\frac{\alpha}{\pi\rho}\right) e^{-\rho t} + k, \tag{5}$$

where π and k are two constants that remain to be determined. Find the specific values for these two unknown constants π and k, expressed in terms of the parameters α, z, ρ, and s_0, that allow the general solutions shown in (4) and (5) to also satisfy the initial condition

$$s(0) = s_0$$

and the transversality condition

$$\lim_{T\to\infty} \pi(T)s(T) = 0.$$

c. Finally, combine your results from parts (a) and (b), above, to answer the question: what condition must be satisfied by the parameters α, z, ρ to allow the optimal level of consumption to grow at a positive rate forever?

2. Consumption, Saving, and Labor Supply Under Certainty

Let A_t denote a consumer's bank account balance at the beginning of each period $t = 0, 1, 2, \ldots$. During each period t, the consumer works h_t hours; if w_t denotes the hourly wage received by the consumer, then the consumer earns $w_t h_t$ in total labor income during that period. Let c_t denote the consumer's consumption during period t, and let

$$s_t = A_t + w_t h_t - c_t \tag{6}$$

be a measure of gross (total) savings during period t. If the bank account pays interest at the constant rate $r > 0$, then the consumer's bank account balance evolves according to the constraint

$$(1+r)s_t \geq A_{t+1} \tag{7}$$

for all $t = 0, 1, 2, \ldots$. Suppose the consumer is allowed to borrow by choosing negative values for A_t for $t = 1, 2, 3, \ldots$, but that the initial bank account balance $A_0 = 0$ is taken as given. Finally, let the consumer's preferences be described by the utility function

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) - d(h_t)],$$

where the discount factor β lies between zero and one, $0 < \beta < 1$, the sub-utility function u for consumption is strictly increasing and strictly concave, so that $u'(c) > 0$ and $u''(c) < 0$ for all values of c, and the "disutility" function d for hours worked is strictly increasing and strictly convex, so that $d'(h) > 0$ and $d''(h) > 0$ for all values of h.

To solve the consumer's problem using dynamic programming, it is convenient to use (6) to substitute out for c_t in the utility function. The consumer's problem can then be stated as: choose sequences for the flow (control) variables $\{h_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=0}^{\infty}$ and the stock (state) variable $\{A_t\}_{t=1}^{\infty}$ to maximize the objective function

$$\sum_{t=0}^{\infty} \beta^t [u(A_t + w_t h_t - s_t) - d(h_t)],$$

subject to the constraint shown in (7) for all $t = 0, 1, 2, \ldots$, taking the initial condition $A_0 = 0$ as given.

a. Write down the Bellman equation for the dynamic programming formulation of the consumer's problem. Then write down the first-order conditions for h_t and s_t and the envelope condition for A_t that help describe the solution to this problem. In doing this, you can assume that nonnegativity constraints on c_t and h_t will never bind, so that they can be safely ignored when setting up and solving the problem.

b. Together with the the binding constraints (6) and (7) with equality, the three optimality conditions you just derived in part (a), above, form a system of five equations in the five unknowns: four unknown variables c_t, h_t, s_t, and A_t and the unknown value function $v(\cdot; t)$ that is defined recursively by the Bellman equation itself. Use the constraints (6) and (7), together with one of the optimality conditions from part (a), to reduce this system to one that consists of just two equations describing the optimal behavior of consumption c_t and hours worked h_t over time.

c. Suppose finally that the discount factor β and the interest rate r happen to be such that $\beta(1+r) = 1$. Use your results from part (b), above, together with the assumptions about the functions u and d also stated above, to answer the following two questions. First, what is happening to consumption over time: is it rising, falling, or staying constant? Second, suppose that the wage rate w_t for some period t is higher than the wage rate w_{t+1} for the next period $t+1$: will hours worked h_t during period t be higher, lower, or the same as hours worked h_{t+1} during period $t+1$?

3. Two-Sector Stochastic Growth

This problem asks you use dynamic programming to solve a two-sector version of the stochastic growth model in which physical capital in both sectors depreciates fully between periods; as in the one-sector case, the assumption of full depreciation allows an explicit solution for the value function to be found via the guess-and-verify method. In this version of the model, the representative consumer has preferences defined over contingency plans for consumption c_{1t} of good 1 and consumption c_{2t} of good 2 for each period $t = 0, 1, 2, \ldots$ as described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\theta \ln(c_{1t}) + (1-\theta)\ln(c_{2t})] \tag{8}$$

where ln denotes the natural logarithm and the discount factor β and the preference parameter θ both lie between zero and one: $0 < \beta < 1$ and $0 < \theta < 1$.

By allocating k_{1t} units of capital to sector 1 and k_{2t} units of capital to sector 2 during each period $t = 0, 1, 2, \ldots$, the representative consumer produces $z_{1t}k_{1t}^{\alpha}$ units of good 1 and $z_{2t}k_{2t}^{\alpha}$ of good 2, which he or she then divides up into the amounts c_{1t} and c_{2t} to be consumed and the amount k_{1t+1} and k_{2t+1} to be carried as capital into the following period $t+1$, subject to the constraints

$$z_{1t}k_{1t}^{\alpha} \geq c_{1t} + k_{1t+1} \tag{9}$$

and

$$z_{2t}k_{2t}^{\alpha} \geq c_{2t} + k_{2t+1}, \tag{10}$$

which must hold for all periods $t = 0, 1, 2, \ldots$ and all possible realizations of the sector-specific productivity shocks z_{1t} and z_{2t}. These shocks are assumed to follow first-order autoregressive processes in their natural logarithms, so that

$$\ln(z_{1t+1}) = \rho \ln(z_{1t}) + \varepsilon_{1t+1} \tag{11}$$

and

$$\ln(z_{2t+1}) = \rho \ln(z_{2t}) + \varepsilon_{2t+1}, \tag{12}$$

where the innovations ε_{1t+1} and ε_{2t+1} are serially uncorrelated and uncorrelated with each other and satisfy $E_t\varepsilon_{1t+1} = E_t\varepsilon_{2t+1} = 0$ for all $t = 0, 1, 2, \ldots$. Note than (9) and (10) imply that only output of good 1 at t can be used as capital in sector 1 at $t+1$ and only output of good 2 at t can be used as capital in sector 2 at $t+1$. The technological parameter α and the persistence parameter ρ both lie between zero and one, with $0 < \alpha < 1$ and $0 < \rho < 1$, and, for simplicity, are assumed to be equal across sectors. The timing of the realization of the productivity shocks is assumed to be such that when c_{1t}, c_{2t}, k_{1t+1}, and k_{2t+1} are chosen, the values and z_{1t} and z_{2t} are known but the values of z_{1t+1} and z_{2t+1} are still viewed as random. Note finally that the assumptions made above imply that $E_t[\ln(z_{1t+1})] = \rho \ln(z_{1t})$ and $E_t[\ln(z_{2t+1})] = \rho \ln(z_{2t})$ for all $t = 0, 1, 2, \ldots$.

Thus, the representative consumer's problem can now be stated as: choose contingency plans for c_{1t} and c_{2t} for $t = 0, 1, 2, \ldots$ and k_{1t} and k_{2t} for $t = 1, 2, 3, \ldots$ to maximize the expected utility function in (8) subject to the constraints in (9) and (10) for all periods $t = 0, 1, 2, \ldots$ and all possible realizations of z_{1t} and z_{2t}, taking the initial capital stocks k_{10} and k_{20} and the initial realizations of the shocks z_{10} and z_{20} as given.

a. After guessing that the value function for this problem takes the form

$$v(k_{1t}, k_{2t}, z_{1t}, z_{2t}) = E + F_1 \ln(k_{1t}) + F_2 \ln(k_{2t}) + G_1 \ln(z_{1t}) + G_2 \ln(z_{2t}),$$

where E, F_1, F_2, G_1 and G_2 are unknown constants, write down the Bellman equation for the representative consumer's problem. Note that you can simplify the Bellman equation by recognizing that under the assumptions made above, $k_{1t+1} = z_{1t}k_{1t}^{\alpha} - c_{1t}$ and $k_{2t+1} = z_{2t}k_{2t}^{\alpha} - c_{2t}$ are both known at time t and $E_t[\ln(z_{1t+1})] = \rho \ln(z_{1t})$ and $E_t[\ln(z_{2t+1})] = \rho \ln(z_{2t})$. Then, write down the first-order conditions for c_{1t} and c_{2t} and the envelope conditions for k_{1t} and k_{2t} that help describe the consumer's optimal choices.

b. Use your results from part (a), above, to find solutions that show how the optimal values of c_{1t}, c_{2t}, k_{1t+1}, and k_{2t+1} depend on the realized values of z_{1t} and z_{2t} of the productivity shocks, the beginning-of-period capital stocks k_{1t} and k_{2t}, and the model's parameters β, θ, α, and ρ.

c. Macroeconomists have noted that consumer expenditures on different goods tend to rise and fall together over the business cycle and that most if not all firms will choose to invest more during booms and less during recessions. Use your results from part (b), above, to answer the following question, keeping in mind that under the assumption that the innovations ε_{1t+1} and ε_{2t+1} in (11) and (12) are uncorrelated with each other, the sector-specific shocks z_{1t} and z_{2t} will show no tendency to move together either. Can this model explain these features of the business cycle: that consumptions of different goods and investments in different sectors tend to rise and fall together over time?

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2014

Due Friday, December 19, 11:00am

1. Exhaustible Resources, Technological Change, and Economic Growth

The representative consumer chooses continuously differentiable functions $n(t)$ for $t \in [0,\infty)$ and $s(t)$ for $t \in (0,\infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \ln(e^{zt} n(t)^\alpha)\, dt \tag{3}$$

subject to the constraint

$$-n(t) \geq \dot{s}(t) \tag{2}$$

for all $t \in [0,\infty)$, taking the initial stock $s(0) = s_0 > 0$ of the natural resource as given.

a. In present-value form, the maximized Hamiltonian for this problem is

$$H(s(t),\pi(t);t) = \max_{n(t)} e^{-\rho t} \ln(e^{zt} n(t)^\alpha) - \pi(t)n(t).$$

The maximum principle implies that the solution to the dynamic optimization problem is characterized by the first-order condition

$$\frac{\alpha e^{-\rho t}}{n(t)} - \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_s(s(t),\pi(t);t) = 0$$

and

$$\dot{s}(t) = H_\pi(s(t),\pi(t);t) = -n(t).$$

b. The pair of differential equations from part (a), above, have solutions of the general form

$$\pi(t) = \pi \tag{4}$$

and

$$s(t) = \left(\frac{\alpha}{\pi\rho}\right) e^{-\rho t} + k. \tag{5}$$

where π and k are two constants that remain to be determined. Since (5) implies that

$$s(0) = \frac{\alpha}{\pi\rho} + k$$

and since (4) and (5) imply that

$$\lim_{T\to\infty} \pi(T)s(T) = \lim_{T\to\infty} \left[\left(\frac{\alpha}{\rho}\right) e^{-\rho T} + \pi k\right] = \pi k,$$

the initial and transversality conditions require π and k to satisfy

$$\frac{\alpha}{\pi\rho} + k = s_0$$

and

$$\pi k = 0.$$

These last two equations pin down the specific values

$$k = 0$$

and

$$\pi = \frac{\alpha}{\rho s_0}.$$

c. Since the results from above imply that

$$\pi(t) = \pi = \frac{\alpha}{\rho s_0},$$

the first-order condition for $n(t)$ implies that

$$n(t) = \frac{\alpha e^{-\rho t}}{\pi} = \rho s_0 e^{-\rho t}.$$

This solution for $n(t)$ implies that usage of the natural resource input will decline to zero as the stock of the natural resource is gradually depleted. Consumption, however, is determined by the production function as

$$c(t) = e^{zt} n(t)^{\alpha} = (\rho s_0)^{\alpha} e^{(z-\alpha\rho)t}.$$

This solution shows that so long as the rate of technological progress is fast enough so that

$$z > \alpha\rho$$

is satisfied, consumption will grow at a positive rate forever.

2. Consumption, Saving, and Labor Supply Under Certainty

The representative consumer chooses sequences for the flow (control) variables $\{h_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=0}^{\infty}$ and the stock (state) variable $\{A_t\}_{t=1}^{\infty}$ to maximize the objective function

$$\sum_{t=0}^{\infty} \beta^t [u(A_t + w_t h_t - s_t) - d(h_t)],$$

subject to the constraint

$$(1+r)s_t \geq A_{t+1} \tag{7}$$

for all $t = 0, 1, 2, \ldots$, taking the initial condition $A_0 = 0$ as given.

a. The Bellman equation for this problem is

$$v(A_t;t) = \max_{h_t,s_t} u(A_t + w_t h_t - s_t) - d(h_t) + \beta v[(1+r)s_t; t+1].$$

The first-order conditions for h_t and s_t are

$$w_t u'(A_t + w_t h_t - s_t) - d'(h_t) = 0$$

and

$$-u'(A_t + w_t h_t - s_t) + \beta(1+r)v'[(1+r)s_t; t+1]$$

and the envelope condition for A_t is

$$v'(A_t;t) = u'(A_t + w_t h_t - s_t).$$

b. Use the binding constraints

$$c_t = A_t + w_t h_t - s_t$$

and

$$A_{t+1} = (1+r)s_t$$

to rewrite the three optimality conditions from part (a), above, as

$$w_t u'(c_t) = d'(h_t),$$

$$u'(c_t) = \beta(1+r)v'(A_{t+1}; t+1),$$

and

$$v'(A_t;t) = u'(c_t).$$

The first of these three equations can be rewritten as

$$w_t = \frac{d'(h_t)}{u'(c_t)},$$

indicating that the optimizing consumer sets the marginal rate of substitution between consumption and labor equal to the wage. The other two equations can be combined to yield

$$\frac{1}{1+r} = \frac{\beta u'(c_{t+1})}{u'(c_t)},$$

which shows that the optimizing consumer sets the intertemporal marginal rate of substitution equal to the slope of the intertemporal budget constraint. Together, these last two optimality conditions summarize how consumption and hours worked behave over time.

c. If the discount factor β and the interest rate r happen to be such that $\beta(1+r) = 1$, the second optimality condition from part (b), above, requires consumption to be constant over time. Since consumption is constant, the first optimality condition from part (b) requires that w_t and $d'(h_t)$ move together from period to period. Since d is assumed to be convex, this means that w_t and h_t will also move together. Thus, if $w_t > w_{t+1}$, the consumer will find it optimal to set $h_t > h_{t+1}$, working more hours during period t when the wage is higher.

3. Two-Sector Stochastic Growth

The representative consumer chooses contingency plans for c_{1t} and c_{2t} for $t = 0, 1, 2, \dots$ and k_{1t} and k_{2t} for $t = 1, 2, 3, \dots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\theta \ln(c_{1t}) + (1-\theta)\ln(c_{2t})] \tag{8}$$

subject to the constraints

$$z_{1t}k_{1t}^{\alpha} \geq c_{1t} + k_{1t+1} \tag{9}$$

and

$$z_{2t}k_{2t}^{\alpha} \geq c_{2t} + k_{2t+1}, \tag{10}$$

for all periods $t = 0, 1, 2, \dots$ and all possible realizations of z_{1t} and z_{2t}, taking the initial capital stocks k_{10} and k_{20} and the initial realizations of the shocks z_{10} and z_{20} as given. The sector-specific shocks follow the autoergressive processes

$$\ln(z_{1t+1}) = \rho \ln(z_{1t}) + \varepsilon_{1t+1} \tag{11}$$

and

$$\ln(z_{2t+1}) = \rho \ln(z_{2t}) + \varepsilon_{2t+1}, \tag{12}$$

where the innovations ε_{1t+1} and ε_{1t+2} are serially uncorrelated and uncorrelated with each other and satisfy $E_t\varepsilon_{1t+1} = E_t\varepsilon_{1t+2} = 0$ for all $t = 0, 1, 2, \dots$.

a. Using the conjectured form

$$v(k_{1t}, k_{2t}, z_{1t}, z_{2t}) = E + F_1 \ln(k_{1t}) + F_2 \ln(k_{2t}) + G_1 \ln(z_{1t}) + G_2 \ln(z_{2t}),$$

for the value function, together with the facts that $k_{1t+1} = z_{1t}k_{1t}^{\alpha} - c_{1t}$ and $k_{2t+1} = z_{2t}k_{2t}^{\alpha} - c_{2t}$ are both known at time t and $E_t[\ln(z_{1t+1})] = \rho\ln(z_{1t})$ and $E_t[\ln(z_{2t+1})] = \rho\ln(z_{2t})$, the Bellman equation for this problem can be written as

$$\begin{aligned} & E + F_1 \ln(k_{1t}) + F_2 \ln(k_{2t}) + G_1 \ln(z_{1t}) + G_2 \ln(z_{2t}) \\ = \; & \max_{c_{1t}, c_{2t}} \theta \ln(c_{1t}) + (1-\theta)\ln(c_{2t}) + \beta E + \beta F_1 \ln(z_{1t}k_{1t}^{\alpha} - c_{1t}) \\ & + \beta F_2 \ln(z_{2t}k_{2t}^{\alpha} - c_{2t}) + \beta\rho G_1 \ln(z_{1t}) + \beta\rho G_2 \ln(z_{2t}). \end{aligned}$$

The first-order conditions for c_{1t} and c_{2t} are then

$$\frac{\theta}{c_{1t}} - \frac{\beta F_1}{z_{1t}k_{1t}^{\alpha} - c_{1t}} = 0$$

and

$$\frac{1-\theta}{c_{2t}} - \frac{\beta F_2}{z_{2t}k_{2t}^{\alpha} - c_{2t}} = 0,$$

and the envelope conditions for k_{1t} and k_{2t} are

$$\frac{F_1}{k_{1t}} = \frac{\alpha\beta F_1 z_{1t}k_{1t}^{\alpha-1}}{z_{1t}k_{1t}^{\alpha} - c_{1t}}$$

and

$$\frac{F_2}{k_{2t}} = \frac{\alpha\beta F_2 z_{2t} k_{2t}^{\alpha-1}}{z_{2t}k_{2t}^{\alpha} - c_{2t}}.$$

b. Rewrite the first order conditions as

$$c_{1t} = \left(\frac{\theta}{\theta + \beta F_1}\right) z_{1t}k_{1t}^{\alpha}$$

$$c_{2t} = \left(\frac{1-\theta}{1-\theta + \beta F_2}\right) z_{2t}k_{2t}^{\alpha},$$

and substitute these expressions into the envelope conditions to obtain

$$F_1 z_{1t}k_{1t}^{\alpha} - F_1\left(\frac{\theta}{\theta + \beta F_1}\right) z_{1t}k_{1t}^{\alpha} = \alpha\beta F_1 z_{1t}k_{1t}^{\alpha}$$

and

$$F_2 z_{2t}k_{2t}^{\alpha} - F_2\left(\frac{1-\theta}{1-\theta + \beta F_2}\right) z_{2t}k_{2t}^{\alpha} = \alpha\beta F_2 z_{2t}k_{2t}^{\alpha},$$

which imply that

$$\frac{\theta}{\theta + \beta F_1} = \frac{1-\theta}{1-\theta+\beta F_2} = 1 - \alpha\beta.$$

In light of these last equalities, the first-order conditions for c_{1t} and c_{2t} imply that

$$c_{1t} = (1-\alpha\beta) z_{1t}k_{1t}^{\alpha}$$

and

$$c_{2t} = (1-\alpha\beta) z_{2t}k_{2t}^{\alpha}.$$

The binding constraints for k_{1t+1} and k_{2t+1} then imply that

$$k_{1t+1} = \alpha\beta z_{1t}k_{1t}^{\alpha}$$

and

$$k_{2t+1} = \alpha\beta z_{2t}k_{2t}^{\alpha}.$$

These results show that it is optimal, in both sectors, to consume the fixed fraction $1-\alpha\beta$ of output and save the remaining fraction $\alpha\beta$ regardless of the values of the parameters θ and ρ.

c. The solutions for c_{1t}, c_{2t}, k_{1t+1}, and k_{2t+1} derived in part (b), above, show that consumption of good 1 and investment in sector 1 depend only on the shock to productivity in sector 1, and consumption of good 2 and investment in sector 2 depend only on the shock to productivity in sector 2. Therefore, if the sector-specific shocks are assumed to be uncorrelated, then the model predicts that consumptions of the two goods and investment in the two sectors will be uncorrelated as well. Unless the assumptions are changed to allow the sector-specific shocks to be correlated with each other, this model will not be able to explain the co-movement of spending on different goods and investment in different sectors observed by macroeconomists.

Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2015

Thursday, October 29, 1:30 - 2:45pm

This exam has three questions on four pages; before you begin, please check to make sure that your copy has all three questions and all four pages. Each part of each question is worth five points. Hence, question 1 is worth 20 points, question 2 is worth 10 points, and question 3 is worth 15 points.

1. Long-Run and Short-Run Profit Maximization (20 points)

Consider a firm that produces output y with capital k and labor n according to the technology described by

$$3k^{1/3}n^{1/3} \geq y \tag{1.1}$$

The firm sells each unit of output at the price p, rents each unit of capital at rate r, and hires each unit labor at the wage w. Hence, it chooses y, k, and n to maximize profits

$$py - rk - wn, \tag{1.2}$$

subject to the constraint just shown in (1.1). In answering each of the questions below, you can assume that the prices p, k, and n are such that the constraint (1.1) is always binding when the firm chooses inputs and outputs optimally.

a. Find solutions that show how the firm's optimal choices y^*, k^*, and n^* depend on r, w, and p. *Note:* You can derive these solutions by setting up the Lagrangian for the constrained problem – maximize (1.2) subject to (1.1) – or, if it's easier, you can substitute the binding constraint (1.1) into the expression (1.2) for profits and solve the unconstrained problem

$$\max_{k,n} 3pk^{1/3}n^{1/3} - rk - wn,$$

in which case you'll have to use the binding constraint again to find the optimal choice of y. Either way, you'll arrive at the same solutions for y^*, k^*, and n^*.

b. Now consider a "short-run" version of the same problem, where the capital stock is fixed at some value $\bar{k}$ and the firm solves

$$\max_{y,n} py - r\bar{k} - wn \text{ subject to } 3\bar{k}^{1/3}n^{1/3} \geq y. \tag{1.3}$$

Find solutions that show how the firm's optimal short-run choices y^s and n^s depend on w, p, and $\bar{k}$. *Note:* Again, you can find these solutions by setting up the Lagrangian for the constrained problem shown in (1.3) or, if you find it easier, by substituting the constraint into the objective function, solving the unconstrained problem

$$\max_n 3p\bar{k}^{1/3}n^{1/3} - r\bar{k} - wn,$$

and using the binding constraint to find the optimal choice of y. Either way, you'll arrive at the same solutions for y^s and n^s.

c. Use your solution for $y^* = y^*(r, w, p)$ from part (a) and your solution for $y^s(w, p, \bar{k})$ from part (b) to derive expressions for

$$y_3^*(r, w, p) = \frac{\partial y^*(r, w, p)}{\partial p}$$

and

$$y_2^s(w, p, \bar{k}) = \frac{\partial y^s(w, p, \bar{k})}{\partial p}$$

measuring the firm's optimal long-run and short-run responses of output supplied to changes in the output price p.

d. In general, it is not possible to compare the magnitudes of $y_3^*(r, w, p)$ and $y_2^s(w, p, \bar{k})$, since the former depends on r and the latter depends on $\bar{k}$. Suppose, however, that the fixed stock of capital in the short run just happens to equal the value that is optimal in the long run, so that $\bar{k} = k^*(r, w, p)$. Substitute your solution for $k^*(r, w, p)$ from part (a) into your solution for $y_2^s(w, p, \bar{k})$ from part (c) in order to answer the following question: which is larger, the long-run supply response measured by

$$y_3^*(r, w, p) = \frac{\partial y^*(r, w, p)}{\partial p}$$

or the short-run supply response measured by

$$y_2^s(w, p, \bar{k}) = \frac{\partial y^s(w, p, \bar{k})}{\partial p}$$

when, in the short run, capital is fixed at $\bar{k} = k^*(r, w, p)$.

2. Hotelling's Lemma (10 points)

Consider a firm that produces output y with labor n according to the technology described by

$$n^\alpha \geq y,$$

where $0 < \alpha < 1$. Let w denote the real wage, measured in units of the firm's own output, so that the firm can be depicted as choosing labor input to maximize profits:

$$\max_n n^\alpha - wn.$$

a. Find a solution that shows how the firm's optimal choice of n^* depends on the technological parameter α and the real wage w.

b. Define $V(w)$ as the maximum value function associated with the firm's problem:

$$V(w) = \max_n n^\alpha - wn.$$

Find an expression that shows how the derivative $V'(w)$ of this maximum value function depends on the technological parameter α and the real wage w. *Note:* There are a number of ways to obtain this expression – one is to use the envelope theorem; another is to substitute your solution for n^* into the definition of $V(w)$ and differentiate both sides of the resulting expression by w. You can use whichever approach you find most convenient.

3. Monopolistic Behavior with Learning-By-Doing (15 points)

Consider a monopolistic producer that faces the demand curve

$$p(Q_t) = A_1 - (A_2/2)Q_t, \tag{3.1}$$

where $p(Q_t)$ is the price at which it can sell Q_t units of output, and $A_1 > 0$ and $A_2 > 0$ are both positive parameters. Assume that the firm's cost of producing Q_t units of output is given by

$$c(Q_t) = C_1Q_t + (C_2/2)Q_t^2 - C_3K_tQ_t, \tag{3.2}$$

where $C_1 > 0$, $C_2 > 0$, and $C_3 > 0$ are all positive parameters. The first two terms in (3.2) imply that costs are increasing and convex in the quantity produced. The last term is introduced to capture the effects of learning-by-doing. Let K_t stand for the quantity of accumulated knowledge, and assume that it evolves according to

$$(1-\delta)K_t + Q_t \geq K_{t+1}, \tag{3.3}$$

so that technical knowledge at $t+1$ depends on period t output Q_t (because of learning-by-doing). In (3.3) the parameter δ, satisfying $0 < \delta < 1$, implies that technical knowledge is gradually depreciates away, or is forgotten, unless replenished through new production during each period.

Thus, the monopolist takes the initial stock of knowledge K_0 as given and chooses sequences of values $\{Q_t\}_{t=0}^\infty$ for production and $\{K_t\}_{t=1}^\infty$ for technical knowledge in order to maximize the present discounted value of its profits over an infinite horizon,

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t [p(Q_t)Q_t - c(Q_t)],$$

where profits each period equal revenues $p(Q_t)Q_t$ minus costs $c(Q_t)$ and $r > 0$ is the interest rate. Using (3.1) and (3.2), the monopolist's objective function can be written out more explicitly as

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t [B_1Q_t - (B_2/2)Q_t^2 + C_3K_tQ_t], \tag{3.4}$$

where the new parameters $B_1 = A_1 - C_1$ and $B_2 = A_2 + C_2$ have been defined to keep the statement of the problem more compact, subject to the knowledge accumulation constraint (3.3) for all $t = 0, 1, 2, \ldots$, which you can assume will always bind at the optimum.

a. As a first step in characterizing the solution to this problem using the maximum principle, define (write down) the maximized Hamiltonian for the monopolist's problem.

b. Next, write down the first-order condition and pair of difference equations that, according to the maximum principle, help characterize the solution to the monopolist's problem.

c. Finally, find expressions for steady-state values of output and technical knowledge Q^* and K^* that allow the optimality conditions you derived in part (b), above, to be satisfied with $Q_t = Q^*$ and $K_t = K^*$ for all $t = 0, 1, 2, \ldots$. *Note:* This last part is a bit tedious, so you might not want to start working on it until you're happy with your answers to all of the previous questions. Although there are a number of ways to find the steady-state values, perhaps the easiest is to start by observing that (3.3) requires Q^* and K^* to satisfy $K^* = Q^*/\delta$. Then, use the first-order condition that your derived for Q_t to substitute out for the additional variable, which we denoted by π_{t+1} in class, that you introduced into the problem when setting up the maximized Hamiltonian, in the difference equation linking π_{t+1} and π_t. You'll then have a second equation involving Q^* and K^* that you can use together with $K^* = Q^*/\delta$ to solve for these two steady-state values.

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2015

Thursday, October 29, 1:30 - 2:45pm

1. Long-Run and Short-Run Profit Maximization (20 points)

a. In the long run, the firm solves

$$\max_{k,n} 3pk^{1/3}n^{1/3} - rk - wn.$$

The first-order conditions for this problem are

$$p(k^*)^{-2/3}(n^*)^{1/3} - r = 0$$

and

$$p(k^*)^{1/3}(n^*)^{-2/3} - w = 0.$$

To solve for k^* and n^*, divide the first-order condition for k^* by the first-order condition for n^* to obtain

$$n^* = (r/w)k^*,$$

and substitute this expression into the first-order condition for k^* to find

$$k^*(r, w, p) = r^{-2}w^{-1}p^3.$$

Substitute this solution for k^* back into $n^* = (r/w)k^*$ to obtain

$$n^*(r, w, p) = r^{-1}w^{-2}p^3.$$

Finally, use the production function $y = 3k^{1/3}n^{1/3}$ to solve for

$$y^*(r, w, p) = 3w^{-1}r^{-1}p^2.$$

b. In the short run, the firm solves

$$\max_{n} 3p\bar{k}^{1/3}n^{1/3} - r\bar{k} - wn.$$

The first-order condition for this problem is

$$p\bar{k}^{1/3}(n^s)^{-2/3} - w = 0,$$

which leads directly to the solution

$$n^s(w, p, \bar{k}) = w^{-3/2}p^{3/2}\bar{k}^{1/2}.$$

Substituting this expression into the production function $y = 3\bar{k}^{1/3}n^{1/3}$ yields

$$y^s(w, p, \bar{k}) = 3w^{-1/2}p^{1/2}\bar{k}^{1/2}.$$

c. Using the solution for $y^* = y^*(r, w, p)$ from part (a) and the solution for $y^s(w, p, \bar{k})$ from part (b), the firm's supply response to a change in the output price is measured by

$$y_3^*(r, w, p) = \frac{\partial y^*(r, w, p)}{\partial p} = 6w^{-1}r^{-1}p$$

in the long run and

$$y_2^s(w, p, \bar{k}) = \frac{\partial y^s(w, p, \bar{k})}{\partial p} = (3/2)w^{-1/2}p^{-1/2}\bar{k}^{1/2}$$

in the short run.

d. In general, it is not possible to compare the magnitudes of $y_3^*(r, w, p)$ and $y_2^s(w, p, \bar{k})$, since the former depends on r and the latter depends on $\bar{k}$. If it happens that $\bar{k} = k^*(r, w, p)$, however, the solution

$$k^*(r, w, p) = r^{-2}w^{-1}p^3$$

from part (a) can be combined with the solution

$$y_2^s(w, p, \bar{k}) = \frac{\partial y^s(w, p, \bar{k})}{\partial p} = (3/2)w^{-1/2}p^{-1/2}\bar{k}^{1/2}$$

from part (c) to obtain

$$y_2^s(w, p, k^*(r, w, p)) = (3/2)w^{-1/2}p^{-1/2}(r^{-2}w^{-1}p^3)^{1/2} = (3/2)w^{-1}r^{-1}p.$$

Comparing this last expression to the solution

$$y_3^*(r, w, p) = \frac{\partial y^*(r, w, p)}{\partial p} = 6w^{-1}r^{-1}p$$

from part (c) illustrates a version of the Le Chatelier Principle, by showing that under these conditions, the output response in the long run is larger.

2. Hotelling's Lemma (10 points)

a. When the firm solves

$$\max_n n^\alpha - wn,$$

the first-order condition

$$\alpha(n^*)^{\alpha-1} - w = 0$$

leads directly to the solution

$$n^*(w) = \left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} w^{1/(\alpha-1)}.$$

b. When $V(w)$ is defined as the maximum value function associated with the firm's problem,

$$V(w) = \max_n n^\alpha - wn,$$

the envelope theorem implies that

$$V'(w) = -n^*(w) = -\left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} w^{1/(\alpha-1)}.$$

This condition states a simple version of Hotelling's Lemma: the derivative of the profit function with respect to the real wage coincides with -1 times the labor demand curve.

3. Monopolistic Behavior with Learning-By-Doing (15 points)

a. The monopolist chooses sequences of values $\{Q_t\}_{t=0}^\infty$ for production and $\{K_t\}_{t=1}^\infty$ for technical knowledge in order to maximize

$$\sum_{t=0}^\infty \left(\frac{1}{1+r}\right)^t [B_1 Q_t - (B_2/2)Q_t^2 + C_3 K_t Q_t],$$

subject to the constraint

$$(1-\delta)K_t + Q_t \geq K_{t+1}$$

for all $t = 0, 1, 2, \ldots$. The maximized Hamiltonian for this problem is

$$H(K_t, \pi_{t+1}; t) = \max_{Q_t} \left(\frac{1}{1+r}\right)^t [B_1 Q_t - (B_2/2)Q_t^2 + C_3 K_t Q_t] + \pi_{t+1}(Q_t - \delta K_t).$$

b. According to the maximum principle, the solution to the monopolist's dynamic optimization is characterized by the first-order condition

$$\left(\frac{1}{1+r}\right)^t (B_1 - B_2 Q_t + C_3 K_t) + \pi_{t+1} = 0$$

for the value of Q_t that solves the static optimization problem on the right-hand side of the maximized Hamiltonian and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_K(K_t, \pi_{t+1}; t) = -\left(\frac{1}{1+r}\right)^t C_3 Q_t + \delta \pi_{t+1}$$

and

$$K_{t+1} - K_t = H_\pi(K_t, \pi_{t+1}; t) = Q_t - \delta K_t,$$

where the envelope theorem provides the expressions shown for the partial derivatives H_K and H_π of the maximized Hamiltonian.

c. The first-order condition from part (b), above, implies that

$$\pi_{t+1} = -\left(\frac{1}{1+r}\right)^t (B_1 - B_2 Q_t + C_3 K_t)$$

and

$$\pi_t = -\left(\frac{1}{1+r}\right)^{t-1} (B_1 - B_2 Q_{t-1} + C_3 K_{t-1}).$$

Substituting these equations into the difference equation linking π_{t+1} to π_t and yields

$$\begin{aligned} &\left(\frac{1}{1+r}\right)^{t-1} (B_1 - B_2 Q_{t-1} + C_3 K_{t-1}) \\ = \quad & -\left(\frac{1}{1+r}\right)^t C_3 Q_t + (1-\delta)\left(\frac{1}{1+r}\right)^t (B_1 - B_2 Q_t + C_3 K_t) \end{aligned}$$

or, after multiplying through by $(1+r)^t$,

$$(1+r)(B_1 - B_2 Q_{t-1} + C_3 K_{t-1}) = -C_3 Q_t + (1-\delta)(B_1 - B_2 Q_t + C_3 K_t).$$

This difference equation requires the steady-state values Q^* and K^* to satisfy

$$(1+r)(B_1 - B_2 Q^* + C_3 K^*) = -C_3 Q^* + (1-\delta)(B_1 - B_2 Q^* + C_3 K^*),$$

while the knowledge accumulation constraint requires

$$K^* = Q^*/\delta.$$

Substituting this second steady-state condition into the first yields

$$(1+r)(B_1 - B_2 Q^* + C_3 Q^*/\delta) = -C_3 Q^* + (1-\delta)(B_1 - B_2 Q^* + C_3 Q^*/\delta)$$

or

$$(r+\delta)B_1 = \left[(r+\delta)B_2 - \left(\frac{r+2\delta}{\delta}\right) C_3\right] Q^*,$$

which leads to the solutions

$$Q^* = \frac{(r+\delta)B_1}{(r+\delta)B_2 - \left(\frac{r+2\delta}{\delta}\right) C_3}$$

and

$$K^* = \frac{(r+\delta)B_1}{\delta(r+\delta)B_2 - (r+2\delta)C_3}.$$

These solutions reveal that as the learning-by-doing parameter C_3 increases, so does steady-state output and the stock of technical knowledge.

Final Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2015

Due Friday, December 18, 11:00am

This exam has two questions on five pages; please check to make sure that your copy has all five pages. Each part of each question is worth five points. Hence, question 1 with six parts is worth 30 points, and question 2 with three parts is worth 15 points, for a total of 45 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Investment-Specific Technological Change (30 points)

In the simplest version of the Ramsey model, without any technological change, the economy converges to a steady state in which consumption and capital both remain constant over time. This problem will guide you through an analysis of an extended version of the Ramsey model, in which technological change allows consumption and capital to continue growing forever. Different from the model presented by Acemoglu (pp. 307-312), where technological change is "labor augmenting" and leads to a path along which consumption and capital both grow at the same rate in the long run, technological progress here is "investment-specific" and leads to a path along which consumption and capital grow at different rates in the long run.

As in the standard Ramsey model that we studied in class, output $y(t)$ is produced with capital $k(t)$ during each period $t \in [0, \infty)$ according to the aggregate production function

$$y(t) = k(t)^{\alpha}, \tag{1.1}$$

where α lies between zero and one. As we discussed in class, the production function in (1.1) can be viewed as a special case of a more general, Cobb-Douglas production function according to which output is produced with both capital and labor, under the assumption that aggregate labor supply is constant and normalized to equal one.

During each period $t \in [0, \infty)$, a social planner divides output up into an amount $c(t)$ to be consumed and an amount $i(t)$ to be invested:

$$k(t)^{\alpha} \geq c(t) + i(t). \tag{1.2}$$

The idea that technological change is "investment specific" is that many technological innovations, like improvements in the internal combustion engine or microprocessor, can only be exploited when those innovations are used to create better capital goods, which is only possible when new investment takes place. A simple way to incorporate this idea into the Ramsey model is to modify the capital accumulation constraint to appear as

$$x(t)i(t) - \delta k(t) \geq \dot{k}(t), \tag{1.3}$$

where, just as in the standard model, δ measures the depreciation rate for capital and also lies between zero and one, and where, as usual, $\dot{k}(t)$ denotes the derivative of the function $k(t)$ with respect to t. The new variable $x(t)$ in (1.3) increases over time at the constant rate g, so that

$$\frac{\dot{x}(t)}{x(t)} = g > 0, \tag{1.4}$$

and captures the fact that, as technological innovation takes place, a given amount of investment translates into a larger amount of productive capital. Combining (1.2) and (1.3) yields the single constraint

$$x(t)[k(t)^\alpha - c(t)] - \delta k(t) \geq \dot{k}(t), \tag{1.5}$$

which differs from the analogous constraint in the model without technological change due only to the presence of the new variable $x(t)$ on the left-hand side. Thus, in the special case where $g = 0$ in (1.4) and $x(0) = 1$, there is no investment-specific technological change and this extended model collapses to the simpler one without growth that we studied in class.

Assume that the social planner takes the initial capital stock $k(0)$ as given, and chooses continuously differentiable functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize the additively time-separable utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t)) \, \mathrm{d}t,$$

where $\rho > 0$ is the discount rate and the single-period utility function has the natural log form, subject to the constraint shown in (1.5), which must hold for all $t \in [0, \infty)$.

a. Define (write down) the maximized Hamiltonian for this problem in present-value form, using $\pi(t)$ to denote the multiplier corresponding to the constraint (1.5) for period t.

b. Next, write down the first-order condition for $c(t)$ and the pair of differential equations for $\pi(t)$ and $k(t)$ that, according to the maximum principle, help characterize the solution to the social planner's problem.

c. As a first step in characterizing the model's solution, use the first-order condition for $c(t)$ that you derived in part (b), above, to eliminate the variable $\pi(t)$ from the two differential equations that you also derived in part (b). In order to accomplish this, you might find it helpful to use the product rule for differentiation, which implies, for example, that

$$\frac{\mathrm{d}\pi(t)x(t)c(t)}{\mathrm{d}t} = \dot{\pi}(t)x(t)c(t) + \pi(t)\dot{x}(t)c(t) + \pi(t)x(t)\dot{c}(t).$$

The pair of differential equations that you end up with will involve the three functions $c(t)$, $k(t)$, and $x(t)$ as well as their time derivatives $\dot{c}(t)$, $\dot{k}(t)$, and $\dot{x}(t)$ and, together with (1.4) describing the constant rate of investment-specific technological change, form a system of three differential equations in those three functions.

d. The system of differential equations that you derived in part (c), above, is difficult to analyze for two reasons. First, because it involves three functions, $c(t)$, $k(t)$, and $x(t)$, it is not possible to draw a two-dimensional phase diagram to illustrate the properties of the solution to this system. Second, because consumption $c(t)$, capital $k(t)$, and the technological variable $x(t)$ can all grow forever, the system will not converge to a steady state in which these variables are constant. Both of these problems can be solved by defining two new variables as

$$c^*(t) = \frac{c(t)}{x(t)^{\alpha/(1-\alpha)}} \text{ and } k^*(t) = \frac{k(t)}{x(t)^{1/(1-\alpha)}}, \tag{1.6}$$

that scale the original consumption and capital stock by the ever-increasing value of $x(t)$. Note that because $\alpha < 1$, the exponent on $x(t)$ in the expression for $c^*(t)$ is smaller than the exponent on $x(t)$ in the expression for $k^*(t)$. Note also that these definitions can be rewritten so as to express the original variables in terms of the new, scaled variables:

$$c(t) = x(t)^{\alpha/(1-\alpha)}c^*(t) \text{ and } k(t) = x(t)^{1/(1-\alpha)}k^*(t). \tag{1.7}$$

The chain and product rules for differentiation can then be used to derive the implied expressions for

$$\begin{aligned} \dot{c}(t) &= \left(\frac{\alpha}{1-\alpha}\right) x(t)^{\alpha/(1-\alpha)} \left[\frac{\dot{x}(t)}{x(t)}\right] c^*(t) + x(t)^{\alpha/(1-\alpha)}\dot{c}^*(t) \\ &= \left(\frac{\alpha g}{1-\alpha}\right) x(t)^{\alpha/(1-\alpha)}c^*(t) + x(t)^{\alpha/(1-\alpha)}\dot{c}^*(t) \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \dot{k}(t) &= \left(\frac{1}{1-\alpha}\right) x(t)^{1/(1-\alpha)} \left[\frac{\dot{x}(t)}{x(t)}\right] k^*(t) + x(t)^{1/(1-\alpha)}\dot{k}^*(t) \\ &= \left(\frac{g}{1-\alpha}\right) x(t)^{1/(1-\alpha)}k^*(t) + x(t)^{1/(1-\alpha)}\dot{k}^*(t) \end{aligned} \tag{1.9}$$

where (1.4) has been used to eliminate reference to $\dot{x}(t)/x(t)$ moving from the first to the second lines in each of these two expressions. Use the expressions in (1.7)-(1.9) to rewrite the two differential equations you obtained in part (c), above, in terms of the scaled functions $c^*(t)$ and $k^*(t)$. When you are finished, you should have a system of two differential equations linking $\dot{c}^*(t)$ and $\dot{k}^*(t)$ to $c^*(t)$ and $k^*(t)$ that does not involve the function $x(t)$ or its time derivative $\dot{x}(t)$; to eliminate all references to $x(t)$ and $\dot{x}(t)$, you will probably have to use (1.4) in a manner similar to how, in (1.8) and (1.9), $\dot{x}(t)/x(t)$ was replaced by the constant parameter g.

e. Draw a phase diagram to illustrate the behavior of solutions to the system of two differential equations involving $c^*(t)$ and $k^*(t)$ that you derived in part (d), above. To do this, put $k^*(t)$ on the x-axis and $c^*(t)$ on the y-axis. Start by drawing the $\dot{k}^*(t) = 0$ and $\dot{c}^*(t) = 0$ isoclines, and then use the differential equations to add arrows showing the direction of trajectories for $c^*(t)$ and $k^*(t)$ in various areas of the graph. Finally, draw in the stable manifold showing trajectories that converge to the steady state in which $k^*(t)$ and $c^*(t)$ are both constant.

f. The phase diagram that you drew in answering part (e), above, should show that, starting from any initial value of $k^*(0)$ implied by any initial value of $k(0)$, there is a unique value of $c^*(0)$ and hence a unique value of $c(0)$ that places the transformed system that you derived in part (d), above, on its stable manifold, along which $k^*(t)$ and $c^*(t)$ converge to their constant, steady-state values. Choices for initial consumption that are "too high" place the economy on trajectories that lead to negative values for capital in finite time, whereas choices for initial consumption that are "too low" place the economy on paths that violate the transversality condition, which for this version of the model can be written either in terms of the original functions or the scaled functions and is given by

$$\lim_{T\to\infty}\left[\frac{e^{-\rho T}}{x(T)c(T)}\right]k(T) = \lim_{T\to\infty}\left[\frac{e^{-\rho T}}{c^*(T)}\right]k^*(T) = 0.$$

The expressions in (1.7) then imply that even after the scaled functions $c^*(t)$ and $k^*(t)$ reach their steady-state values, the original functions $c(t)$ and $k(t)$ for continue to grow because of the effects of investment-specific technological change. To finish deriving this model's implications, use these expressions in (1.7), together with the assumption in (1.4) that $x(t)$ grows at the constant rate $g > 0$, to derive solutions for the constant growth rate of consumption $\dot{c}(t)/c(t)$ and the constant growth rate of capital $\dot{k}(t)/k(t)$ that prevail once $c^*(t)$ and $k^*(t)$ have reached their steady-state levels. Then answer the question: which grows faster in this steady state, consumption $c(t)$ or the capital stock $k(t)$?

2. Production and Inventories with Uncertain Demand (15 points)

Consider a discrete-time, infinite-horizon model of a firm with sales S_t that fluctuate stochastically according to

$$S_t = \bar{S} + \varepsilon_{t+1}, \tag{2.1}$$

where $\bar{S}$ measures the constant expected level of sales during each period t and ε_{t+1} is a serially uncorrelated random shock with mean $E_t\varepsilon_{t+1} = 0$ and variance $E_t\varepsilon_{t+1}^2 = \sigma^2$ for all $t = 0, 1, 2, \ldots$. The seemingly unusual timing of variables in equation (2.1), with sales S_t during period t depending on the shock ε_{t+1} dated $t+1$, is meant to capture the idea that the firm must make its decision of how many units of output Q_t to produce during period t before seeing the shock to its sales.

Let I_t denote the firm's stock of inventories at the beginning of each period $t = 0, 1, 2, \ldots$. If negative, I_t measures a backlog of orders that the firm must make up for with higher production later on. Now the timing of events can be described explicitly, as follows: Given the value of I_t at the beginning of each period t, the firm chooses how many new units of output Q_t to produce during period t. Then, the shock ε_{t+1} to sales is realized, so that the firm enters period $t+1$ with the inventory/order backlog variable equal to

$$I_{t+1} = I_t + Q_t - S_t,$$

or, using (2.1),

$$I_{t+1} = I_t + Q_t - \bar{S} - \varepsilon_{t+1}. \tag{2.2}$$

During each period $t = 0, 1, 2, \ldots$, the firm faces two types of costs. The first is a quadratic cost that the firm incurs whenever its period-t production Q_t deviates from the "normal" value $\bar{S}$; the second is another quadratic cost that the firm incurs either when it has to carry inventories $I_t > 0$ across periods or when it has a backlog of sales $I_t < 0$ so that it must delay shipments to its customers. The firm's managers take the initial value of I_0 as given, and choose contingency plans for production Q_t, $t = 0, 1, 2, \ldots$, and inventories/backlogs I_t, $t = 1, 2, 3, \ldots$, subject to the constraint shown in (2.2) for all periods $t = 0, 1, 2, \ldots$ and all possible realizations of ε_{t+1}, in order to minimize the expected, present discounted sum of these costs; equivalently, the firm chooses the contingency plans to maximize

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[-\left(\frac{c_1}{2}\right)(Q_t - \bar{S})^2 - \left(\frac{c_2}{2}\right) I_t^2\right]$$

where $c_1 > 0$ and $c_2 > 0$ are positive parameters governing the size of the two quadratic costs and r is the constant interest rate.

a. Write down the Bellman equation for the firm's problem, after guessing that the value function takes the time-invariant form

$$v(I_t; t) = v(I_t) = \gamma_0 + \left(\frac{\gamma_1}{2}\right) I_t^2.$$

b. Using this guess for the value function, write down the first-order condition for the firm's optimal choice of Q_t and the envelope condition for I_t.

c. Use your results from above to derive the "algebraic Riccati equation" that would allow you to solve for the value of the unknown coefficient γ_1 from the value function given specific values for the parameters c_1, c_2, and r from the original optimization problem. *Note:* All you need to do for this part of the problem is to write down the Riccati equation for γ_1, you don't actually have to solve it, nor do you need to write down the analogous equation for the other undetermined coefficient γ_0 from the value function, which depends on the variance σ^2 of the shock to sales as well as on the parameters c_1, c_2, and r that determine γ_0.

Solutions to Final Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2015

Due Friday, December 18, 11:00am

1. Investment-Specific Technological Change (30 points)

The social planner takes the initial capital stock $k(0)$ as given, and chooses continuously differentiable functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t))\,\mathrm{d}t$$

subject to the constraint

$$x(t)[k(t)^\alpha - c(t)] - \delta k(t) \geq \dot{k}(t),$$

which must hold for all $t \in [0, \infty)$.

a. The maximized Hamiltonian for this problem is

$$H(k(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)\{x(t)[k(t)^\alpha - c(t)] - \delta k(t)\}.$$

b. According to the maximum principle, the solution to the social planner's problem is characterized by the first-order condition

$$\frac{e^{-\rho t}}{c(t)} - \pi(t)x(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_k(k(t), \pi(t); t) = -\pi(t)[\alpha x(t)k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = x(t)[k(t)^\alpha - c(t)] - \delta k(t).$$

c. To eliminate the function $\pi(t)$ from the system of equations describing the solution to the social planner's problem, rewrite the first-order condition as

$$e^{-\rho t} = c(t)\pi(t)x(t)$$

and differentiate both sides with respect to t to obtain

$$-\rho e^{-\rho t} = \dot{c}(t)\pi(t)x(t) + c(t)\dot{\pi}(t)x(t) + c(t)\pi(t)\dot{x}(t).$$

Substitute the first-order condition back into the left-hand side and substitute the differential equation for $\dot{\pi}(t)$ into the right-hand side to get

$$-\rho c(t)\pi(t)x(t) = \dot{c}(t)\pi(t)x(t) - \pi(t)[\alpha x(t)k(t)^{\alpha-1} - \delta]c(t)x(t) + c(t)\pi(t)\dot{x}(t),$$

which can be simplified to read

$$c(t)\dot{x}(t) + \dot{c}(t)x(t) = [\alpha x(t)k(t)^{\alpha-1} - \delta - \rho]c(t)x(t).$$

This last equation, together with

$$\dot{k}(t) = x(t)[k(t)^{\alpha} - c(t)] - \delta k(t)$$

and

$$\dot{x}(t) = gx(t),$$

forms a system of three differential equations involving the three functions $c(t)$, $k(t)$, and $x(t)$.

d. Define the scaled functions

$$c^*(t) = \frac{c(t)}{x(t)^{\alpha/(1-\alpha)}} \text{ and } k^*(t) = \frac{k(t)}{x(t)^{1/(1-\alpha)}},$$

so that

$$c(t) = x(t)^{\alpha/(1-\alpha)}c^*(t),$$
$$k(t) = x(t)^{1/(1-\alpha)}k^*(t),$$
$$\dot{c}(t) = \left(\frac{\alpha g}{1-\alpha}\right)x(t)^{\alpha/(1-\alpha)}c^*(t) + x(t)^{\alpha/(1-\alpha)}\dot{c}^*(t),$$

and

$$\dot{k}(t) = \left(\frac{g}{1-\alpha}\right)x(t)^{1/(1-\alpha)}k^*(t) + x(t)^{1/(1-\alpha)}\dot{k}^*(t).$$

Substituting these various expressions into the first differential equation from part (c), above, and using $\dot{x}(t) = gx(t)$ as well, yields

$$\dot{c}^*(t) = \left[\alpha k^*(t)^{\alpha-1} - \delta - \rho - \left(\frac{g}{1-\alpha}\right)\right]c^*(t).$$

Substituting these same expressions into the second differential equation from part (c) yields

$$\dot{k}^*(t) = k^*(t)^{\alpha} - c^*(t) - \left[\delta + \left(\frac{g}{1-\alpha}\right)\right]k^*(t).$$

These last two differential equations involve the two functions $c^*(t)$ and $k^*(t)$ alone.

e. After defining the new parameter

$$\delta^* = \delta + \left(\frac{g}{1-\alpha}\right),$$

the two differential equations from part (d), above, can be rewritten as

$$\dot{c}^*(t) = [\alpha k^*(t)^{\alpha-1} - \delta^* - \rho]c^*(t).$$

and

$$\dot{k}^*(t) = k^*(t)^{\alpha} - c^*(t) - \delta^* k^*(t).$$

These equations take exactly the same form as the differential equations for $c(t)$ and $k(t)$ in the original Ramsey model, except that δ^* replaces δ and $c^*(t)$ and $k^*(t)$ replace $c(t)$ and $k(t)$. Hence, the phase diagram for this system looks exactly the same as the phase diagram for the original Ramsey model that we drew in class.

f. The expressions linking $\dot{c}(t)$ and $c(t)$ to $\dot{c}^*(t)$ and $c^*(t)$ introduced in part (d), above, imply that

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha g}{1-\alpha} + \frac{\dot{c}^*(t)}{c^*(t)}.$$

Thus, in this model's steady state, where $c^*(t)$ is constant so that $\dot{c}^*(t) = 0$, consumption continues growing at the constant rate

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha g}{1-\alpha}.$$

Similarly, the expressions linking $\dot{k}(t)$ and $k(t)$ to $\dot{k}^*(t)$ and $k^*(t)$ introduced in part (d), above, imply that

$$\frac{\dot{k}(t)}{k(t)} = \frac{g}{1-\alpha} + \frac{\dot{k}^*(t)}{k^*(t)}$$

and, hence, in the model's steady state where $k^*(t)$ is constant so that $\dot{k}^*(t) = 0$, capital continues to grow at the constant rate

$$\frac{\dot{k}(t)}{k(t)} = \frac{g}{1-\alpha}.$$

Since $\alpha < 1$, these solutions imply that capital grows faster than consumption along the steady-state growth path. For a detailed discussion of evidence consistent with these implications of the model, see Jeremy Greenwood, Zvi Hercowitz, and Per Krusell, "Long-Run Implications of Investment-Specific Technological Change," *American Economic Review* 87 (June 1997), pp. 342-362.

2. Production and Inventories with Uncertain Demand (15 points)

The firm's managers take the initial value of I_0 as given, and choose contingency plans for production Q_t, $t = 0, 1, 2, \ldots$, and inventories/backlogs I_t, $t = 1, 2, 3, \ldots$ to maximize

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \left[-\left(\frac{c_1}{2}\right) (Q_t - \bar{S})^2 - \left(\frac{c_2}{2}\right) I_t^2 \right]$$

subject to the constraint

$$I_{t+1} = I_t + Q_t - \bar{S} - \varepsilon_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of ε_{t+1}, where the shock to sales is serially uncorrelated with $E_t \varepsilon_{t+1} = 0$ and $E_t \varepsilon_{t+1}^2 = \sigma^2$.

a. Using the guess that the value function takes the time-invariant form

$$v(I_t; t) = v(I_t) = \gamma_0 + \left(\frac{\gamma_1}{2}\right) I_t^2,$$

the Bellman equation for the firm's problem is

$$\begin{aligned} \gamma_0 + \left(\frac{\gamma_1}{2}\right) I_t^2 &= \max_{Q_t} \; -\left(\frac{c_1}{2}\right) (Q_t - \bar{S})^2 - \left(\frac{c_2}{2}\right) I_t^2 \\ &\quad + \left(\frac{1}{1+r}\right) \gamma_0 + \left(\frac{1}{1+r}\right) \left(\frac{\gamma_1}{2}\right) E_t (I_t + Q_t - \bar{S} - \varepsilon_{t+1})^2. \end{aligned}$$

b. Still using the guess for the value function, the first-order condition for Q_t is

$$-c_1(Q_t - \bar{S}) + \left(\frac{1}{1+r}\right) \gamma_1 E_t (I_t + Q_t - \bar{S} - \varepsilon_{t+1}) = 0$$

or, more simply,

$$c_1(Q_t - \bar{S}) = \left(\frac{\gamma_1}{1+r}\right) (I_t + Q_t - \bar{S}).$$

Meanwhile, the envelope condition for I_t is

$$\gamma_1 I_t = -c_2 I_t + \left(\frac{1}{1+r}\right) \gamma_1 E_t (I_t + Q_t - \bar{S} - \varepsilon_{t+1}),$$

or, again more simply,

$$\gamma_1 I_t = -c_2 I_t + \left(\frac{\gamma_1}{1+r}\right) (I_t + Q_t - \bar{S}).$$

c. The first-order condition from part (b), above, implies that

$$Q_t - \bar{S} = \left[\frac{\gamma_1}{(1+r)c_1 - \gamma_1} \right] I_t.$$

Substituting this expression for $Q_t - \bar{S}$ into the envelope condition yields

$$\gamma_1 I_t = -c_2 I_t + \left[\frac{\gamma_1 c_1}{(1+r)c_1 - \gamma_1} \right] I_t.$$

Since this last condition must hold for all possible values of I_t, it requires that

$$\gamma_1 = -c_2 + \frac{\gamma_1 c_1}{(1+r)c_1 - \gamma_1},$$

which is the algebraic Riccati equation that provides the solution for unknown coefficient γ_1 from the value function given specific values for the parameters c_1, c_2, and r from the original optimization problem. Although it was not necessary to do so in order to answer part (c) of this problem, substituting the expression

$$Q_t - \bar{S} = \left[\frac{\gamma_1}{(1+r)c_1 - \gamma_1} \right] I_t.$$

back into the Bellman equation and simplifying yields the expression

$$\gamma_0 = \left(\frac{\gamma_1}{2r} \right) \sigma^2,$$

which shows how the remaining coefficient γ_0 from the value function depends on the solution for γ_1 from the Riccati equation as well as the parameters r and σ^2.

Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2016

Thursday, October 27, 1:30 - 2:45pm

This exam has three questions on four pages; before you begin, please check to make sure that your copy has all three questions and all four pages. Each part of each question will receive equal weight in determining your overall exam score.

1. The Permanent Income Hypothesis with Multiple Consumption Goods

Consider a consumer who must decide how much to save versus spend on two goods during two periods, $t = 0$ and $t = 1$. Let x_0 and x_1 denote his or her consumption of the first good ("good x") during periods $t = 0$ and $t = 1$, let y_0 and y_1 denote his or her consumption of the second good ("good y") during periods $t = 0$ and $t = 1$, and let s denote his or her savings at $t = 0$, which earn interest at the gross rate $1 + r$ between $t = 0$ and $t = 1$.

If the consumer receives income I_0 during period $t = 0$ and I_1 during period $t = 1$, his or her budget constraints can be written as

$$I_0 \geq p_0 x_0 + q_0 y_0 + s$$

for period $t = 0$ and

$$I_1 + (1 + r)s \geq p_1 x_1 + q_1 y_1$$

for period $t = 1$, where p_t denotes the price of good x and q_t the price of good y during each period $t = 0$ and $t = 1$. Assume that the consumer's preferences are described by the utility function

$$\alpha \ln(x_0) + (1 - \alpha)\ln(y_0) + \beta[\alpha \ln(x_1) + (1 - \alpha)\ln(y_1)], \tag{1.1}$$

where ln denotes the natural logarithm and the parameter α measuring how much the consumer likes good x relative to good y and the discount factor β both lie between zero and one: $0 < \alpha < 1$ and $0 < \beta < 1$.

As useful first step in setting up the consumer's optimization problem, divide the budget constraint for period $t = 1$ by $1 + r$ and add it to the budget constraint for period $t = 0$ to obtain the single "lifetime" budget constraint

$$I \geq p_0 x_0 + q_0 y_0 + \frac{p_1 x_1}{1 + r} + \frac{q_1 y_1}{1 + r}, \tag{1.2}$$

where

$$I = I_0 + \frac{I_1}{1 + r}$$

measures the present value of the consumer's income received over the two periods.

Now the consumer's problem can be stated as: choose x_0, y_0, x_1, and y_1 to maximize utility (1.1) subject to the constraint (1.2).

a. Define (write down) the Lagrangian for the consumer's problem, using λ to denote the multiplier on the constraint (1.2). Then, write down the first-order conditions, constraints, nonnegativity condition, and complementary slackness condition that, according to the Kuhn-Tucker theorem, must be satisfied by the values x_0^*, y_0^*, x_1^*, and y_1^* that solve this problem together with the associated value λ^* of the Lagrange multiplier.

b. Since the utility function in (1.1) is strictly increasing in consumption of all four goods, the budget constraint (1.2) will bind (hold with equality) when x_0^*, y_0^*, x_1^*, and y_1^* are chosen optimally and the prices p_0, p_1, q_0 and q_1 are all strictly positive. Use this fact, together with the first-order conditions you derived in part (a), above, to find solutions for x_0^*, y_0^*, x_1^*, and y_1^* in terms of the prices p_0, p_1, q_0 and q_1 and parameters α, β, r and I.

c. Finally, assume that the discount factor and interest rate are related via $\beta(1+r) = 1$, and use your solutions from part (b), above, to obtain expressions for optimal spending in period $t = 0$,

$$p_0 x_0^* + q_0 y_0^*,$$

and period $t = 1$,

$$p_1 x_1^* + q_1 y_1^*.$$

Does total spending rise, fall, or stay constant across the two periods?

2. The Envelope Theorem

Consider a consumer with income I who purchases c_1 and c_2 units of "goods 1 and 2" at prices p_1 and p_2 in perfectly competitive markets in order to maximize utility $U(c_1, c_2)$ subject to the budget constraint $I \geq p_1 c_1 + p_2 c_2$. The solutions $c_1^* = c_1^*(p_1, p_2, I)$ and $c_2^* = c_2^*(p_1, p_2, I)$, expressed as functions of the prices p_1 and p_2 and income I, describe the consumer's Marshallian demand curves for the two goods. The consumer's indirect utility function $v(p_1, p_2, I)$, meanwhile, is defined by

$$v(p_1, p_2, I) = \max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

a. For any given set of parameter values p_1, p_2, and I, let $\lambda^*(p_1, p_2, I)$ denote the value of the Lagrange multiplier associated with the values of $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$ that solve the consumer's problem. Use the envelope theorem to derive expressions for each of the three partial derivatives of the indirect utility function,

$$v_1(p_1, p_2, I) = \frac{\partial v(p_1, p_2, I)}{\partial p_1},$$

$$v_2(p_1, p_2, I) = \frac{\partial v(p_1, p_2, I)}{\partial p_2},$$

and

$$v_3(p_1, p_2, I) = \frac{\partial v(p_1, p_2, I)}{\partial I}$$

in terms of the functions $\lambda^*(p_1, p_2, I)$, $c_1^*(p_1, p_2, I)$, $c_2^*(p_1, p_2, I)$.

b. Use your results from part (a) above to derive the Marshallian demand curves for goods 1 and 2 in the special case where consumer's indirect utility function is known to be

$$v(p_1, p_2, I) = \ln(I) - \alpha \ln(p_1) - (1 - \alpha) \ln(p_2),$$

where ln denotes the natural logarithm.

c. Still focusing on the special case from part (b), above:

What happens to the optimal choice $c_1^*(p_1, p_2, I)$ if income I doubles, while both prices p_1 and p_2 remain unchanged: does $c_1^*(p_1, p_2, I)$ rise, fall, or stay the same?

What happens to the optimal choice $c_1^*(p_1, p_2, I)$ if the price p_1 of good 1 doubles, while both income I and the price p_2 of good 2 remain unchanged: does $c_1^*(p_1, p_2, I)$ rise, fall, or stay the same?

What happens to the optimal choice $c_1^*(p_1, p_2, I)$ if income I and the two prices p_1 and p_2 all double at the same time: does $c_1^*(p_1, p_2, I)$ rise, fall, or stay the same?

3. Equilibrium Asset Pricing: A Special Case

Consider a special case of the model of stock prices and consumption developed by Lucas (1978) in which a representative consumer has additively-time separable utility

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t), \tag{3.1}$$

where the discount factor satisfies $0 < \beta < 1$ and the single-period utility function takes the natural log form.

This consumer finances his or her consumption by trading equity shares in fruit trees. Each share in each tree provides a dividend in the form of d_t pieces of fruit during each period $t = 0, 1, 2, \ldots$, where fruit is the economy's only consumption good. Let s_t denote the number of shares carried by the representative consumer into each period $t = 0, 1, 2, \ldots$, and let p_t denote the price of each share in each tree during each period $t = 0, 1, 2, \ldots$.

Then, as sources of funds during each period $t = 0, 1, 2, \ldots$, the representative consumer has his or her dividend payments $d_t s_t$ and the total value $p_t s_t$ of the shares carried into the period. And as uses of funds during each period $t = 0, 1, 2, \ldots$, the consumer has his or her consumption c_t and the value $p_t s_{t+1}$ of the shares that he or she will carry into period $t + 1$. The representative consumer therefore faces the budget constraint

$$(d_t + p_t)s_t \geq c_t + p_t s_{t+1}$$

for all $t = 0, 1, 2, \ldots$. For the analysis that follows, it is useful to divide both sides of this budget constraint through by p_t and rearrange the resulting terms to write

$$\frac{d_t s_t - c_t}{p_t} \geq s_{t+1} - s_t. \tag{3.2}$$

a. The representative consumer therefore takes s_0 as given, and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize the utility function in (3.1) subject to the budget constraint in (3.2), which as noted above must hold for all $t = 0, 1, 2, \ldots$. To start the process of solving the consumer's problem using the maximum principle, write down the maximized Hamiltonian for this problem. *Note:* For this problem, you can use either the present value or the current value formulation of the Hamiltonian (in class and in Problem Set 7, we used the present value formulation). But, whichever form you choose here, in part (a), make sure that the optimality conditions you write down below, in part (b), are consistent with your choice.

b. Next, use your definition of the maximized Hamiltonian from part (a), above, to derive the first-order condition for c_t and the pair of difference equations for s_t and the associated multiplier (in class, we labelled the multiplier as π_{t+1}, but you can use whatever notation is consistent with your answer to part (a), above) that, according to the maximum principle, help characterize the solution to the representative consumer's problem.

c. Finally, suppose that there is one share per tree and one tree per consumer in the economy as a whole. Then, in equilibrium, it must be that $s_t = 1$ for all $t = 0, 1, 2, \ldots$. In this case, the representative consumer's budget constraint (3.2) implies that $c_t = d_t$ must also hold in equilibrium for all $t = 0, 1, 2, \ldots$. In Problem Set 7, we saw that in such an equilibrium, the dividend yield on shares is constant over time, with

$$\frac{d_t}{p_t} = \frac{d_{t-1}}{p_{t-1}} = y$$

for all $t = 0, 1, 2, \ldots$. By substituting these equilibrium conditions into the optimality conditions you wrote down previously, in answering part (b), above, solve for the equilibrium constant dividend yield y in terms of the consumer's discount factor β.

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2016

Thursday, October 27, 1:30 - 2:45pm

1. The Permanent Income Hypothesis with Multiple Consumption Goods

The consumer chooses x_0, y_0, x_1, and y_1 to maximize utility

$$\alpha \ln(x_0) + (1-\alpha)\ln(y_0) + \beta[\alpha \ln(x_1) + (1-\alpha)\ln(y_1)], \tag{1.1}$$

subject to the budget constraint

$$I \geq p_0 x_0 + q_0 y_0 + \frac{p_1 x_1}{1+r} + \frac{q_1 y_1}{1+r}. \tag{1.2}$$

a. With the Lagrangian for the consumer's problem defined as

$$\begin{aligned} L(x_0, y_0, x_1, y_1, \lambda) &= \alpha \ln(x_0) + (1-\alpha)\ln(y_0) + \beta[\alpha \ln(x_1) + (1-\alpha)\ln(y_1)] \\ &\quad + \lambda \left(I - p_0 x_0 - q_0 y_0 - \frac{p_1 x_1}{1+r} - \frac{q_1 y_1}{1+r} \right), \end{aligned}$$

the Kuhn-Tucker theorem implies that there exists a value of the Lagrange multiplier λ^* that, together with the values x_0^*, y_0^*, x_1^*, and y_1^* that solve the consumer's problem satisfy the first-order conditions

$$L_1(x_0^*, y_0^*, x_1^*, y_1^*, \lambda^*) = \frac{\alpha}{x_0^*} - \lambda^* p_0 = 0,$$

$$L_2(x_0^*, y_0^*, x_1^*, y_1^*, \lambda^*) = \frac{1-\alpha}{y_0^*} - \lambda^* q_0 = 0,$$

$$L_3(x_0^*, y_0^*, x_1^*, y_1^*, \lambda^*) = \frac{\beta\alpha}{x_1^*} - \frac{\lambda^* p_1}{1+r} = 0,$$

and

$$L_4(x_0^*, y_0^*, x_1^*, y_1^*, \lambda^*) = \frac{\beta(1-\alpha)}{y_1^*} - \frac{\lambda^* q_1}{1+r} = 0,$$

the constraint

$$L_5(x_0^*, y_0^*, x_1^*, y_1^*, \lambda^*) = I - p_0 x_0^* - q_0 y_0^* - \frac{p_1 x_1^*}{1+r} - \frac{q_1 y_1^*}{1+r} \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^* \left(I - p_0 x_0^* - q_0 y_0^* - \frac{p_1 x_1^*}{1+r} - \frac{q_1 y_1^*}{1+r} \right) = 0.$$

b. The first-order conditions from part (a), above, imply that

$$x_0^* = \frac{\alpha}{\lambda^* p_0},$$

$$y_0^* = \frac{1-\alpha}{\lambda^* q_0},$$

$$x_1^* = \frac{\beta\alpha(1+r)}{\lambda^* p_1},$$

and

$$y_1^* = \frac{\beta(1-\alpha)(1+r)}{\lambda^* q_1}.$$

Substituting these expressions into the budget constraint, which will bind at the optimum, yields

$$I = \frac{1+\beta}{\lambda^*}$$

or

$$\lambda^* = \frac{1+\beta}{I}.$$

Substituting this solution for λ^* back into the previous expressions for optimal consumptions yields

$$x_0^* = \frac{\alpha I}{(1+\beta)p_0},$$

$$y_0^* = \frac{(1-\alpha)I}{(1+\beta)q_0},$$

$$x_1^* = \frac{\beta\alpha(1+r)I}{(1+\beta)p_1},$$

and

$$y_1^* = \frac{\beta(1-\alpha)(1+r)I}{(1+\beta)q_1}.$$

c. When $\beta(1+r) = 1$, the solutions for optimal consumptions derived above simplify and imply

$$p_0 x_0^* = \frac{\alpha I}{1+\beta},$$

$$q_0 y_0^* = \frac{(1-\alpha)I}{1+\beta},$$

$$p_1 x_1^* = \frac{\alpha I}{1+\beta},$$

and

$$q_1 y_1^* = \frac{(1-\alpha)I}{1+\beta}.$$

so that total spending in period $t = 0$ is

$$p_0 x_0^* + q_0 y_0^* = \frac{I}{1+\beta}$$

and total spending in period $t = 1$ is

$$p_1 x_1^* + q_1 y_1^* = \frac{I}{1+\beta}.$$

These last two expressions have an interpretation consistent with the permanent income hypothesis: the consumer smooths out perfectly his or her total spending over time.

2. The Envelope Theorem

Let $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$ denote the Marshallian demand curves describing the solutions to the consumer's utility maximization problem

$$\max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2,$$

and let $\lambda^*(p_1, p_2, I)$ denote the corresponding value of the Lagrange multiplier. The consumer's indirect utility function $v(p_1, p_2, I)$ is defined by

$$v(p_1, p_2, I) = \max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

a. Since the complementary slackness condition implied by the Kuhn-Tucker theorem requires that

$$\lambda^*(p_1, p_2, I)[I - p_1 c_1^*(p_1, p_2, I) - p_2 c_2^*(p_1, p_2, I)],$$

the indirect utility function can be evaluated as

$$v(p_1, p_2, I) = U[c_1^*(p_1, p_2, I), c_2^*(p_1, p_2, I)] + \lambda^*(p_1, p_2, I)[I - p_1 c_1^*(p_1, p_2, I) - p_2 c_2^*(p_1, p_2, I)].$$

The envelope theorem then implies

$$v_1(p_1, p_2, I) = \frac{\partial v(p_1, p_2, I)}{\partial p_1} = -\lambda^*(p_1, p_2, I) c_1^*(p_1, p_2, I),$$

$$v_2(p_1, p_2, I) = \frac{\partial v(p_1, p_2, I)}{\partial p_2} = -\lambda^*(p_1, p_2, I) c_2^*(p_1, p_2, I)$$

and

$$v_3(p_1, p_2, I) = \frac{\partial v(p_1, p_2, I)}{\partial I} = \lambda^*(p_1, p_2, I).$$

b. The results from part (a), above, reveal that Roy's identity, which says that

$$c_1^*(p_1, p_2, I) = -\frac{v_1(p_1, p_2, I)}{v_3(p_1, p_2, I)}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{v_2(p_1, p_2, I)}{v_3(p_1, p_2, I)}$$

is just an application or special case of the envelope theorem. Since the indirect utility function

$$v(p_1, p_2, I) = \ln(I) - \alpha \ln(p_1) - (1 - \alpha) \ln(p_2)$$

is such that

$$v_1(p_1, p_2, I) = -\frac{\alpha}{p_1},$$
$$v_2(p_1, p_2, I) = -\frac{\alpha}{p_2},$$

and

$$v_3(p_1, p_2, I) = \frac{1}{I},$$

it follows that the Marshallian demand curves in this case are

$$c_1^*(p_1, p_2, I) = \frac{\alpha I}{p_1}$$

and

$$c_2^*(p_1, p_2, I) = \frac{(1 - \alpha)I}{p_2}.$$

c. The solution for $c_1^*(p_1, p_2, I)$ from part (b) above implies that

Consumption of good 1 rises (in fact, doubles) when income I doubles, while both prices p_1 and p_2 remain unchanged.

Consumption of good 1 falls (in fact, is cut in half) when the price p_1 of good 1 doubles, while both income I and the price p_2 of good 2 remain unchanged.

Consumption of good 1 stays the same when income I and both price p_1 and p_2 all double at the same time.

3. Equilibrium Asset Pricing: A Special Case

The representative consumer takes s_0 as given, and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize the utility

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t), \tag{3.1}$$

subject to the budget constraint

$$\frac{d_t s_t - c_t}{p_t} \geq s_{t+1} - s_t, \tag{3.2}$$

which must hold for all $t = 0, 1, 2, \ldots$.

a. In present value form, the maximized Hamiltonian for the consumer's problem is

$$H(s_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \ln(c_t) + \pi_{t+1} \left(\frac{d_t s_t - c_t}{p_t} \right).$$

b. According to the maximum principle, the solution to the consumer's problem is characterized by the first-order condition

$$\frac{\beta^t}{c_t} - \frac{\pi_{t+1}}{p_t} = 0$$

and the pair of difference equation

$$\pi_{t+1} - \pi_t = -H_s(s_t, \pi_{t+1}; t) = -\pi_{t+1} \left(\frac{d_t}{p_t} \right)$$

and

$$s_{t+1} - s_t = H_\pi(s_t, \pi_{t+1}; t) = \frac{d_t s_t - c_t}{p_t}.$$

c. In equilibrium, $s_{t+1} = s_t = 1$, $d_t = c_t$, and $d_t/p_t = y$ for all $t = 0, 1, 2, \ldots$. Substituting these equilibrium conditions into the first-order condition for c_t and the difference equation for π_{t+1} derived in part (b), above, yields

$$\pi_{t+1} = \beta^t / y$$

and

$$\pi_{t+1} - \pi_t = -\pi_{t+1} y$$

for all $t = 0, 1, 2, \ldots$. Substituting the first of these equations into the second yields

$$\beta^t / y - \beta^{t-1} / y = -\beta^t.$$

Finally, dividing through by β^{t-1} and solving for y yields

$$y = (1 - \beta)/\beta.$$

Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2016

Due Wednesday, December 21, 11:00am

This exam has three questions on five pages; please check to make sure that your copy has all five pages. Each part of each question is worth five points. Hence, question 1 with two parts is worth 10 points, question 2 with four parts is worth 20 points, and question 3 with three parts is worth 15 points, for a total of 45 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Life-Cycle Consumption, Saving, and Wage Growth

This problem asks you to use the maximum principle to characterize the solution to a continuous-time dynamic optimization problem with a finite horizon.

A consumer works and consumes during all periods $t \in [0, T]$. Let $w(t)$ denote the consumer's labor income during each period $t \in [0, T]$ and assume that, starting from an initial level $w(0) = w_0$, this labor income grows at the constant rate $g > 0$, so that

$$w(t) = w_0 e^{gt}$$

for all $t \in [0, T]$. Next, let $c(t)$ denote consumption during each period $t \in [0, T]$, and suppose the consumer has preferences described by a utility function of the form

$$\int_0^T e^{-\rho t} \ln(c(t))dt, \tag{1.1}$$

where the discount rate $\rho > 0$ reflects the consumer's impatience and the single-period utility function takes the natural log form. Finally, let $A(t)$ denote the consumer's "assets," or bank account balance, at each date $t \in [0, T]$ and assume that the consumer's savings earn interest at the constant rate $r > 0$, so that

$$e^{gt} w_0 + rA(t) - c(t) \geq \dot{A}(t), \tag{1.2}$$

for all $t \in [0, T]$. The consumer starts with no assets, $A(0) = 0$. He or she is allowed to borrow by choosing negative values of $A(t)$ at any date $t \in (0, T)$, but must repay whatever is borrowed by the end of the horizon at T; this last requirement is reflected in the constraint

$$A(T) \geq 0. \tag{1.3}$$

The consumer's dynamic optimization problem can now be stated as: choose continuously differentiable functions $c(t)$, $t \in [0, T]$, and $A(t)$, $t \in (0, T]$, to maximize the utility function in (1.1) subject to $A(0) = 0$ given, the constraint (1.2) for all $t \in [0, T]$, and the constraint (1.3) on the terminal value $A(T)$.

a. Write down an expression defining the maximized Hamiltonian for the consumer's problem. Then, using your definition, write down the first-order condition and the pair of differential equations that, according to the maximum principle, help characterize the solution to the consumer's problem. *Note:* In answering this question, you can work with either the present-value or current-value formulation of the Hamiltonian; but, whichever form you choose, make sure you write down the optimality conditions corresponding to that form.

b. Your answers to part (a), above, should imply that the optimal growth rate of consumption, $\dot{c}(t)/c(t)$, is constant over time. Use those answers to derive an expression that shows how that constant rate of consumption growth depends on the parameters of the model: w_0, g, r, and ρ. Does your solution for the constant rate of consumption growth depend on the parameters w_0 and g describing the initial level and growth rate of labor income? Can you explain why or why not?

2. Consumption and Hours Worked Under Certainty

This problem asks you to use dynamic programming to characterize the solution to a discrete-time dynamic optimization problem with an infinite horizon in which there are no elements of uncertainty, so that optimal decisions are made under conditions of perfect foresight.

Let A_t denote a consumer's bank account balance at the beginning of each period $t = 0, 1, 2, \ldots$. During each period $t = 0, 1, 2, \ldots$, the consumer works h_t hours; if w_t denotes the hourly wage received by the consumer, then he or she earns $w_t h_t$ in total labor income during that period. Let c_t denote the consumer's consumption during period t, and let

$$s_t = A_t + w_t h_t - c_t \tag{2.1}$$

be a measure of gross (total) savings during period t. If the bank account pays interest at the constant rate $r > 0$, then the consumer's bank account balance evolves according to the constraint

$$(1 + r)s_t \geq A_{t+1} \tag{2.2}$$

for all $t = 0, 1, 2, \ldots$. The consumer takes his or her initial bank account balance A_0 as given. He or she can borrow by choosing negative values for A_t for $t = 1, 2, 3, \ldots$, but faces a no-Ponzi-game constraint specified in more detail below that rules out unlimited borrowing and implies that he or she faces an infinite-horizon, present value budget constraint. Finally, suppose that the consumer's preferences are described by the utility function

$$\sum_{t=0}^{\infty} \beta^t u[c_t - (1/\gamma)h_t^{\gamma}], \tag{2.3}$$

where the discount factor β lies between zero and one, $0 < \beta < 1$, the additional preference parameter $\gamma > 1$ is greater than one and helps measure the rate at which the disutility of labor rises with hours worked, and the single-period utility function u, which as you will notice is defined over the "composite" $c - (1/\gamma)h^\gamma$ of consumption and hours worked, is strictly increasing and strictly concave, so that $u'[c - (1/\gamma)h^\gamma] > 0$ and $u''[c - (1/\gamma)h^\gamma] < 0$ for all values of c and h.

To solve this consumer's problem using dynamic programming, it is convenient to use (2.1) to substitute out for c_t in the utility function. The consumer's problem can then be stated as: choose sequences for the flow (control) variables $\{h_t\}_{t=0}^\infty$ and $\{s_t\}_{t=0}^\infty$ and the stock (state) variable $\{A_t\}_{t=1}^\infty$ to maximize the objective function

$$\sum_{t=0}^{\infty} \beta^t u[A_t + w_t h_t - s_t - (1/\gamma)h_t^\gamma],$$

subject to the constraint shown in (2.2) for all $t = 0, 1, 2, \ldots$, taking the initial condition A_0 as given.

a. Write down the Bellman equation for the dynamic programming formulation of the consumer's problem. Then write down the first-order conditions for h_t and s_t and the envelope condition for A_t that help describe the solution to this problem. In doing this, you can assume that the nonnegativity constraints on the choice variables will never bind, so that they can be safely ignored when setting up and solving the problem, but that the constraint in (2.2) always binds at the optimum, so that it can be substituted directly into the Bellman equation.

b. Your results from part (a), above, should reveal that the preferences described by the special utility function introduced in equation (2.3) have the implication that hours worked h_t at each date t depend only on the wage rate w_t and the preference parameter γ. Use your results to find this expression linking the optimal choice of h_t to w_t and γ.

c. Next, assume that the wage rate is constant, with $w_t = w$ for all $t = 0, 1, 2, \ldots$, and that the consumer's discount factor and the constant interest rate are related via $\beta(1 + r) = 1$. Use these assumptions, together with your results from parts (a) and (b), above, to answer the following question: is optimal consumption c_t increasing, decreasing, or constant over time?

d. Continuing to assume that $w_t = w$ for all $t = 0, 1, 2, \ldots$ and $\beta(1 + r) = 1$, suppose, as well, that the preference parameter $\gamma = 2$. Recall that, in class, we saw how (2.1) and (2.2) could be combined, starting from $t = 0$ and iterating forward, to obtain the consumer's infinite-horizon, or present value, budget constraint

$$A_0 + \sum_{t=0}^{\infty} \frac{w_t h_t}{(1+r)^t} \geq \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}$$

assuming that the consumer faces a no-Ponzi-game constraint of the specific form

$$\lim_{T\to\infty} \frac{A_T}{(1+r)^T} \geq 0.$$

It can also be shown that the transversality condition that is necessary for a solution to the consumer's problem implies that the non-negative limit specified by the no-Ponzi-game constraint is equal to zero, so that the infinite horizon budget constraint holds with equality at the optimum:

$$A_0 + \sum_{t=0}^{\infty} \frac{w_t h_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}.$$

Use these conditions, together with your results from parts (b) and (c), above, to obtain an expression showing how the consumer's optimal choice of consumption c_0 at $t = 0$ depends on the initial value of A_0, the constant wage rate w, and the interest rate $1+r$ or the discount factor β. *Note:* Since you are still assuming that $\beta(1+r) = 1$, there are a number of fully equivalent ways to incorporate β and or $1+r$ into your solution for c_0; you are free to choose whichever way is most convenient.

3. Saving with a Random Return

This problem asks you to use dynamic programming to characterize the solution to a dynamic, stochastic optimization problem in discrete time with an infinite horizon. The problem is one of those few for which you can solve explicitly for the value function that enters into both sides of the Bellman equation.

A consumer finances his or her consumption over the infinite horizon $t = 0, 1, 2, \ldots$ from returns on his or her savings, which vary stochastically over time. Let A_t denote the consumer's "assets," or financial wealth, at the beginning of period $t = 0, 1, 2, \ldots$. During each period, the consumer divides this wealth up into an amount c_t to be consumed and an amount s_t to be saved, where

$$A_t = c_t + s_t \tag{3.1}$$

for all $t = 0, 1, 2, \ldots$. At the end of period t, after s_t has been chosen, the random, gross return R_{t+1} on savings is determined, so that the consumer enters the next period $t+1$ with financial wealth A_{t+1} satisfying

$$R_{t+1} s_t \geq A_{t+1} \tag{3.2}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of R_{t+1} between each pair of dates t and $t+1$. For simplicity, assume that R_{t+1} is independently and identically distributed over time, with

$$E \ln(R_{t+1}) = \ln(R)$$

for all $t = 0, 1, 2, \ldots$, where ln denotes the natural log and R is a constant. Finally, assume that the consumer seeks to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where E_0 denotes the expectation based on time $t = 0$ information, β is a discount factor between zero and one, and the single-period utility function $\ln(c)$ takes the natural log form.

As a first step in solving this consumer's problem via dynamic programming, it is convenient to use (3.1) to substitute out for c_t in the utility function. The consumer's problem can then be stated as: choose contingency plans for control variable s_t, $t = 0, 1, 2, \ldots$, and the state variable A_t, $t = 1, 2, 3, \ldots$ to maximize the objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t),$$

subject to the constraint shown in (3.2) for $t = 0, 1, 2, \ldots$ and all possible realizations of R_{t+1}, taking the initial condition A_0 as given.

a. After guessing that the value function for the consumer's problem takes the time-invariant form

$$v(A_t) = F + G \ln(A_t),$$

where F and G are constant coefficients to be determined, write down the Bellman equation for the dynamic programming formulation of the consumer's problem. *Note:* To put the Bellman equation in its simplest form, it may help to use the fact that since s_t is chosen based on all information that is available at time t, $E_t \ln(s_t) = \ln(s_t)$, where E_t denotes the expectation at time t.

b. Next, write down the first-order condition for s_t and the envelope condition for A_t that help characterize the solution to the consumer's problem.

c. Use your results from part (b) above to find solutions that show how the consumer's optimal choices for c_t and s_t during period t depend on the consumer's wealth A_t and the discount factor β. *Note:* To find these solutions, you'll have to use (3.1) again to find c_t given A_t and s_t. You may also have to solve for the undetermined coefficient G in terms of the discount factor β. Although it is also possible to solve for the other undetermined coefficient F, finding this solution for F is not necessary to get the solutions for c_t and s_t.

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2016

Due Wednesday, December 21, 11:00am

1. Life-Cycle Consumption, Saving, and Wage Growth

The consumer chooses continuously differentiable functions $c(t)$, $t \in [0, T]$, and $A(t)$, $t \in (0, T]$, to maximize the utility function

$$\int_0^T e^{-\rho t} \ln(c(t)) dt, \tag{1.1}$$

subject to $A(0) = 0$ given,

$$e^{gt} w_0 + rA(t) - c(t) \geq \dot{A}(t), \tag{1.2}$$

for all $t \in [0, T]$, and

$$A(T) \geq 0. \tag{1.3}$$

a. In present-value form, the maximized Hamiltonian for the consumer's problem is defined by

$$H(A(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)[e^{gt} w_0 + rA(t) - c(t)].$$

According to the maximum principle, the solution to the consumer's problem is characterized by the first-order condition

$$\frac{e^{-\rho t}}{c(t)} - \pi(t) = 0$$

for all $t \in [0, T]$ and the pair of differential equations

$$\dot{\pi}(t) = -H_A(A(t), \pi(t); t) = -r\pi(t)$$

and

$$\dot{A}(t) = H_\pi(A(t), \pi(t); t) = e^{gt} w_0 + rA(t) - c(t)$$

for all $t \in [0, T]$.

b. To solve for the constant rate of consumption growth, rewrite the first-order condition as

$$e^{-\rho t} = \pi(t)c(t)$$

and differentiate with respect to t to obtain

$$-\rho e^{-\rho t} = \dot{\pi}(t)c(t) + \pi(t)\dot{c}(t).$$

Next, substitute the first-order condition into the left-hand side and the differential equation for $\dot{\pi}(t)$ into the right-hand side of this last expression to obtain

$$-\rho\pi(t)c(t) = -r\pi(t)c(t) + \pi(t)\dot{c(t)}.$$

Finally, divide through by $\pi(t)c(t)$ and rearrange to get

$$\frac{\dot{c}(t)}{c(t)} = r - \rho.$$

This solution indicates that the optimal consumption growth rate rises when a higher interest rate r provides the consumer with an incentive to save more and therefore postpone consumption, and that the optimal consumption growth rate falls when a higher discount rate makes the consumer less patient. The optimal consumption growth rate does not depend on the parameters w_0 and g describing the initial level and growth rate of labor income. Instead, the ability to borrow means that the consumer adjusts the optimal level of consumption based on the present discounted value of his or her lifetime labor income and the sets the growth rate of consumption based on a comparison between the interest rate and the discount rate.

2. Consumption and Hours Worked Under Certainty

The consumer chooses sequences $\{h_t\}_{t=0}^{\infty}$, $\{s_t\}_{t=0}^{\infty}$, and $\{A_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t u[A_t + w_t h_t - s_t - (1/\gamma)h_t^{\gamma}],$$

subject to the constraint

$$(1+r)s_t \geq A_{t+1} \tag{2.2}$$

for all $t = 0, 1, 2, \ldots$, taking A_0 as given.

a. The Bellman equation for the consumer's problem is

$$v(A_t; t) = \max_{h_t, s_t} \; u[A_t - w_t h_t - s_t - (1/\gamma)h_t^{\gamma}] + \beta v[(1+r)s_t; t+1].$$

The first-order condition for h_t is

$$(w_t - h_t^{\gamma-1})u'[A_t - w_t h_t - s_t - (1/\gamma)h_t^{\gamma}] = 0,$$

the first-order condition for s_t is

$$-u'[A_t - w_t h_t - s_t - (1/\gamma)h_t^{\gamma}] + \beta(1+r)v'[(1+r)s_t; t+1] = 0,$$

and the envelope condition for A_t is

$$v'(A_t; t) = u'[A_t - w_t h_t - s_t - (1/\gamma)h_t^{\gamma}].$$

b. Since the single-period utility function is strictly increasing, the first-order condition for h_t requires that

$$w_t = h_t^{\gamma-1}.$$

This condition provides the solution for hours worked in terms of the wage rate w_t and the preference parameter γ:

$$h_t = w_t^{1/(\gamma-1)}.$$

c. To determine the behavior of consumption, use the constraints (2.1) and (2.2) to rewrite the first-order condition for s_t and the envelope condition for A_t from part (a) as

$$u'[c_t - (1/\gamma)h_t^\gamma] = \beta(1+r)v(A_{t+1}; t+1)$$

and

$$v'(A_t; t) = u'[c_t - (1/\gamma)h_t^\gamma].$$

Since the envelope condition must hold for all $t = 0, 1, 2, \ldots$, it also implies that

$$v'(A_{t+1}; t+1) = u'[c_{t+1} - (1/\gamma)h_{t+1}^\gamma].$$

Substituting this last expression into the first-order condition for s_t yields

$$u'[c_t - (1/\gamma)h_t^\gamma] = \beta(1+r)u'[c_{t+1} - (1/\gamma)h_{t+1}^\gamma]$$

or, since $\beta(1+r) = 1$,

$$u'[c_t - (1/\gamma)h_t^\gamma] = u'[c_{t+1} - (1/\gamma)h_{t+1}^\gamma].$$

Since the single-period utility function is strictly concave, this last condition requires

$$c_t - (1/\gamma)h_t^\gamma = c_{t+1} - (1/\gamma)h_{t+1}^\gamma$$

and since, with a constant wage, hours worked are constant according to the solution from part (b),

$$c_t = c_{t+1}$$

for all $t = 0, 1, 2, \ldots$. Evidently, under all of these assumptions, optimal consumption is constant over time.

d. With $w_t = w$ for all $t = 0, 1, 2, \ldots$ and with $\gamma = 2$, the solution from part (b), above, implies that

$$h_t = w$$

for all $t = 0, 1, 2, \ldots$. Substituting the constant wage, the constant solution for hours worked, and the constant level of consumption into the infinite-horizon budget constraint

$$A_0 + \sum_{t=0}^{\infty} \frac{w_t h_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}$$

yields

$$A_0 + \sum_{t=0}^{\infty} \frac{w^2}{(1+r)^t} = c_0 \sum_{t=0}^{\infty} \beta^t,$$

using $\beta = 1/(1+r)$ on the right-hand side. Although there are many other, equivalent ways of writing this expression, the solution

$$c_0 = (1-\beta)\left[A_0 + \sum_{t=0}^{\infty} \frac{w^2}{(1+r)^t}\right]$$

shows that the optimal c_0 is determined as a fixed fraction $1-\beta$ of total wealth

$$A_0 + \sum_{t=0}^{\infty} \frac{w^2}{(1+r)^t},$$

equal to the sum of the consumer's initial financial wealth plus the optimally-chosen present value of lifetime labor income. Note that because $\beta(1+r) = 1$, this solution for c_0 can be written even more simply as

$$c_0 = (1-\beta)A_0 + w^2.$$

3. Saving with a Random Return

The consumer chooses contingency plans for s_t, $t = 0, 1, 2, \ldots$, A_t, $t = 1, 2, 3, \ldots$ to maximize the objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t),$$

subject to the constraint

$$R_{t+1} s_t \geq A_{t+1} \tag{3.2}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of R_{t+1}.

a. After guessing that the value function takes the time-invariant form

$$v(A_t) = F + G\ln(A_t),$$

the Bellman equation for the consumer's problem can be written

$$F + G\ln(A_t) = \max_{s_t} \; \ln(A_t - s_t) + \beta F + \beta G E_t \ln(R_{t+1}) + \beta G \ln(s_t),$$

using the fact that s_t is chosen based on time t information to eliminate the conditional expectation E_t that would otherwise appear before the last term on the right.

b. The first-order condition for s_t is

$$-\frac{1}{A_t - s_t} + \frac{\beta G}{s_t} = 0,$$

and the envelope condition for A_t is

$$\frac{G}{A_t} = \frac{1}{A_t - s_t}.$$

c. Rearrange the first-order condition to obtain

$$s_t = \left(\frac{\beta G}{1 + \beta G}\right) A_t,$$

and substitute this expression into the envelope condition to get

$$GA_t - G\left(\frac{\beta G}{1 + \beta G}\right) A_t = A_t,$$

which can be solved for

$$G = \frac{1}{1 - \beta}.$$

This last expression confirms that G is constant, and can be substituted back into the first-order condition for s_t to obtain

$$s_t = \beta A_t.$$

Finally, using the constraint (3.1),

$$c_t = A_t - s_t = (1 - \beta)A_t.$$

Evidently, it is optimal for the consumer to save the constant fraction β of his or her financial wealth and consume the remaining fraction $1 - \beta$. And while it is not necessary to solve for F in order to obtain these results, substituting the solutions for s_t and c_t back into the Bellman equation and using the fact that $E_t \ln(R_{t+1}) = \ln(R)$ yields

$$F = \left(\frac{1}{1 - \beta}\right) \left[\ln(1 - \beta) + \left(\frac{\beta}{1 - \beta}\right) \ln(R) + \left(\frac{\beta}{1 - \beta}\right) \ln(\beta)\right],$$

confirming that F is constant as well.

Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2017

Thursday, October 26, 1:30 - 2:45pm

This exam has three questions on three pages; before you begin, please check to make sure that your copy has all three questions and all three pages. Note, also, that each question has three parts. Each question will receive equal weight in determining your overall exam score.

1. Cost Minimization

Consider a firm that produces output y using capital k and labor l according to the technology described by

$$k^{1/3}l^{1/3} \geq y.$$

Assume that this firm rents capital and hires labor in perfectly competitive markets, where r denotes the rental rate for capital and w the wage rate for labor. The firm's managers choose the mix of inputs to minimize the cost of producing at least y units of output; that is, they solve

$$\min_{k,l} rk + wl \text{ subject to } k^{1/3}l^{1/3} \geq y.$$

a. Write down the Lagrangian for the firm's cost minimization problem. Then, write down write down the first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values k^* and l^* that solve this problem along with the corresponding value of the Lagrange multiplier.

b. Use your results from part (a), above, to derive expressions showing how k^* and l^* depend on the factor prices r and w and the output requirement y. That is, find expressions for the firm's conditional factor demand curves. *Note:* in obtaining these expressions, you can assume that the parameters of the problem are such that the constraint binds at the optimum.

c. Finally, with the minimum cost function defined as

$$C(r,w,y) = \min_{k,l} rk + wl \text{ subject to } k^{1/3}l^{1/3} \geq y,$$

find expressions for the partial derivatives $C_1(r,w,y)$ and $C_2(r,w,y)$ of the minimum cost function with respect to each of the two factor prices r and w.

2. General Equilibrium

Consider an economy with two goods – goods one and two – and two types of agents – types A and B. Markets for both goods are perfectly competitive: agents of both types can buy or sell as many units of each good as they wish at prices p_1 for good one and p_2 for good two that they take as given. Both agent types have identical preferences, described by the utility function

$$\ln(c_1^i) + \ln(c_2^i)$$

where c_1^i and c_2^i denote consumption of goods 1 and 2 by the agent of type $i = A$ or $i = B$. The agent types, differ, however, in their endowments: each type A agent is endowed with 4 units of good 1 and 0 units of good 2, while each type B agent is endowed with 0 units of good 1 and 2 units of good 2.

a. Find solutions to the problem

$$\max_{c_1^A, c_2^A} \ln(c_1^A) + \ln(c_2^A) \text{ subject to } 4p_1 \geq p_1 c_1^A + p_2 c_2^A$$

that show how each type A agent's optimal consumptions depend on the prices p_1 and p_2.

b. Likewise, find solutions to the problem

$$\max_{c_1^B, c_2^B} \ln(c_1^B) + \ln(c_2^B) \text{ subject to } 2p_2 \geq p_1 c_1^B + p_2 c_2^B$$

that show how each type B agent's optimal consumptions depend on the prices p_1 and p_2.

c. Assume now that there an equal number of type A and B agents, so that, in competitive equilibrium, market-clearing requires that

$$4 = c_1^A + c_1^B$$

for good 1 and

$$2 = c_2^A + c_2^B$$

for good 2. In equilibrium, only the relative price p_2/p_1 will be determined uniquely, so take good 1 as the numeraire, by setting $p_1 = 1$ and re-interpreting p_2 as the price of good 2 in units of good 1. Use your results from part (a) and (b) above, together with the market-clearing conditions for the two goods, to find the equilibrium value of p_2.

3. Optimal Growth

Consider a discrete-time version of the Ramsey model, in which the representative consumer has preferences over consumption c_t at each date $t = 0, 1, 2, \ldots$ described by

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1 - \sigma} \right), \tag{3.1}$$

where the discount factor lies between zero and one, with $0 < \beta < 1$, and the constant coefficient of relative risk aversion is positive, with $\sigma > 0$. As usual, let k_t denote the economy-wide capital stock at the beginning of each period $t = 0, 1, 2, \ldots$, assume that output is produced with capital according to the production function k_t^α, where $0 < \alpha < 1$, and let the δ, with $0 < \delta < 1$, denote the depreciation rate of capital. Then, given $k_0 > 0$, the capital stock evolves over time according to

$$k_t^\alpha - \delta k_t - c_t \geq k_{t+1} - k_t \tag{3.2}$$

for all $t = 0, 1, 2, \ldots$.

a. Since the two welfare theorems of economics apply, we can characterize equilibrium resource allocations by solving the social planner's problem: choose sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function (3.1) subject to the constraints $k_0 > 0$ given and (3.2) for all $t = 0, 1, 2, \ldots$. In order to apply the maximum principle to the social planner's problem, start by defining the maximized Hamiltonian for this problem. Then, write down the first-order condition and the pair of difference equations that, according to the maximum principle, help characterize the solution to the problem. *Note:* For this problem, you can use either the present value or the current value formulation of the Hamiltonian; but, whichever form you choose, make sure that the optimality conditions you write down are consistent with that choice.

b. The first-order and difference equations you derived in answering part (a), above, form a system of three equations in three unknowns: the optimal choices of c_t and k_t together with the associated value of the multiplier that you introduced when defining the maximized Hamiltonian. Use one of the optimality conditions to eliminate the multiplier, reducing the system to one consisting of two difference equations involving consumption c_t and the capital stock k_t alone.

c. The pair of difference equations that you derived in answering part (b), above, can be used to find unique sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ that satisfy the difference equations, the initial condition $k_0 > 0$ given, and the transversality condition

$$\lim_{T \to \infty} \beta^T c_T^{-\sigma} k_T = 0.$$

It is not possible to find these sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ analytically, but you can use your two difference equations to characterize analytically the model's unique, nontrivial steady state, in which consumption and the capital stock remain forever constant at values c^* and k^*. By substituting the steady-state conditions $c_{t+1} = c_t = c^*$ and $k_{t+1} = k_t = k^*$ into your difference equations, find expressions that show how the steady-state values c^* and k^* depend on the model's parameters: σ, β, α, and δ.

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2017

Thursday, October 26, 1:30 - 2:45pm

1. Cost Minimization

The firm's managers solve

$$\min_{k,l} rk + wl \text{ subject to } k^{1/3}l^{1/3} \geq y.$$

a. Letting λ denote the Lagrange multiplier on the constraint, the Lagrangian for the firm's problem can be defined as

$$L(k, l, \lambda) = rk + wl - \lambda(k^{1/3}l^{1/3} - y).$$

With the Lagrangian defined in this way, the Kuhn-Tucker theorem implies that there exists a value λ^* that, together with the values k^* and l^* that solve the firm's problem, satisfy the first-order conditions

$$r - (1/3)\lambda^*(k^*)^{-2/3}(l^*)^{1/3} = 0$$

and

$$w - (1/3)\lambda^*(k^*)^{1/3}(l^*)^{-2/3} = 0$$

the constraint

$$(k^*)^{1/3}(l^*)^{1/3} \geq y,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*[(k^*)^{1/3}(l^*)^{1/3} - y] = 0.$$

b. Divide the first-order condition for k by the first-order condition for l to obtain

$$\frac{r}{w} = \frac{l^*}{k^*}$$

or $l^* = (r/w)k^*$. Then, substitute this last expression into the constraint, which will bind at the optimum so long as the factor prices r and w are strictly positive, to obtain

$$(r/w)^{1/3}(k^*)^{2/3} = y$$

or

$$k^*(r, w, y) = (w/r)^{1/2}y^{3/2},$$

which is the conditional demand curve for capital. Plugging this solution for k^* back into the relation $l^* = (r/w)k^*$ derived earlier yields

$$l^*(r, w, y) = (r/w)^{1/2} y^{3/2},$$

which is the conditional demand curve for labor.

c. The complementary slackness condition from part (a) implies that the minimum cost function defined by

$$C(r, w, y) = \min_{k,l} rk + wl \text{ subject to } k^{1/3} l^{1/3} \geq y,$$

can be evaluated as

$$C(r, w, y) = rk^*(r, w, y) + wl^*(r, w, y) - \lambda^*(r, w, y)[k^*(r, w, y)^{1/3} l^*(r, w, y)^{1/3} - y].$$

The envelope theorem then implies that

$$C_1(r, w, y) = k^*(r, w, y) = (w/r)^{1/2} y^{3/2},$$

and

$$C_2(r, w, y) = l^*(r, w, y) = (r/w)^{1/2} y^{3/2}.$$

Alternatively, these same expressions can be derived by evaluating the minimum cost function as

$$C(r, w, y) = rk^*(r, w, y) + wl^*(r, w, y) = 2r^{1/2} w^{1/2} y^{3/2}$$

and differentiating with respect to r and w.

2. General Equilibrium

There are two goods – goods one and two – and two types of agents – types A and B. Markets for both goods are perfectly competitive: agents of both types can buy or sell as many units of each good as they wish at prices p_1 for good one and p_2 for good two that they take as given. Both agent types have identical preferences, described by the utility function

$$\ln(c_1^i) + \ln(c_2^i)$$

where c_1^i and c_2^i denote consumption of goods 1 and 2 by the agent of type $i = A$ or $i = B$. The agent types, differ, however, in their endowments: each type A agent is endowed with 4 units of good 1 and 0 units of good 2, while each type B agent is endowed with 0 units of good 1 and 2 units of good 2.

a. Each type A agent solves

$$\max_{c_1^A, c_2^A} \ln(c_1^A) + \ln(c_2^A) \text{ subject to } 4p_1 \geq p_1 c_1^A + p_2 c_2^A.$$

To find the solution to this problem, set up the Lagrangian

$$L(c_1^A, c_2^A, \lambda^A) = \ln(c_1^A) + \ln(c_2^A) + \lambda^A(4p_1 - p_1c_1^A - p_2c_2^A)$$

and differentiate partially with respect to each choice variable to obtain the first-order conditions

$$\frac{1}{c_1^{A*}} - \lambda^{A*}p_1 = 0$$

and

$$\frac{1}{c_2^{A*}} - \lambda^{A*}p_2 = 0.$$

Note, also, that since the utility function is strictly increasing, the agent's budget constraint will also bind:

$$4p_1 = p_1c_1^{A*} + p_2c_2^{A*}.$$

Rewrite the first-order conditions as

$$c_1^{A*} = \frac{1}{\lambda^{A*}p_1}$$

and

$$c_2^{A*} = \frac{1}{\lambda^{A*}p_2}$$

and substitute these expressions into the binding constraint to obtain

$$4p_1 = p_1\left(\frac{1}{\lambda^{A*}p_1}\right) + p_2\left(\frac{1}{\lambda^{A*}p_2}\right) = \frac{2}{\lambda^{A*}}$$

or

$$\lambda^{A*} = \frac{1}{2p_1}.$$

Finally, substitute this solution for λ^{A*} back into the first-order conditions to get

$$c_1^{A*} = 2$$

and

$$c_2^{A*} = 2\left(\frac{p_1}{p_2}\right),$$

which show how each type A agent's optimal consumptions depend on the prices p_1 and p_2.

b. Likewise, each type B agent solves

$$\max_{c_1^B, c_2^B} \ln(c_1^B) + \ln(c_2^B) \text{ subject to } 2p_2 \geq p_1c_1^B + p_2c_2^B$$

To find the solution to this problem, set up the Lagrangian

$$L(c_1^B, c_2^B, \lambda^B) = \ln(c_1^B) + \ln(c_2^B) + \lambda^B(2p_2 - p_1c_1^A - p_2c_2^A)$$

and differentiate partially with respect to each choice variable to obtain the first-order conditions

$$\frac{1}{c_1^{B*}} - \lambda^{B*} p_1 = 0$$

and

$$\frac{1}{c_2^{B*}} - \lambda^{B*} p_2 = 0.$$

Note, again, that since the utility function is strictly increasing, the agent's budget constraint will also bind:

$$2p_2 = p_1 c_1^{B*} + p_2 c_2^{B*}.$$

Rewrite the first-order conditions as

$$c_1^{B*} = \frac{1}{\lambda^{B*} p_1}$$

and

$$c_2^{B*} = \frac{1}{\lambda^{B*} p_2}$$

and substitute these expressions into the binding constraint to obtain

$$2p_2 = p_1 \left(\frac{1}{\lambda^{B*} p_1} \right) + p_2 \left(\frac{1}{\lambda^{B*} p_2} \right) = \frac{2}{\lambda^{B*}}$$

or

$$\lambda^{B*} = \frac{1}{p_2}.$$

Finally, substitute this solution for λ^{B*} back into the first-order conditions to get

$$c_1^{B*} = \frac{p_2}{p_1}$$

and

$$c_2^{B*} = 1,$$

which show how each type B agent's optimal consumptions depend on the prices p_1 and p_2.

c. Assume, now that there an equal number of type A and B agents, so that, in competitive equilibrium, market-clearing requires that

$$4 = c_1^A + c_1^B$$

for good 1 and

$$2 = c_2^A + c_2^B$$

for good 2. In equilibrium, only the relative price p_2/p_1 will be determined uniquely, so take good 1 as the numeraire, by setting $p_1 = 1$ and re-interpreting p_2 as the price of good 2 in units of good 1. The solutions $c_1^{A*} = 2$ from part (a) and $c_1^B = p_2/p_1$ can then be substituted into the market-clearing condition for good 1 to obtain

$$4 = 2 + p_2,$$

showing that, in equilibrium, $p_2 = 2$.

Walras' Law states that, in general equilibrium, if supply equals demand in the market for good 1, supply must also equal demand in the market for good 2. We can check that this is true, by substituting the solutions $c_2^{A*} = 2(p_1/p_2)$ from part (a) and $c_2^{B*} = 1$ from part (b) into the market-clearing condition for good 2 to obtain

$$2 = \frac{2}{p_2} + 1,$$

which is also satisfied when $p_2 = 2$.

3. Optimal Growth

The social planner's problem is to find sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right), \tag{3.1}$$

subject to the constraints $k_0 > 0$ given and

$$k_t^{\alpha} - \delta k_t - c_t \geq k_{t+1} - k_t \tag{3.2}$$

for all $t = 0, 1, 2, \ldots$.

a. With the maximized present-value Hamiltonian for this problem defined as

$$H(k_t, \pi_{t+1}; t) = \max_{c_t} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) + \pi_{t+1}(k_t^{\alpha} - \delta k_t - c_t),$$

the maximum principle implies that the solution must satisfy the first-order condition

$$\beta^t c_t^{-\sigma} - \pi_{t+1} = 0$$

for all $t = 0, 1, 2, \ldots$, and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_k(k_t, \pi_{t+1}; t) = -\pi_{t+1}(\alpha k_t^{\alpha-1} - \delta)$$

for all $t = 1, 2, 3, \ldots$ and

$$k_{t+1} - k_t = H_{\pi}(k_t, \pi_{t+1}; t) = k_t^{\alpha} - \delta k_t - c_t.$$

for all $t = 0, 1, 2, \ldots$

b. Use the first-order condition to rewrite the difference equation for π_t as

$$\beta^t c_t^{-\sigma}(\alpha k_t^{\alpha-1} + 1 - \delta) = \beta^{t-1} c_{t-1}^{-\sigma}$$

or

$$c_{t+1}^{\sigma} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta) c_t^{\sigma}$$

for all $t = 0, 1, 2, \ldots$. Together with the difference equation

$$k_{t+1} - k_t = k_t^{\alpha} - \delta k_t - c_t,$$

for all $t = 0, 1, 2, \ldots$, the initial condition $k_0 > 0$ given, and the transversality condition

$$\lim_{T\to\infty} \beta^T c_T^{-\sigma} k_T = 0,$$

this difference equation determines the unique sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=1}^{\infty}$ that solve the social planner's problem.

c. Although it is not possible to analytically characterize the full solution to the model, it is possible to find expressions for the values of consumption and capital in the unique, nontrivial steady state. To do this, substitute the steady-state conditions $c_{t+1} = c_t = c^*$ and $k_{t+1} = k_t = k^*$ into the pair of difference equations from part (b), above, to obtain

$$(c^*)^{\sigma} = \beta[\alpha(k^*)^{\alpha-1} + 1 - \delta](c^*)^{\sigma}$$

and

$$k^* - k^* = (k^*)^{\alpha} - \delta k^* - c^*.$$

The first of these two steady-state conditions leads to the solution

$$k^* = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{1/(\alpha-1)}.$$

for steady-state capital and the second leads to the solution

$$c^* = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{\alpha/(\alpha-1)} - \delta\left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{1/(\alpha-1)}$$

for consumption. Notably, the model's steady state does not depend at all on the parameter σ; in discrete time as in continuous time, the value for this parameter affects the speed of transition from the initial condition k_0 to the steady state but not the steady state itself.

Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2017

Due Monday, December 18 at 12 noon

This exam has two questions on five pages; please check to make sure that your copy has all five pages. Each part of each question is worth five points. Hence, question 1 with four parts is worth 20 points and question 2 with five parts is worth 25 points, for a total of 45 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Human Capital Accumulation and Economic Growth

This problem asks you to consider a more general version of the model of human capital accumulation and economic growth presented earlier in problem set 11, recast in continuous time and relaxing the previous assumptions of log utility and complete depreciation of physical capital.

In this version of the model, a representative consumer divides his or her time into an amount $u(t)$ devoted to human capital accumulation and an amount $1 - u(t)$ devoted to producing goods for consumption and investment in physical capital. The stocks $k(t)$ and $h(t)$ of physical and human capital evolve according to

$$k(t)^\alpha \{[1 - u(t)]h(t)\}^{1-\alpha} - \delta k(t) - c(t) \geq \dot{k}(t) \tag{1.1}$$

and

$$\gamma u(t)h(t) \geq \dot{h}(t), \tag{1.2}$$

for all $t \in [0, \infty)$. Equation (1) shows that output is produced with physical capital $k(t)$ and skilled labor $[1 - u(t)]h(t)$ according to a Cobb-Douglas production function with capital's share α satisfying $0 < \alpha < 1$. The parameter δ, also satisfying $0 < \delta < 1$, is the depreciation rate for physical capital. Therefore, (1.1) implies that output is used to replace depreciated capital, for consumption $c(t)$, or to add to the capital stock. Equation (1.2), with $\gamma > 0$, describes how human capital gets built up through time $u(t)$ allocated to schooling, training, and scientific exploration.

The representative household takes the initial stocks $k(0)$ and $h(0)$ of physical and human capital as given and chooses continuously differentiable functions $c(t)$ and $u(t)$ for $t \in [0, \infty)$ and $k(t)$ and $h(t)$ for $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1 - \sigma} \right] \mathrm{d}t.$$

where the discount rate $\rho > 0$ and the constant coefficient of relative risk aversion (also the inverse of the intertemporal elasticity of substitution) $\sigma > 0$ are both positive, subject to the constraints in (1.1) and (1.2) for all $t \in [0, \infty)$.

Because this problem is cast in continuous time, the maximum principle provides the easiest way to characterize its solution. With two stock variables and two flow variables, the problem is slightly more complicated than those we've considered before; still, as the following questions will help you confirm, our previous analysis generalizes in a natural way to accommodate these extra model features.

Start by defining the Hamiltonian in current-value form as

$$\begin{aligned}\tilde{H}(c(t), u(t), k(t), h(t), \theta_1(t), \theta_2(t)) \quad &= \quad \frac{c(t)^{1-\sigma} - 1}{1 - \sigma} \\ &\quad + \theta_1(t)\{k(t)^\alpha[1 - u(t)]^{1-\alpha} h(t)^{1-\alpha} - \delta k(t) - c(t)\} \\ &\quad + \theta_2(t)\gamma u(t) h(t).\end{aligned}$$

Define the maximized current-value Hamiltonian accordingly, as

$$H(k(t), h(t), \theta_1(t), \theta_2(t)) = \max_{c(t), u(t)} \tilde{H}(c(t), u(t), k(t), h(t), \theta_1(t), \theta_2(t)). \tag{1.3}$$

a. Write down the first-order conditions for the values of $c(t)$ and $u(t)$ that maximize the current-value Hamiltonian as shown in (1.3). The maximum principle implies that these first-order conditions coincide with those that identify the values of $c(t)$ and $u(t)$ that solve the dynamic problem.

b. The maximum principle also implies that the multipliers $\theta_1(t)$ and $\theta_2(t)$ appearing in the definition of the current-value Hamiltonian will evolve over time according to

$$\dot{\theta}_1(t) = \rho\theta_1(t) - H_k(k(t), h(t), \theta_1(t), \theta_2(t))$$

and

$$\dot{\theta}_2(t) = \rho\theta_2(t) - H_h(k(t), h(t), \theta_1(t), \theta_2(t)),$$

where $H_k(k(t), h(t), \theta_1(t), \theta_2(t))$ and $H_h(k(t), h(t), \theta_1(t), \theta_2(t))$ denote the partial derivatives of the maximized current-value Hamiltonian with respect to each of its first two arguments. Use the envelope theorem to find expressions for these two partial derivatives, which you can then substitute into the differential equations for $\dot{\theta}_1(t)$ and $\dot{\theta}_2(t)$ in order to recover the first-order conditions for the values of $k(t)$ and $h(t)$ that solve the dynamic problem.

c. You can now use the two first order conditions that you derived in part (a) and the two differential equations that you derived in part (b) to characterize the model's balanced growth path, along which consumption and the two capital stocks all grow at the same rate g, with

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{\dot{h}(t)}{h(t)} = g.$$

In fact, it only requires a few more steps to find the economy's long-run growth rate g. First, use the first-order condition for $u(t)$ that you derived in part (a) to obtain an expression for $\theta_1(t)$ that you can substitute into the differential equation for $\dot{\theta}_2(t)$ that you derived in part (b). Then, use the resulting expression to show that $\dot{\theta}_2(t)/\theta_2(t)$ is constant, with

$$\frac{\dot{\theta}_2(t)}{\theta_2(t)} = \eta$$

where η depends on the model's parameters: α, δ, γ, ρ, and σ. *Note:* You'll find that η depends on some, but not all, of these parameters.

d. It turns out that $\theta_1(t)$ and $\theta_2(t)$ change at the same rate along the balanced growth path, so that, in fact,

$$\frac{\dot{\theta}_1(t)}{\theta_1(t)} = \frac{\dot{\theta}_2(t)}{\theta_2(t)} = \eta.$$

Use this fact, together with the first-order condition for $c(t)$ that you derived in part (a) and the expression for η that you derived in part (d) to find an expression that shows how the economy's long-run growth rate g depends on the parameters: α, δ, γ, ρ, and σ. *Note:* Once again, you'll find that g depends on some, but not all, of these parameters.

2. Stock Prices and Consumption

This problem asks you to consider a stochastic version of the general equilibrium model of stock prices and consumption presented earlier in problem set 7. That previous model is extended, here, by allowing the dividend d_t paid by the stock to vary randomly over time and by generalizing the form of the utility function.

In this version of the model, the representative consumer carries s_t shares of stock into each period $t = 0, 1, 2, \ldots$. Each share pays the dividend d_t at the beginning of period t. Thus, as sources of funds during period t, the consumer has dividends of total value $d_t s_t$ as well as shares of value $p_t s_t$, where p_t is the price of each share of stock. And as uses of funds, the consumer has his or her consumption c_t and the value $p_t s_{t+1}$ of the shares that he or she will carry into the following period $t+1$. The consumer therefore faces the budget constraint

$$(d_t + p_t)s_t \geq c_t + p_t s_{t+1} \tag{2.1}$$

for all $t = 0, 1, 2, \ldots.$

The dividend payment d_t made by each share of stock is assumed to fluctuate randomly; as we will see, this assumption implies that, in equilibrium, the share price p_t will fluctuate randomly as well. We will assume that, from the consumer's point of view, when consumption c_t and new share purchases s_{t+1} are chosen during period t, d_t and p_t are already known, but d_{t+1} and p_{t+1} are still random. Under these conditions, the consumer can be described as taking s_0, d_0, and p_0 as given, and choosing contingency plans for c_t for $t = 0, 1, 2, \ldots$ and

s_t for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2.2}$$

subject to the constraint in (2.1) for all $t = 0, 1, 2, \ldots$, where, in (2.2), the discount factor satisfies $0 < \beta < 1$, the single-period utility function is strictly increasing and strictly concave, and E_0 denotes the expectation as of period $t = 0$.

To solve this consumer's problem via dynamic programming, it is helpful to observe, first, that since the consumer's utility function is strictly increasing, the constraint (2.1) will always bind at the optimum. We can use it, therefore, to substitute out for c_t in (2.2), and then take s_t as the state variable and s_{t+1} as the control variable. If d_t and p_t both follow Markov processes, the Bellman equation corresponding to the consumer's problem can be written as

$$v(s_t, d_t, p_t) = \max_{s_{t+1}} u[(d_t + p_t)s_t - p_t s_{t+1}] + \beta E_t[v(s_{t+1}, d_{t+1}, p_{t+1})], \tag{2.3}$$

where E_t denotes the expectation as of period t.

a. Based on the formulation of the Bellman equation shown in (2.3), write down the first-order condition for the consumer's optimal choice of s_{t+1} and the envelope condition for s_t.

b. Next, use the envelope condition to eliminate reference to the derivative of the value function that appears in the first-order condition that you derived in part (a). This will yield a single optimality condition that makes no reference to the unknown value function. *Note:* Although not essential, you might find it helpful to use the binding constraint (2.1) to simplify both the first-order and envelope conditions from part (a) before combining them here.

c. Assume now that the single-period utility function takes the specific form

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma},$$

where $\sigma > 0$ denotes the constant coefficient of relative risk aversion. In equilibrium, if there is one share of stock per consumer in the economy, then $s_t = 1$ and $c_t = d_t$ must hold as market-clearing conditions for shares and goods in each period $t = 0, 1, 2, \ldots$. Use these equilibrium conditions, together with the specific form for the utility function, to rewrite your optimality condition from part (b) as one that relates an expression involving the share price p_t and dividend d_t at time t to the discount factor β and the expectation at time t of another expression involving the share price p_{t+1} and dividend d_{t+1} at time $t + 1$.

d. The expression that you derived in part (c) can be used to determine how, in equilibrium, fluctuations in the share price p_t are driven by exogenous, stochastic fluctuations in dividends. As a first step in doing so, let $q_t = p_t/d_t$ denote the price-dividend ratio (the inverse of the "dividend yield") at each date $t = 0, 1, 2, \ldots$. Then, rewrite the equilibrium condition from part (c) as one that relates an expression involving q_t and d_t at time t to the discount factor β and the expectation at time t of another expression involving q_{t+1} and d_{t+1} at time $t + 1$.

e. Finally, suppose that dividend *growth*

$$x_{t+1} = \frac{d_{t+1}}{d_t}$$

follows a two-state Markov chain, taking on two possible values, x_1 or x_2, at each date $t = 0, 1, 2, \ldots$. Let

$$\pi_{ij} = \text{Prob}(x_{t+1} = x_j | x_t = x_i),$$

denote the probability that x_{t+1} will equal x_j, $j = 1$ or $j = 2$, given that x_t equals x_i, $i = 1$ or $i = 2$, and assume that

$$\pi_{11} = \pi_{22} = \phi$$

and

$$\pi_{12} = \pi_{21} = 1 - \phi.$$

This specification, with $\phi > 0.5$, will imply positive serial correlation in the dividend process, since if $x_t = x_1$, the probability that x_{t+1} will equal x_1 is higher than the probability that given that x_{t+1} will equal x_2 and similarly, if $x_t = x_2$, the probability that x_{t+1} will equal x_2 is higher than the probability that given that x_{t+1} will equal x_1. Under these conditions, the equilibrium value of q_t will depend only on the value of x_t, with

$$q_t = \begin{cases} q_1 & \text{if } x_t = x_1 \\ q_2 & \text{if } x_t = x_2 \end{cases}$$

and, likewise,

$$q_{t+1} = \begin{cases} q_1 & \text{if } x_{t+1} = x_1 \\ q_2 & \text{if } x_{t+1} = x_2 \end{cases}.$$

To show this, start by rewriting the equilibrium condition you derived in part (d) as one that relates q_t to the discount factor β and the expectation at time t of an expression involving q_{t+1} and x_{t+1} at time $t + 1$. Then, for the case where $x_t = x_1$ and $q_t = q_1$, use the probabilities $\pi_{11} = \phi$ and $\pi_{12} = 1 - \phi$ to write out the expectation in that expression in terms of β, σ, ϕ, x_1, x_2, q_1 and q_2. Likewise, for the case where $x_t = x_2$ and $q_t = q_2$, use the probabilities $\pi_{21} = 1 - \phi$ and $\pi_{22} = \phi$ to write out the expectation in the same expression in terms of β, σ, ϕ, x_1, x_2, q_1 and q_2. The result will be a system of two linear equations that determine q_1 and q_2 as functions of β, σ, ϕ, x_1, and x_2. *Note:* To answer this part of the equation, you only need to derive these two equations. In principle, however, you could solve them simply by inverting a 2×2 matrix to find q_1 and q_2 in terms of the parameters.

Solutions to Final Exam

ECON 772001 - Math for Economists
Boston College, Department of Economics

Peter Ireland
Fall 2017

Due Monday, December 18 at 12 noon

1. Human Capital Accumulation and Economic Growth

The representative household takes the initial stocks $k(0)$ and $h(0)$ of physical and human capital as given and chooses continuously differentiable functions $c(t)$ and $u(t)$ for $t \in [0, \infty)$ and $k(t)$ and $h(t)$ for $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1-\sigma} \right] \mathrm{d}t.$$

subject to the constraints

$$k(t)^\alpha \{[1 - u(t)]h(t)\}^{1-\alpha} - \delta k(t) - c(t) \geq \dot{k}(t) \tag{1.1}$$

and

$$\gamma u(t)h(t) \geq \dot{h}(t), \tag{1.2}$$

for all $t \in [0, \infty)$.

Define the current-value Hamiltonian for this problem as

$$\begin{aligned} \tilde{H}(c(t), u(t), k(t), h(t), \theta_1(t), \theta_2(t)) \quad = \quad & \frac{c(t)^{1-\sigma} - 1}{1-\sigma} \\ & + \theta_1(t)\{k(t)^\alpha [1 - u(t)]^{1-\alpha} h(t)^{1-\alpha} - \delta k(t) - c(t)\} \\ & + \theta_2(t)\gamma u(t)h(t), \end{aligned}$$

and the maximized current-value Hamiltonian, accordingly, as

$$H(k(t), h(t), \theta_1(t), \theta_2(t)) = \max_{c(t), u(t)} \tilde{H}(c(t), u(t), k(t), h(t), \theta_1(t), \theta_2(t)). \tag{1.3}$$

a. The maximum principle implies that the values of $c(t)$ and $u(t)$ that solve the dynamic problem also satisfy the first-order conditions for the values of $c(t)$ and $u(t)$ that maximize the current-value Hamiltonian as shown in (1.3). These first-order conditions are

$$c(t)^{-\sigma} - \theta_1(t) = 0$$

and

$$-\theta_1(t)(1-\alpha)k(t)^\alpha [1 - u(t)]^{-\alpha} h(t)^{1-\alpha} + \theta_2(t)\gamma h(t) = 0.$$

b. The maximum principle also implies that the multipliers $\theta_1(t)$ and $\theta_2(t)$ appearing in the definition of the current-value Hamiltonian will satisfy

$$\dot{\theta}_1(t) = \rho\theta_1(t) - H_k(k(t), h(t), \theta_1(t), \theta_2(t))$$

and

$$\dot{\theta}_2(t) = \rho\theta_2(t) - H_h(k(t), h(t), \theta_1(t), \theta_2(t)),$$

where $H_k(k(t), h(t), \theta_1(t), \theta_2(t))$ and $H_h(k(t), h(t), \theta_1(t), \theta_2(t))$ denote the partial derivatives of the maximized current-value Hamiltonian with respect to each of its first two arguments. Using the envelope theorem to find expressions for the two partial derivatives, these differential equations can be written more specifically as

$$\dot{\theta}_1(t) = \rho\theta_1(t) - \theta_1(t)\{\alpha k(t)^{\alpha-1}[1-u(t)]^{1-\alpha}h(t)^{1-\alpha} - \delta\}$$

and

$$\dot{\theta}_2(t) = \rho\theta_2(t) - \theta_1(t)\{(1-\alpha)k(t)^{\alpha}[1-u(t)]^{1-\alpha}h(t)^{-\alpha}\} - \theta_2(t)\gamma u(t).$$

c. We can now use the two first order conditions derived in part (a) and the two differential equations derived in part (b) to characterize the model's balanced growth path, along which consumption and the two capital stocks all grow at the same rate g, with

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{\dot{h}(t)}{h(t)} = g.$$

To begin, note that the first-order condition for $u(t)$ from part (a) implies that

$$\theta_1(t) = \left\{\frac{\gamma[1-u(t)]^{\alpha}h(t)^{\alpha}}{(1-\alpha)k(t)^{\alpha}}\right\}\theta_2(t).$$

Then use this expression to substitute out for $\theta_1(t)$ in the differential equation for $\dot{\theta}_2(t)$ from part (b) to obtain

$$\dot{\theta}_2(t) = (\rho - \gamma)\theta_2(t),$$

implying that

$$\frac{\dot{\theta}_2(t)}{\theta_2(t)} = \eta = \rho - \gamma.$$

d. It turns out that $\theta_1(t)$ and $\theta_2(t)$ change at the same rate along the balanced growth path, so that, in fact,

$$\frac{\dot{\theta}_1(t)}{\theta_1(t)} = \frac{\dot{\theta}_2(t)}{\theta_2(t)} = \eta = \rho - \gamma.$$

Using this fact, the first-order condition for $c(t)$ from part (a) can be used to solve for the economy's long-run growth rate g. To obtain this solution, rewrite the first-order condition as

$$\theta_1(t) = c(t)^{-\sigma}$$

and differentiate both sides with respect to time to obtain

$$\dot{\theta}_1(t) = -\sigma c(t)^{-\sigma-1}\dot{c}(t)$$

or

$$\frac{\dot{\theta}_1(t)}{\theta_1(t)} = -\frac{\sigma c(t)^{-\sigma-1}\dot{c}(t)}{c(t)^{-\sigma}} = -\sigma\left[\frac{\dot{c}(t)}{c(t)}\right].$$

Evidently,

$$g = \frac{\dot{c}(t)}{c(t)} = -\frac{1}{\sigma}\left[\frac{\dot{\theta}_1(t)}{\theta_1(t)}\right] = \frac{\gamma-\rho}{\sigma},$$

so that the economy's long-run growth rate depends positively on the productivity parameter γ for the accumulation of human capital, negatively on the discount rate ρ, and positively on the intertemporal elasticity of substitution $1/\sigma$.

Although these last steps are not required to answer any of the exam questions, the remaining optimality conditions can be used to characterize still further the balanced growth path. The binding constraint for human capital accumulation, for example, implies that the allocation of time across production and human capital accumulation is constant, with

$$u(t) = u = \frac{\gamma-\rho}{\gamma\sigma},$$

revealing that for an interior solution with $0 < u < 1$, the parameter restrictions

$$\gamma \geq \rho$$

and

$$\sigma \geq 1 - \frac{\rho}{\gamma}$$

must be satisfied. The differential equation for $\dot{\theta}_1(t)$ from part (b) implies that the constant ratio of human to physical capital is

$$\frac{h(t)}{k(t)} = \left(\frac{\gamma+\delta}{\alpha}\right)^{1/(1-\alpha)}\left(\frac{\gamma\sigma}{\gamma\sigma-\gamma+\rho}\right)$$

and the binding capital accumulation constraint implies that the constant ratio of consumption to physical capital is

$$\frac{c(t)}{k(t)} = \frac{\gamma+\delta}{\alpha} - \delta - \frac{\gamma-\rho}{\sigma}.$$

2. Stock Prices and Consumption

The representative consumer takes s_0, d_0, and p_0 as given, and chooses contingency plans for c_t for $t = 0, 1, 2, \ldots$ and s_t for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2.2}$$

subject to the constraint

$$(d_t + p_t)s_t \geq c_t + p_t s_{t+1} \tag{2.1}$$

for all $t = 0, 1, 2, ...$

If d_t and p_t both follow Markov processes, the Bellman equation corresponding to the consumer's problem can be written as

$$v(s_t, d_t, p_t) = \max_{s_{t+1}} u[(d_t + p_t)s_t - p_t s_{t+1}] + \beta E_t[v(s_{t+1}, d_{t+1}, p_{t+1})]. \tag{2.3}$$

a. Based on the Bellman equation shown in (2.3), the first-order condition for the consumer's optimal choice of s_{t+1} is

$$-p_t u'[(d_t + p_t)s_t - p_t s_{t+1}] + \beta E_t[v_1(s_{t+1}, d_{t+1}, p_{t+1})] = 0$$

and the envelope condition for s_t is

$$v_1(s_t, d_t, p_t) = (d_t + p_t)u'[(d_t + p_t)s_t - p_t s_{t+1}].$$

b. Using the binding constraint (2.1), the first-order and envelope conditions from part (a) can be written more compactly as

$$p_t u'(c_t) = \beta E_t[v_1(s_{t+1}, d_{t+1}, p_{t+1})]$$

and

$$v_1(s_t, d_t, p_t) = (d_t + p_t)u'(c_t).$$

Since the envelope condition must hold for all $t = 0, 1, 2, \ldots$, it also implies that

$$v_1(s_{t+1}, d_{t+1}, p_{t+1}) = (d_{t+1} + p_{t+1})u'(c_{t+1}).$$

Substituting this last relation into the first-order condition yields

$$p_t u'(c_t) = \beta E_t[(d_{t+1} + p_{t+1})u'(c_{t+1})],$$

an optimality condition that makes no reference to the unknown value function.

c. When single-period utility function takes the specific form

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma},$$

the optimality condition from part (b) becomes

$$p_t c_t^{-\sigma} = \beta E_t[(d_{t+1} + p_{t+1})c_{t+1}^{-\sigma}].$$

Imposing the equilibrium conditions $c_t = d_t$ and $c_{t+1} = d_{t+1}$ then yields

$$p_t d_t^{-\sigma} = \beta E_t(d_{t+1}^{1-\sigma} + p_{t+1} d_{t+1}^{-\sigma}).$$

d. Since the definition of the price-dividend ratio $q_t = p_t/d_t$ implies that $p_t = q_t d_t$, the equilibrium condition from part (c) also implies

$$q_t d_t^{1-\sigma} = \beta E_t[d_{t+1}^{1-\sigma}(1 + q_{t+1})].$$

e. Finally, using

$$x_{t+1} = \frac{d_{t+1}}{d_t}$$

to denote the growth rate of dividends, the equilibrium condition from part (d) implies

$$q_t = \beta E_t[x_{t+1}^{1-\sigma}(1 + q_{t+1})].$$

When dividend growth follows a two-state Markov chain, taking on two possible values $x_t = x_1$ or $x_t = x_2$, with

$$\pi_{ij} = \text{Prob}(x_{t+1} = x_j | x_t = x_i)$$

given by

$$\pi_{11} = \pi_{22} = \phi$$

and

$$\pi_{12} = \pi_{21} = 1 - \phi.$$

This equilibrium condition implies that q_t will depend only on the value of x_t, with

$$q_t = \begin{cases} q_1 & \text{if } x_t = x_1 \\ q_2 & \text{if } x_t = x_2 \end{cases}$$

and

$$q_{t+1} = \begin{cases} q_1 & \text{if } x_{t+1} = x_1 \\ q_2 & \text{if } x_{t+1} = x_2 \end{cases}.$$

To see this, note that when $x_t = x_1$, the equilibrium condition requires that

$$q_1 = \beta\phi x_1^{1-\sigma}(1 + q_1) + \beta(1-\phi)x_2^{1-\sigma}(1 + q_2)$$

and when $x_t = x_2$, the equilibrium condition requires that

$$q_2 = \beta(1-\phi)x_1^{1-\sigma}(1 + q_1) + \beta\phi x_2^{1-\sigma}(1 + q_2).$$

Rewriting these two equations in matrix form as

$$\begin{bmatrix} 1 - \beta\phi x_1^{1-\sigma} & -\beta(1-\phi)x_2^{1-\sigma} \\ -\beta(1-\phi)x_1^{1-\sigma} & 1 - \beta\phi x_2^{1-\sigma} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \beta\phi x_1^{1-\sigma} + \beta(1-\phi)x_2^{1-\sigma} \\ \beta(1-\phi)x_1^{1-\sigma} + \beta\phi x_2^{1-\sigma} \end{bmatrix}$$

and inverting the 2×2 matrix on the left-hand side leads to the solutions

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 - \beta\phi x_1^{1-\sigma} & -\beta(1-\phi)x_2^{1-\sigma} \\ -\beta(1-\phi)x_1^{1-\sigma} & 1 - \beta\phi x_2^{1-\sigma} \end{bmatrix}^{-1} \begin{bmatrix} \beta\phi x_1^{1-\sigma} + \beta(1-\phi)x_2^{1-\sigma} \\ \beta(1-\phi)x_1^{1-\sigma} + \beta\phi x_2^{1-\sigma} \end{bmatrix}$$

for q_1 and q_2 in terms of β, σ, ϕ, x_1, and x_2.

Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2018

Thursday, October 25, 1:30 - 2:45pm

This exam has four questions on four pages; before you begin, please check to make sure that your copy has all four questions and all four pages. Each question has two parts, and each part of each question is worth five points, for a total of 10 points per question and 40 points overall.

1. Utility Maximization

Consider the problem solved by a consumer who uses his or her income I to purchase in perfectly competitive markets c_1 units of good 1 at the price of p_1 per unit and c_2 units of good 2 at the price of p_2 per unit to maximize utility

$$U(c_1, c_2) = \ln(c_1) - (1/2)(c_2 - \bar{c})^2,$$

with $\bar{c} > 0$, subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2.$$

In characterizing the solution to this problem, below, assume that the optimal choices for consumption will turn out to be strictly positive so that it is not necessary to explicitly impose the nonnegativity constraints $c_1 \geq 0$ and $c_2 \geq 0$ when setting up the Lagrangian or taking the first-order conditions.

a. Write down the Lagrangian for the consumer's utility maximization problem. Then, write down the first-order conditions, constraints, nonnegativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve this problem along with the corresponding value of the Lagrange multiplier.

b. Next, specialize the problem by setting $I = 10$, $p_1 = 2$, $p_2 = 1$, and $\bar{c} = 10$, and use your results from part (a), above, to find numerical values for the optimal choices c_1^* and c_2^*.

2. Roy's Identity

Consider the problem faced by another consumer who uses her or her income I to purchase in perfectly competitive markets c_1 units of good 1 at the price of p_1 per unit and c_2 units of good 2 at the price of p_2 per unit to maximize the utility function

$$U(c_1, c_2) = \left(\frac{\sigma}{\sigma - 1}\right) \ln \left[\left(\frac{1}{2}\right)\left(c_1^{\frac{\sigma-1}{\sigma}}\right) + \left(\frac{1}{2}\right)\left(c_2^{\frac{\sigma-1}{\sigma}}\right)\right]$$

subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2,$$

where σ, a positive parameter not equal to one ($\sigma > 0$ and $\sigma \neq 1$), measures the consumer's constant elasticity of substitution between the two goods.

a. It turns out that the indirect utility function for this problem, defined as the maximized level of utility for any given parameter configuration (p_1, p_2, I), takes the form

$$V(p_1, p_2, I) = \ln(I) - \left(\frac{1}{1-\sigma}\right) \ln\left(p_1^{1-\sigma} + p_2^{1-\sigma}\right) + \left(\frac{\sigma}{1-\sigma}\right) \ln(2)$$

Use this expression for the indirect utility function, together with Roy's identity (equvialently, the envelope theorem), to derive formulas for the Marshallian demand curves $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$, describing how the consumer's optimal choices of c_1 and c_2 depend on the prices p_1 and p_2 as well as his or her income I. *Note:* You can also find the Marshallian demand curves by solving the consumer's utility maximization problem, but given the indirect utility function, using Roy's identity should be easier.

b. In the limiting case where $\sigma \to 1$, so that the elasticity of substitution equals one, the utility and indirect utility functions from the problem above simplify to

$$U(c_1, c_2) = (1/2)\ln(c_1) + (1/2)\ln(c_2)$$

and

$$V(p_1, p_2, I) = \ln(I) - (1/2)\ln(p_1) - (1/2)\ln(p_2) - \ln(2).$$

What are the Marshallian demand curves $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$ now? Again, you can answer this question by applying Roy's identity to the indirect utility function, by solving the consumer's utility maximization problem, or just by remembering what the solution looks like from problem set 1.

3. The Ramsey Model

Consider a version of the Ramsey model in which the representative consumer's preferences over consumption $c(t)$ at each date $t \in [0, \infty)$ are described by a general utility function $u(c(t))$ that is strictly increasing and strictly concave, instead of taking on the specific logarithmic form as we assumed in class. For this version of the model, the social planner's problem becomes one of choosing functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize utility over the infinite horizon

$$\int_0^\infty e^{-\rho t} u(c(t))\,\mathrm{d}t,$$

where $\rho > 0$ is the consumer's discount rate, subject to the capital accumulation constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t),$$

for all $t \in [0, \infty)$, where the parameter from the aggregate production function satisfies $0 < \alpha < 1$ and the depreciation rate for capital satisfies $\delta > 0$, taking the initial capital stock $k(0)$ as given.

a. Write down an expression for the maximized current value Hamiltonian for the social planner's problem, using $\theta(t)$ to denote the multiplier on the capital accumulation constraint. Then, write down the first-order condition for $c(t)$ and the differential equations for $\theta(t)$ and $k(t)$ that, according to the maximum principle, must be satisfied by the solution to the social planner's problem.

b. Just as in the case with log utility, this version of the model has a unique nontrivial steady state, in which $c(t) = c^*$, $k(t) = k^*$, and $\theta(t) = \theta^*$ are all equal to constants, so that $\dot{k}(t) = \dot{\theta}(t) = 0$. Use the optimality conditions you derived in part (a) above, to solve for the steady-state values c^* and k^* for consumption and the capital stock in terms of the model's parameters: ρ, α, and δ.

4. Natural Resource Depletion

Let c_t denote society's consumption of an exhaustible natural resource at each date $t = 0, 1, 2, \ldots$, and suppose that a representative consumer gets utility from this resource as described by

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1 - \sigma} \right),$$

where the discount factor lies between zero and one, with $0 < \beta < 1$, and the parameter describing the curvature of the single-period utility function is strictly positive, with $\sigma > 0$.

Let s_t denote the stock of the resource that remains at the beginning of period $t = 0, 1, 2, \ldots$, and consider a social planner who takes the initial stock s_0 of the exhaustible resource as given and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize the representative consumer's utility subject to the constraints

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, \ldots$, which indicate that since the resource is nonrenewable, consumption c_t at each date t subtracts from the stock s_t that remains at the beginning of t to determine the stock s_{t+1} that remains at the beginning of $t + 1$.

a. As we discussed in class, discrete-time dynamic optimization problems like this one can be solved using either the method of Lagrange multipliers or the maximum principle. Use whichever method you find easiest or most convenient to derive a set of optimality conditions that, together with the initial condition s_0 given and the transversality condition, which for this problem is

$$\lim_{T \to \infty} \beta^T c_T^{-\sigma} s_{T+1} = 0,$$

describe the evolution of the optimal choices for c_t and s_t.

b. Use your optimality conditions from part (a), above, to derive an expression that shows how the optimal growth rate of consumption, c_{t+1}/c_t, depends on the preference parameters β and σ.

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2018

Thursday, October 25, 1:30 - 2:45pm

1. Utility Maximization

The consumer chooses c_1 and c_2 to maximize the utility function

$$U(c_1, c_2) = \ln(c_1) - (1/2)(c_2 - \bar{c})^2,$$

subject to the budget constraint

$$I \geq p_1 c_1 + p_2 c_2.$$

a. With the Lagrangian for this problem defined as

$$L(c_1, c_2, \lambda) = \ln(c_1) - (1/2)(c_2 - \bar{c})^2 + \lambda(I - p_1 c_1 - p_2 c_2),$$

the Kuhn-Tucker theorem implies that the optimal choices c_1^* and c_2^* and the associated value λ^* of the Lagrange multiplier must satisfy the first-order conditions

$$\frac{1}{c_1^*} - \lambda^* p_1 = 0$$

and

$$-(c_2^* - \bar{c}) - \lambda^* p_2 = 0,$$

the constraint

$$I \geq p_1 c_1^* + p_2 c_2^*,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*(I - p_1 c_1^* - p_2 c_2^*) = 0.$$

b. Since the utility function is strictly increasing in c_1, we know in advance that $\lambda^* > 0$ and the budget constraint will always bind. Based on these observations, we can substitute the parameter settings $I = 10$, $p_1 = 2$, $p_2 = 1$, and $\bar{c} = 10$ into the first-order conditions and constraint to obtain the system of three equations in the three unknowns c_1^*, c_2^*, and λ^*:

$$\frac{1}{c_1^*} - 2\lambda^* = 0$$

$$-(c_2^* - 10) - \lambda^* = 0,$$

and

$$10 = 2c_1^* + c_2^*.$$

Rearrange the first-order conditions to read

$$c_1^* = \frac{1}{2\lambda^*}$$

and

$$c_2^* = 10 - \lambda^*,$$

and substitute these expressions into the constraint to obtain

$$10 = \frac{1}{\lambda^*} + 10 - \lambda^*$$

or, more simply

$$(\lambda^*)^2 = 1.$$

Since $\lambda^* > 0$, this last equation requires $\lambda^* = 1$. The optimal consumptions are therefore

$$c_1^* = \frac{1}{2}$$

and

$$c_2^* = 9.$$

2. Roy's Identity

To answer this question, it helps to consider first the general utility maximization problem

$$\max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2.$$

By defining the indirect utility function

$$V(p_1, p_2, I) = \max_{c_1,c_2} U(c_1, c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2,$$

we can use the envelope theorem (in this case, Roy's identity) to recover the Marshallian demand curves as

$$c_1^*(p_1, p_2, I) = -\frac{V_1(p_1, p_2, I)}{V_3(p_1, p_2, I)}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{V_2(p_1, p_2, I)}{V_3(p_1, p_2, I)}.$$

a. For the specific case of the constant elasticity utility function, the indirect utility function is given by

$$V(p_1, p_2, I) = \ln(I) - \left(\frac{1}{1-\sigma}\right) \ln\left(p_1^{1-\sigma} + p_2^{1-\sigma}\right) + \left(\frac{\sigma}{1-\sigma}\right) \ln(2).$$

Therefore, applying Roy's identity in this case yields

$$c_1^*(p_1, p_2, I) = -\frac{V_1(p_1, p_2, I)}{V_3(p_1, p_2, I)} = \frac{Ip_1^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{V_2(p_1, p_2, I)}{V_3(p_1, p_2, I)} = \frac{Ip_2^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}.$$

b. In the limiting case where $\sigma \to 1$, so that the elasticity of substitution equals one, the indirect utility function simplifies to

$$V(p_1, p_2, I) = \ln(I) - (1/2)\ln(p_1) - (1/2)\ln(p_2) - \ln(2),$$

and Roy's identity implies

$$c_1^*(p_1, p_2, I) = -\frac{V_1(p_1, p_2, I)}{V_3(p_1, p_2, I)} = \frac{I}{2p_1}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{V_2(p_1, p_2, I)}{V_3(p_1, p_2, I)} = \frac{I}{2p_2}.$$

3. The Ramsey Model

The social planner takes the initial capital stock $k(0)$ as given and chooses functions $c(t)$ for $t \in [0, \infty)$ and $k(t)$ for $t \in (0, \infty)$ to maximize utility

$$\int_0^\infty e^{-\rho t} u(c(t))\,\mathrm{d}t$$

subject to the capital accumulation constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t),$$

for all $t \in [0, \infty)$.

a. The current value Hamiltonian for the social planner's problem is

$$\bar{H}(c(t), k(t), \theta(t)) = u(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

With the maximized Hamiltonian defined by extension as

$$H(k(t), \theta(t)) = \max_{c(t)} \; u(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)],$$

the maximum principle implies that the solution to the dynamic problem must satisfy the first-order condition

$$u'(c(t)) - \theta(t) = 0$$

and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t)) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\theta(k(t), \theta(t)) = k(t)^\alpha - \delta k(t) - c(t).$$

b. In the unique non-trivial steady state, where $c(t) = c^*$, $k(t) = k^*$, and $\theta(t) = \theta^*$ are all equal to constants, so that $\dot{k}(t) = \dot{\theta}(t) = 0$, the first differential equation, for $\dot{\theta}(t)$, implies that

$$k^* = \left(\frac{\delta + \rho}{\alpha}\right)^{\frac{1}{\alpha - 1}}$$

and the second differential equation, for $\dot{k}(t)$, implies that

$$c^* = \left(\frac{\delta + \rho}{\alpha}\right)^{\frac{\alpha}{\alpha - 1}} - \delta \left(\frac{\delta + \rho}{\alpha}\right)^{\frac{1}{\alpha - 1}}.$$

4. Natural Resource Depletion

The social planner takes s_0 as given and chooses sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ to maximize utility

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1 - \sigma}\right)$$

subject to the constraints

$$s_t - c_t \geq s_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

a. Although this problem can also be solved using the method of Lagrange multipliers, let's use the maximum principle instead. Start by defining the maximized present value Hamiltonian as

$$H(s_t, \pi_{t+1}; t) = \max_{c_t} \ \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1 - \sigma}\right) - \pi_{t+1} c_t.$$

Then the maximum principle implies that the sequences $\{c_t\}_{t=0}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$ that solve the dynamic problem must satisfy the first-order condition

$$\beta^t c_t^{-\sigma} - \pi_{t+1} = 0$$

and the pair of difference equations

$$\pi_{t+1} - \pi_t = -H_s(s_t, \pi_{t+1}; t) = 0$$

and

$$s_{t+1} - s_t = H_\pi(s_t, \pi_{t+1}; t) = -c_t,$$

as well as the initial condition s_0 given and the transversality condition

$$\lim_{T \to \infty} \beta^T c_T^{-\sigma} s_{T+1} = 0.$$

b. Since the first difference equation implies that $\pi_{t+1} = \pi_t$ for all $t = 1, 2, 3, \ldots$, the first-order condition implies that

$$\beta^{t+1} c_{t+1}^{-\sigma} = \beta^t c_t^{-\sigma}$$

for all $t = 0, 1, 2, \ldots$. This result shows that optimal consumption growth is constant, with

$$c_{t+1}/c_t = \beta^{1/\sigma}$$

for all $t = 0, 1, 2, \ldots$. Since the right-hand side of this last expression is less than one, consumption of the natural resource declines over time for all values of σ, generalizing the result that you derived previously for the special case of log utility, with $\sigma = 1$.

Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2018

Thursday, December 6, 1:30 - 2:45pm

This exam has three questions on five pages; please check to make sure that your copy has all three questions and all five pages. Questions 1 and 2 each have three parts; question 3 has two parts. Each part of each question is worth 5 points, for a total of $15 + 15 + 10 = 40$ points overall.

1. Learning By Doing

Consider an industry in which marginal costs of production depend inversely on the total volume of past production, reflecting the effects of learning by doing. Suppose in particular, that these costs are given by

$$c(x(t)) = [1 + x(t)]^{-\gamma}, \tag{1.1}$$

with $\gamma > 0$, where $x(t)$, the total stock of past production, depends on $y(t)$, the flow of output at each date $t \in [0, \infty)$, according to

$$y(t) \geq \dot{x}(t), \tag{1.2}$$

$\dot{x}(t) = dx(t)/dt$, as always, denoting the derivative of the function $x(t)$ with respect to t. Consumer demand for the product is described by the inverse demand curve

$$p(y(t)) = y(t)^{-\alpha}, \tag{1.3}$$

where $0 < \alpha < 1$, so that $-1/\alpha < -1$ measures the constant price elasticity of demand, assumed to be greater than one in absolute value.

If the good is produced and sold in a perfectly competitive market by a large number of firms, each of which ignores the effect that today's production has on future marginal costs, current price will equal market cost:

$$p(y(t)) = c(x(t)) = [1 + x(t)]^{-\gamma} \tag{1.4}$$

Combining (1.4) with (1.1)-(1.3), the dynamics of output under perfect competition are described by

$$y(t) = \dot{x}(t) = [1 + x(t)]^{\gamma/\alpha}. \tag{1.5}$$

Even under perfect competition, learning by doing results in rising output and falling prices over time.

Suppose now that, instead of a large number of perfectly competitive firms, the good is produced by a single monopolist, who takes into account the effects of today's learning by doing on future marginal costs and therefore solves a dynamic optimization problem, choosing continuously differentiable functions $y(t)$ for $t \in [0, \infty)$ and $x(t)$ for $t \in (0, \infty)$ to maximize the present discounted value of profits over the infinite horizon

$$\int_0^\infty e^{-rt}[y(t)p(t) - y(t)c(c(t))]\,\mathrm{d}t$$

or, using the specific functional forms introduced in (1.1) and (1.3)

$$\int_0^\infty e^{-rt}\left\{y(t)^{1-\alpha} - \frac{y(t)}{[1+x(t)]^\gamma}\right\}\,\mathrm{d}t,$$

subject to the constraint in (1.2) for all $t \in [0, \infty)$ and taking the initial condition $x(0) = 0$ as given, where $r > 0$ denotes a constant interest rate.

a. As a first step in solving the monopolist's problem using the maximum principle, write down the maximized Hamiltonian for this problem.

b. Next, write down the first-order condition and the pair of differential equations that, according to the maximum principle, help characterize the solution to the monopolist's problem.

c. If you compare the first-order condition for the monopolist's optimal choice of $y(t)$ that you derived in part (b), above, to the expression for output under perfect competition shown earlier in (1.5), you will notice two differences. First, like any monopolist operating with or without learning-by-doing, the one in this problem accounts for the fact that higher production today results in a lower output price today; that is, the monopolist compares marginal revenue, instead of price, to marginal cost when setting production. Second, unlike the competitive firms, the monopolist in this dynamic problem internalizes the effects that learning-by-doing today has on future marginal costs. Explain briefly (two or three sentences is all is should take) how these two effects on the monopolist's optimal choice of $y(t)$ are reflected in the first-order condition that you derived in part (b).

2. Optimal Growth via Human Capital Accumulation

Consider an economy in which output y_t depends on "quality adjusted" hours worked $k_t h_t$ during each period $t = 0, 1, 2, \ldots$, where h_t denotes the actual number of hours worked and k_t denotes the level of a representative consumer's human capital. Suppose that the consumer can accumulate human capital in any period only by reducing the total number of hours worked, so that

$$h_t = \varphi(k_{t+1}/k_t)$$

for all $t = 0, 1, 2, \ldots$, where φ is strictly positive, strictly decreasing, strictly concave, and continuously differentiable.

Assume, in particular, that

$$y_t = (k_t h_t)^\alpha = [k_t \varphi(k_{t+1}/k_t)]^\alpha, \tag{2.1}$$

where $0 < \alpha < 1$. Since this simple model of growth via human capital accumulation abstracts from physical capital accumulation, consumption equals output in each period $t = 0, 1, 2, \ldots$. Thus, if the representative consumer's preference are described by the additively-time separable utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} \right)$$

where the discount factor lies between zero and one, with $0 < \beta < 1$, and the constant coefficient of relative risk aversion is positive and different from one, with $\sigma > 0$ and $\sigma \neq 1$, then (2.1) implies that the representative consumer's problem can be stated as: given the initial stock of human capital $k_0 > 0$, choose the sequence $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{ \frac{[k_t \varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} \right\}.$$

To solve this problem using dynamic programming, it is most convenient to treat k_t as the period-t state variable and k_{t+1} as the period-t control variable, and then to write the Bellman equation as

$$v(k_t; t) = \max_{k_{t+1}} \frac{[k_t \varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} + \beta v(k_{t+1}; t+1). \tag{2.2}$$

a. For this problem, the value function takes the specific, time-invariant form

$$v(k_t; t) = v(k_t) = \frac{A k_t^{\alpha(1-\sigma)}}{1-\sigma}$$

for all $t = 0, 1, 2, \ldots$, where A is an undetermined coefficient. Substitute this guess for the value function into the Bellman equation (2.2). Then, using the guess, write down the first-order and envelope conditions that characterize the solution to the problem.

b. The optimality conditions you just derived imply that it is optimal for the consumer to choose $k_{t+1} = \theta k_t$ for all $t = 0, 1, 2, \ldots$, so that the growth rate of human capital is constant and equal to a second undetermined coefficient θ. Substitute this conjectured solution into the first-order and envelope conditions that are derived in part (a), above, to obtain a pair of equations that relate the undetermined coefficients A and θ to the model's parameters α, β, and σ.

c. In general, it is not possible to use the pair of equations you derived in part (b), above, to obtain explicit, closed-form expressions for A and θ in terms of α, β, and σ. Since the growth rate of human capital is constant and equal to θ, however, we can conclude in any case that the growth rate of output, y_{t+1}/y_t, will be constant as well. Use the aggregate production function (2.1) to obtain an expression for the constant output growth rate in terms of θ and one or more of the parameters α, β, and σ.

3. Saving with a Random Return

Let A_t denote an infinitely-lived consumer's assets at the beginning of each period $t = 0, 1, 2, \ldots$. Suppose that, during each period t, the consumer divides these assets up into an amount c_t to be consumed and an amount s_t to be saved. Then, between t and $t+1$, the consumer earns a return on his or her savings at the random, gross rate R_{t+1}, which does not become known until after the consumer chooses c_t and s_t during period t. For simplicity, assume that R_{t+1} is independently and identically distributed with

$$E_t[\ln(R_{t+1})] = 0$$

for all $t = 0, 1, 2, \ldots$.

Thus, the consumer takes his or her initial assets A_0 as given and chooses contingency plans for s_t for all $t = 0, 1, 2, \ldots$ and A_t for all $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t),$$

with $0 < \beta < 1$, subject to the constraints

$$R_{t+1} s_t \geq A_{t+1},$$

which must hold for all $t = 0, 1, 2, \ldots$ and all possible realizations of R_{t+1}.

a. In the dynamic programming formulation of this problem, the value function takes the specific form

$$v(A_t) = E + F \ln(A_t),$$

where E and F are undetermined coefficients. Using this guess, write down the Bellman equation for the consumer's problem. Then, write down the first-order and envelope conditions that characterize the solution to the problem.

b. Use the optimality conditions you derived in part (a), above, together with the binding constraint

$$c_t = A_t - s_t$$

to obtain an expression that shows how the optimal choice of c_t will depend, at each date $t = 0, 1, 2, \ldots$, on the consumer's stock of assets A_t and the discount factor β.

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2018

Thursday, December 6, 1:30 - 2:45pm

1. Learning By Doing

The monopolist takes the initial condition $x(0) = 0$ as given, and chooses continuously differentiable functions $y(t)$ for $t \in [0, \infty)$ and $x(t)$ for $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-rt} \left\{ y(t)^{1-\alpha} - \frac{y(t)}{[1 + x(t)]^\gamma} \right\} \mathrm{d}t$$

subject to

$$y(t) \geq \dot{x}(t) \tag{1.2}$$

for all $t \in [0, \infty)$.

a. In present value form, the maximized Hamiltonian for the monopolist's problem is

$$H(x(t), \pi(t); t) = \max_{y(t)} \; e^{-rt} \left\{ y(t)^{1-\alpha} - \frac{y(t)}{[1 + x(t)]^\gamma} \right\} + \pi(t)y(t).$$

b. According to the maximum principle, the solution to the monopolist's problem is characterized by the first-order condition

$$e^{-rt} \left\{ (1 - \alpha)y(t)^{-\alpha} - \frac{1}{[1 + x(t)]^\gamma} \right\} + \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_x(x(t), \pi(t); t) = -\gamma e^{-rt} \left\{ \frac{y(t)}{[1 + x(t)]^{1+\gamma}} \right\}$$

and

$$\dot{x}(t) = H_\pi(x(t), \pi(t); t) = y(t)$$

for all $t \in [0, \infty)$.

c. Under perfect competition, output during each period $t \in [0, \infty)$ is described by

$$y(t) = \dot{x}(t) = [1 + x(t)]^{\gamma/\alpha}. \tag{1.5}$$

In the first-order condition for the dynamic problem, the term inside brackets.

$$(1 - \alpha)y(t)^{-\alpha} - \frac{1}{[1 + x(t)]^\gamma}$$

shows that, without learning-by-doing, the monopolist would set marginal revenue equal to marginal cost by choosing

$$y(t) = (1-\alpha)^{1/\alpha}[1+x(t)]^{\gamma/\alpha}.$$

Since $0 < \alpha < 1$, for any given value of $x(t)$, this choice for $y(t)$ is smaller than under perfect competition. Since the monopolist also takes into account the effects that learning-by-doing today has on future marginal costs, however, the first-order condition dictates that term inside brackets

$$(1-\alpha)y(t)^{-\alpha} - \frac{1}{[1+x(t)]^{\gamma}}$$

be set equal to $-e^{rt}\pi(t) < 0$ instead of zero. This second effect pushes the monopolist in the opposite direction, towards choosing a larger value of $y(t)$.

2. Optimal Growth via Human Capital Accumulation

The representative consumer takes the initial stock of human capital $k_0 > 0$ as given and chooses the sequence $\{k_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{ \frac{[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} \right\}.$$

To solve this problem using dynamic programming, it is most convenient to treat k_t as the period-t state variable and k_{t+1} as the period-t control variable, and then to write the Bellman equation as

$$v(k_t;t) = \max_{k_{t+1}} \frac{[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} + \beta v(k_{t+1};t+1). \tag{2.2}$$

a. For this problem, the value function takes the specific, time-invariant form

$$v(k_t;t) = v(k_t) = \frac{Ak_t^{\alpha(1-\sigma)}}{1-\sigma},$$

where A is an undetermined coefficient. Substituting this guess into the Bellman equation yields

$$\frac{Ak_t^{\alpha(1-\sigma)}}{1-\sigma} = \max_{k_{t+1}} \frac{[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)}}{1-\sigma} + \frac{\beta Ak_{t+1}^{\alpha(1-\sigma)}}{1-\sigma}.$$

The first-order condition for k_{t+1} is then

$$\alpha[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)-1}\varphi'(k_{t+1}/k_t) + \alpha\beta Ak_{t+1}^{\alpha(1-\sigma)-1} = 0$$

and the envelope condition for k_t is

$$\alpha Ak_t^{\alpha(1-\sigma)-1} = \alpha[k_t\varphi(k_{t+1}/k_t)]^{\alpha(1-\sigma)-1}[\varphi(k_{t+1}/k_t) - (k_{t+1}/k_t)\varphi'(k_{t+1}/k_t)].$$

b. Note that the first-order and envelope conditions from part (b), above, can be rewritten as

$$\varphi(k_{t+1}/k_t)^{\alpha(1-\sigma)-1}\varphi'(k_{t+1}/k_t) + \beta A(k_{t+1}/k_t)^{\alpha(1-\sigma)-1} = 0$$

and

$$A = \varphi(k_{t+1}/k_t)^{\alpha(1-\sigma)-1}[\varphi(k_{t+1}/k_t) - (k_{t+1}/k_t)\varphi'(k_{t+1}/k_t)].$$

Besides the constant coefficient A and the constant parameters α, β, and σ, these expressions depend only on the growth rate k_{t+1}/k_t of human capital, suggesting that it is optimal for the consumer to choose $k_{t+1} = \theta k_t$ for all $t = 0, 1, 2, \ldots$, so that this growth rate is constant and equal to θ. The two equations

$$\varphi(\theta)^{\alpha(1-\sigma)-1}\varphi'(\theta) + \beta A\theta^{\alpha(1-\sigma)-1} = 0$$

and

$$A = [\varphi(\theta)]^{\alpha(1-\sigma)-1}[\varphi(\theta) - \theta\varphi'(\theta)]$$

then determine solutions for the undetermined coefficients A and θ.

c. Although it is not possible to solve the pair of equations from part (b), above, to obtain explicit, closed-form expressions for A and θ in terms of the parameters α, β, and σ, the production function

$$y_t = (k_t h_t)^\alpha = [k_t\varphi(k_{t+1}/k_t)]^\alpha \tag{2.1}$$

implies that with human capital growing at the constant rate θ, output will also grow at a constant rate, equal to

$$\frac{y_{t+1}}{y_t} = \frac{[k_{t+1}\varphi(k_{t+2}/k_{t+1})]^\alpha}{[k_t\varphi(k_{t+1}/k_t)]^\alpha} = \frac{[k_{t+1}\varphi(\theta)]^\alpha}{[k_t\varphi(\theta)]^\alpha} = \left(\frac{k_{t+1}}{k_t}\right)^\alpha = \theta^\alpha.$$

3. Saving with a Random Return

The consumer takes his or her initial assets A_0 as given and chooses contingency plans for s_t for all $t = 0, 1, 2, \ldots$ and A_t for all $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t - s_t),$$

with $0 < \beta < 1$, subject to the constraints

$$R_{t+1}s_t \geq A_{t+1},$$

which must hold for all $t = 0, 1, 2, \ldots$ and all possible realizations of R_{t+1}.

a. In the dynamic programming formulation of this problem, the value function takes the specific form

$$v(A_t) = E + F\ln(A_t),$$

where E and F are undetermined coefficients. Using this guess, the Bellman equation for the consumer's problem is

$$E + F\ln(A_t) = \max_{s_t}\ \ln(A_t - s_t) + \beta E + \beta F E_t \ln(R_{t+1}s_t).$$

But because s_t must be chosen based on time-t information, and because the random asset return R_{t+1} is assumed to be distributed such that

$$E_t[\ln(R_{t+1})] = 0$$

for all $t = 0, 1, 2, \ldots$ the Bellman equation can be simplified to

$$E + F\ln(A_t) = \max_{s_t}\ \ln(A_t - s_t) + \beta E + \beta F E_t \ln(s_t).$$

The first-order condition for s_t is therefore

$$-\frac{1}{A_t - s_t} + \frac{\beta F}{s_t} = 0,$$

and the envelope condition for A_t is

$$\frac{F}{A_t} = \frac{1}{A_t - s_t}.$$

b. Although there are a number of ways to use the first-order and envelope conditions from part (a) to obtain this result, perhaps the easiest is to note that, together, they imply

$$\frac{\beta F}{s_t} = \frac{1}{A_t - s_t} = \frac{F}{A_t}$$

and hence

$$s_t = \beta A_t$$

and

$$c_t = A_t - s_t = (1 - \beta)A_t.$$

Evidently, the consumer finds it optimal to always consume an amount equal to the fraction $1 - \beta$ of his or her assets.

Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2019

Tuesday, October 29, 1:30 - 2:45pm

This exam has three questions on four pages; before you begin, please check to make sure that your copy has all three questions and all four pages. Each part of each question is worth ten points. Therefore, questions 1 and 2 with three parts are worth 30 points each and question 3 with four parts is worth 40 points, for a total of 100 points overall.

1. Linear Expenditure System

Consider the problem solved by a consumer who uses his or her income Y to purchase in perfectly competitive markets c_1 units of good 1 at the price of p_1 per unit, c_2 units of good 2 at the price of p_2 per unit, and c units of good 1 at the price of p_3 per unit to maximize utility

$$U(c_1, c_2, c_3) = a_1 \ln(c_1 - x_1) + a_2 \ln(c_2 - x_2) + a_3 \ln(c_3 - x_3),$$

subject to the budget constraint

$$Y \geq p_1c_1 + p_2c_2 + p_3c_3,$$

where the positive parameters a_1, a_2, and a_3 satisfy

$$a_1 + a_2 + a_3 = 1$$

and the positive parameters x_1, x_2, and x_3 are such that

$$Y - p_1x_1 - p_2x_2 - p_3x_3 > 0.$$

a. Write down the Lagrangian for the consumer's utility maximization problem. Then, write down the first-order conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^*, c_2^*, and c_3^* that solve this problem along with the corresponding value of the Lagrange multiplier.

b. Next, use the first-order conditions from part (a) together with the consumer's budget constraint to find the optimal values for c_1^*, c_2^*, and c_3^* in terms of the parameters Y, p_1, p_2, p_3, a_1, a_2, a_3, x_1, x_2, and x_3. Here, you can assume that the budget constraint binds at the optimum; this will be true because utility is strictly increasing in the consumption of all three goods.

c. Finally, use your solutions from part (b) to find expressions that show how optimal expenditures $p_1c_1^*$, $p_2c_2^*$, and $p_3c_3^*$ on each of the three goods depend linearly on income Y and the prices p_1, p_2, and p_3, given the parameters a_1, a_2, a_3, x_1, x_2, and x_3.

2. Le Chatelier's Principle

Consider a firm that rents capital at the competitive rate r and hires labor at the competitive wage w to produce output y according to the technology described by

$$4k^{1/4}l^{1/4} \geq y,$$

which it then sells at the competitive price p.

a. Suppose that, in the "long run," the firm freely chooses both capital and labor inputs and output to maximize profits. This long-run problem can be stated as

$$\max_{k,l,y} yp - rk - wl \text{ subject to } 4k^{1/4}l^{1/4} \geq y.$$

Use the first-order conditions and binding constraint for this problem to find the firm's optimal long-run supply function $y^*(p,r,w)$ and optimal long-run factor demand functions $k^*(p,r,w)$ and $l^*(p,r,w)$. Then, differentiate the supply function by p to obtain an expression for $\partial y^*(p,r,w)/\partial p$, describing how optimal output supply responds to a change in output price over the long run.

b. Next, suppose that in the "short run," the firm takes its capital input $k = \bar{k}$ as fixed, and simply chooses labor input and output to maximize profits. This short-run problem can be stated as

$$\max_{l,y} yp - r\bar{k} - wl \text{ subject to } 4\bar{k}^{1/4}l^{1/4} \geq y.$$

Use the first-order conditions and binding constraint for this problem to find the firm's optimal short-run supply function $y^s(p,w,\bar{k})$ and optimal short-run labor demand function $l^s(p,w,\bar{k})$. Then, differentiate the supply function by p, still holding capital fixed at $\bar{k}$, to obtain an expression for $\partial y^s(p,w,\bar{k})/\partial p$, describing how optimal output supply responds to a change in output price in the short run.

c. In general, it will not be possible to tell which is larger: the adjustment of output to a change in price in the long run or the short run. Suppose, however, that the short-run fixed capital input just happens to equal the long-run optimal capital input. By substituting $\bar{k} = k^*(p,r,w)$ into the expression for $\partial y^s(p,w,\bar{k})/\partial p$ you derived in part (b) and comparing it to the expression for $\partial y^*(p,r,w)/\partial p$ you derived in part (a), show that a version of Le Chatelier's principle holds: adjustment of output to a change in price is larger in the long run.

3. Optimal and Equilibrium Allocations

As we discussed in class, the Ramsey, or neoclassical growth, model describes an economic environment in which the two welfare theorems apply: the competitive equilibrium allocation is Pareto optimal and the Pareto optimal allocation can be supported in a competitive equilibrium.

a. To verify this, start by characterizing optimal allocations in the same way we did in class. Taking the initial capital stock $k(0) > 0$ as given, a social planner chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize a representative consumer's utility,

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t,$$

subject to the capital accumulation constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$, where $0 < \alpha < 1$ is capital's share in a Cobb-Douglas production function when labor input is normalized to equal one in every period, $\delta > 0$ is the depreciation rate for capital, and $\rho > 0$ is a constant discount rate. For this problem, the maximized current-value Hamiltonian takes the form

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

Using this form for the maximized Hamiltonian, write down the first-order condition for $c(t)$ and the pair of differential equations for $\dot{\theta}(t)$ and $\dot{k}(t)$ that characterize Pareto optimal allocations.

b. Next, consider the activities of a representative consumer in a decentralized version of the same economy. At each instant $t \in [0, \infty)$, the consumer rents $k(t)$ units of capital and supplies one unit of labor inelastically to a representative firm, earning total income $r(t)k(t) + w(t)$. Thus, taking the initial capital stock $k(0) > 0$ as given, the consumer chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t))\, \mathrm{d}t,$$

subject to the capital accumulation constraint

$$r(t)k(t) + w(t) - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. For this problem, the maximized current-value Hamiltonian takes the form

$$H(k(t), \theta(t)) = \max_{c(t)} \ \ln(c(t)) + \theta(t)[r(t)k(t) + w(t) - \delta k(t) - c(t)].$$

Using this form for the maximized Hamiltonian, write down the first-order condition for $c(t)$ and the pair of differential equations for $\dot{\theta}(t)$ and $\dot{k}(t)$ that characterize the consumer's optimal choices. *Note:* Strictly speaking, the maximized current-value Hamiltonian for this problem depends on $r(t)$ and $w(t)$ as well as $k(t)$ and $\theta(t)$; because the consumer takes these factor prices as given, however, you'll only need to differentiate H with respect to $k(t)$ and $\theta(t)$ to derive the differential equations describing the consumer's optimal choices.

c. Next, consider the representative firm, which rents $k(t)$ units of capital and hires $n(t)$ units of labor at each instant $t \in [0, \infty)$ to maximize profits

$$k(t)^\alpha n(t)^{1-\alpha} - r(t)k(t) - w(t)n(t),$$

taking $r(t)$ and $w(t)$ as given, where $0 < \alpha < 1$ denotes capital's share in the Cobb-Douglas production function. Write down the first-order conditions for the values of $k(t)$ and $n(t)$ that solve this static, unconstrained optimization problem.

d. In equilibrium, labor demand must equal labor supply, so that $n(t) = 1$, and the representative firm must earn zero profits, so that

$$k(t)^\alpha n(t)^{1-\alpha} = r(t)k(t) + w(t)n(t)$$

or, more simply,

$$k(t)^\alpha = r(t)k(t) + w(t).$$

Combine these equilibrium conditions with the optimality conditions you derived in parts (b) and (c), in order to show that the first-order condition for $c(t)$ and the differential equations describing the evolution of $\dot{\theta}(t)$ and $\dot{k}(t)$ in the decentralized economy are exactly the same as those you derived in part (a) to describe the Pareto optimal allocations.

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2019

Tuesday, October 29, 1:30 - 2:45pm

1. Linear Expenditure System

The consumer solves

$$\max_{c_1,c_2,c_3} a_1 \ln(c_1 - x_1) + a_2 \ln(c_2 - x_2) + a_3 \ln(c_3 - x_3)$$

subject to

$$Y \geq p_1c_1 + p_2c_2 + p_3c_3.$$

where the positive parameters a_1, a_2, and a_3 satisfy

$$a_1 + a_2 + a_3 = 1$$

and the positive parameters x_1, x_2, and x_3 are such that

$$Y - p_1x_1 - p_2x_2 - p_3x_3 > 0.$$

a. With the Lagrangian for the consumer's problem defined as

$$L(c_1, c_2, c_3, \lambda) = a_1 \ln(c_1-x_1)+a_2 \ln(c_2-x_2)+a_3 \ln(c_3-x_3)+\lambda(Y-p_1c_1-p_2c_2-p_3c_3),$$

the Kuhn-Tucker condition implies that there exists a value λ^* that, together with the values c_1^*, c_2^*, and c_3^* that solve this problem, satisfy the first-order conditions

$$\frac{a_1}{c_1^* - x_1} - \lambda^* p_1 = 0,$$

$$\frac{a_2}{c_2^* - x_2} - \lambda^* p_2 = 0,$$

and

$$\frac{a_3}{c_3^* - x_3} - \lambda^* p_3 = 0.$$

b. Rearrange the first-order conditions to obtain

$$c_1^* = x_1 + \frac{a_1}{\lambda^* p_1},$$

$$c_2^* = x_2 + \frac{a_2}{\lambda^* p_2},$$

and

$$c_3^* = x_3 + \frac{a_3}{\lambda^* p_3}.$$

then substitute these expressions into the binding budget constraint to obtain

$$\frac{1}{\lambda^*} = Y - p_1 x_1 - p_2 x_2 - p_3 x_3,$$

using the condition that $a_1 + a_2 + a_3 = 1$. Substituting this result back into the expressions for optimal consumptions yields solutions

$$c_1^* = x_1 + \left(\frac{a_1}{p_1}\right)(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

$$c_2^* = x_2 + \left(\frac{a_2}{p_2}\right)(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

and

$$c_3^* = x_3 + \left(\frac{a_3}{p_3}\right)(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

which can be interpreted as saying: the consumer starts by purchasing the "essential" amounts x_1, x_2, and x_3 of each good, then allocates his or her "discretionary" income $Y - p_1 x_1 - p_2 x_2 - p_3 x_3$ to additional spending on each good in shares a_1, a_2, and a_3.

c. Multiplying the solutions from part (b) for consumption of each good by its price highlights another implication: optimal expenditures depend linearly on income and prices. Specifically,

$$p_1 c_1^* = p_1 x_1 + a_1(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

$$p_2 c_2^* = p_2 x_2 + a_2(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

and

$$p_3 c_3^* = p_3 x_3 + a_3(Y - p_1 x_1 - p_2 x_2 - p_3 x_3),$$

For more about the linear expenditure system and its econometric application, see Richard Stone, "Linear Expenditure Systems and Demand Analysis: An Application to the Pattern of British Demand," *Economic Journal*, September 1954, pp.511-527.

2. Le Chatelier's Principle

a. The firm's long-run problem is

$$\max_{k,l,y} yp - rk - wl \text{ subject to } 4k^{1/4}l^{1/4} \geq y.$$

Associated with this problem, define the Lagrangian as

$$L(k, l, y, \lambda) = yp - rk - wl + \lambda(4k^{1/4}l^{1/4} - y).$$

The first-order conditions

$$r = \lambda^*(k^*)^{-3/4}(l^*)^{1/4},$$

$$w = \lambda^*(k^*)^{1/4}(l^*)^{-3/4},$$

and

$$p = \lambda^*,$$

and can used to find the long-run factor demand functions

$$l^*(p, r, w) = p^2 r^{-1/2} w^{-3/2}$$

and

$$k^*(p, r, w) = p^2 r^{-3/2} w^{-1/2}.$$

Substituting these solutions into the binding constraint yields the long-run output supply function

$$y^*(p, r, w) = 4pr^{-1/2} w^{-1/2},$$

with

$$\frac{\partial y^*(p, r, w)}{\partial p} = 4r^{-1/2} w^{-1/2}.$$

b. The firm's short-run problem is

$$\max_{l,y} yp - r\bar{k} - wl \text{ subject to } 4\bar{k}^{1/4} l^{1/4} \geq y.$$

Associated with this problem, define the Lagrangian as

$$L(l, y, \lambda) = yp - r\bar{k} - wl + \lambda(4\bar{k}^{1/4} l^{1/4} - y).$$

The first-order conditions

$$w = \lambda^s \bar{k}^{1/4}(l^s)^{-3/4},$$

and

$$p = \lambda^s,$$

combine to yield the short-run labor demand function

$$l^s(p, w, \bar{k}) = p^{4/3} w^{-4/3} \bar{k}^{1/3}.$$

Substituting this solution into the binding constraint yields the short-run output supply function

$$y^s(p, w, \bar{k}) = 4p^{1/3} w^{-1/3} \bar{k}^{1/3},$$

with

$$\frac{\partial y^s(p, w, \bar{k})}{\partial p} = (4/3)p^{-2/3} w^{-1/3} \bar{k}^{1/3}.$$

c. Substituting

$$\bar{k} = k^*(p, r, w) = p^2 r^{-3/2} w^{-1/2}$$

into the expression for $\partial y^s(p, w, \bar{k})/\partial p$ just derived yields

$$\left.\frac{\partial y^s(p, w, \bar{k})}{\partial p}\right|_{\bar{k}=k^*} = (4/3)r^{-1/2} w^{-1/2} < 4r^{-1/2} w^{-1/2} = \frac{\partial y^*(p, r, w)}{\partial p},$$

illustrating that a version of Le Chatelier's principle holds: adjustment of output to a change in price is larger on the long run, when capital input can adjust, than in the short-run, while capital is fixed.

3. Optimal and Equilibrium Allocations

a. Taking the initial capital stock $k(0) > 0$ as given, a social planner chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize a representative consumer's utility,

$$\int_0^\infty e^{-\rho t} \ln(c(t))\,\mathrm{d}t,$$

subject to the capital accumulation constraint

$$k(t)^\alpha - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. For this problem, the maximized current-value Hamiltonian takes the form

$$H(k(t), \theta(t)) = \max_{c(t)} \; \ln(c(t)) + \theta(t)[k(t)^\alpha - \delta k(t) - c(t)].$$

The first-order condition

$$\frac{1}{c(t)} - \theta(t) = 0$$

and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t)) = \rho\theta - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = H_\theta(k(t), \theta(t)) = k(t)^\alpha - \delta k(t) - c(t),$$

characterize the Pareto optimal allocations in this economy.

b. Taking the initial capital stock $k(0) > 0$ as given, the representative consumer chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t))\,\mathrm{d}t,$$

subject to the capital accumulation constraint

$$r(t)k(t) + w(t) - \delta k(t) - c(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$. For this problem, the maximized current-value Hamiltonian takes the form

$$H(k(t), \theta(t)) = \max_{c(t)} \; \ln(c(t)) + \theta(t)[r(t)k(t) + w(t) - \delta k(t) - c(t)].$$

The first-order condition

$$\frac{1}{c(t)} - \theta(t) = 0$$

and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_k(k(t), \theta(t)) = \rho\theta - \theta(t)[r(t) - \delta]$$

and

$$\dot{k}(t) = H_\theta(k(t), \theta(t)) = r(t)k(t) + w(t) - \delta k(t) - c(t),$$

characterize the consumer's optimal choices.

c. The representative firm chooses $k(t)$ and $n(t)$ to maximize profits

$$k(t)^{\alpha}n(t)^{1-\alpha} - r(t)k(t) - w(t)n(t),$$

taking $r(t)$ and $w(t)$ as given. The first-order conditions

$$\alpha k(t)^{\alpha-1}n(t)^{1-\alpha} - r(t) = 0$$

and

$$(1-\alpha)k(t)^{\alpha}n(t)^{-\alpha} - w(t) = 0$$

characterize the firm's optimal choices.

d. In equilibrium, labor supply must equal labor demand, so that $n(t) = 1$ and the firm must earn zero profits, so that

$$k(t)^{\alpha}n(t)^{1-\alpha} = r(t)k(t) + w(t)n(t)$$

Substituting $n(t) = 1$ into the firm's first-order condition for $k(t)$ reveals that in equilibrium, the rental rate on capital is related to the marginal product of capital according to

$$r(t) = \alpha k^{\alpha-1}.$$

Substituting $n(t) = 1$ into the zero-profit condition reveals that, as well, income earned by the consumer equals output produced by the firm:

$$r(t)k(t) + w(t) = k(t)^{\alpha}.$$

Substituting these equilibrium conditions into the differential equations describing the solution to the consumer's problem yields the same pair of differential equations

$$\dot{\theta}(t) = \rho\theta - \theta(t)[\alpha k(t)^{\alpha-1} - \delta]$$

and

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t),$$

that characterize the Pareto optimal allocations for this economy. The consumer's first-order condition coincides with the social planner's as well. These observations confirm that the two welfare theorems apply. The equilibrium allocation is Pareto optimal and the optimal allocation can be supported in a competitive equilibrium.

Final Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2019

Due Thursday, December 19, at 12 noon

This exam has two questions on six pages; please check to make sure that your copy has all six pages. Each question has four parts and each part of each question is worth five points, for a total of $2 \times 4 \times 5 = 40$ points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Consumption, Investment, and Capital Accumulation in a Small Open Economy

This question asks you to characterize optimal resource allocations in a "small open economy" version of the Ramsey (neoclassical growth) model. The economy under consideration is "open" because it can borrow from abroad to finance consumption and investment; it is "small" because it takes the worldwide real interest rate as given.

Suppose, in particular, that a representative household in the small open economy can save or borrow by buying or issuing internationally-traded bonds. Each bond sells for one unit of output or consumption at time t and earns or pays interest at the constant global real interest rate r. Let $B(t)$ and $K(t)$ denote the number of bonds and the units of capital owned by the household at each time $t \in [0, \infty)$. Bond holdings $B(t)$ can be positive or negative: if $B(t)$ is positive, the household is saving, or lending to the rest of the world, and if $B(t)$ is negative, the household is borrowing from the rest of the world. As in the closed-economy Ramsey model, the household uses its $K(t)$ units of capital to produce $K(t)^\alpha$ units of output during each period $t \in [0, \infty)$, where $0 < \alpha < 1$. Also as in the closed-economy Ramsey model, this production function can be interpreted as a special case of the more general Cobb-Douglas specification, $K(t)^\alpha L(t)^{1-\alpha}$, where labor is supplied inelastically by the household, so that $L(t) = 1$ holds for all $t \in [0, \infty)$.

Let $C(t)$ and $I(t)$ denote the household's consumption and investment in each period $t \in [0, \infty)$, and assume that the household faces a quadratic cost $(\phi/2)I(t)^2$, with $\phi > 0$, measured in units of output, of adjusting its capital stock during each period $t \in [0, \infty)$. In this model, $I(t)$ can be positive, in which case the household is installing new capital, or negative, in which case it is consuming or selling off existing capital; but, either way, the formulation implies that it will incur adjustment costs in making these changes. Again as in the closed-economy Ramsey model, physical capital depreciates at the constant rate δ, with

$0 < \delta < 1$, at each date $t \in [0, \infty)$.

Given the initial conditions $B(0)$ and $K(0)$, the household's bond holdings and capital then evolve over time according to

$$rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2 \geq \dot{B}(t) \tag{1}$$

and

$$I(t) - \delta K(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0, \infty)$. Equation (1) shows that, during each period $t \in [0, \infty)$, the household receives interest income $rB(t)$ if $B(t) > 0$ or pays the interest expense $rB(t)$ if $B(t) < 0$, which it then combines with income $K(t)^\alpha$ from production in order to finance spending $C(t)$ on consumption and $I(t)$ on investment and to cover the capital adjustment cost $(\phi/2)I(t)^2$. The constraint in (1), which will always bind at the optimum, determines the change $\dot{B}(t) = dB(t)/dt$ in the household's bond holdings over time. Equation (2), meanwhile, shows how new investment $I(t)$ replaces depreciated capital $\delta K(t)$ before adding to the capital stock at each $t \in [0, \infty)$. This constraint, which will also bind at the optimum, determines the change $\dot{K}(t) = dK(t)/dt$ in the household's capital stock over time.

The household's preferences are described by the additively time-separable utility function

$$\int_0^\infty e^{-\rho t} \ln(C(t))\, \mathrm{d}t, \tag{3}$$

where $\rho > 0$ is the constant discount rate. Over the infinite horizon, the household chooses paths for the flow variables $C(t)$ and $I(t)$ for all $t \in [0, \infty)$ and the stock variables $B(t)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function in (3) subject to the constraints in (1) and (2).

Because the household chooses two flow variables and two stock variables, this problem is slightly more complicated than those we studied in class. Nevertheless, our analysis from class can be extended by defining the maximized current value Hamiltonian

$$\begin{aligned} H(B(t), K(t), \theta(t), q(t)) = \max_{C(t), I(t)} \{&\ln(C(t)) \\ &+ \theta(t)[rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2] \\ &+ q(t)[I(t) - \delta K(t)]\} \end{aligned} \tag{4}$$

and observing that, according to the maximum principle, the values of $C(t)$, $I(t)$, $B(t)$, $K(t)$ that solve the dynamic optimization problem, together with the associated values of the multipliers $\theta(t)$ and $q(t)$, must satisfy the first-order conditions for the values of $C(t)$ and $I(t)$ that solve the static, unconstrained optimization problem on the right-hand side of (4) and the differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_B(B(t), K(t), \theta(t), q(t)),$$

$$\dot{q}(t) = \rho q(t) - H_K(B(t), K(t), \theta(t), q(t)),$$

$$\dot{B}(t) = H_\theta(B(t), K(t), \theta(t), q(t)),$$

and

$$\dot{K}(t) = H_q(B(t), K(t), \theta(t), q(t))$$

where the partial derivatives of the maximized current value Hamiltonian can be computed by applying the envelope theorem to the optimization problem in (4).

a. To characterize more sharply the solution to the household's problem, write down the first-order conditions for $C(t)$ and $I(t)$ that solve the static, unconstrained optimization problem on the right-hand side of (4). Then use the envelope theorem to rewrite the differential equations for $\theta(t)$, $q(t)$, $B(t)$, and $K(t)$ so that they, too, describe the solution to the household's dynamic optimization problem.

b. Suppose that $\rho = r$, so that the household's discount rate exactly equals the worldwide real interest rate. Although this might seem like a knife-edged restriction, recall from some of your homework assignments that restrictions like this one can often be reinterpreted as equilibrium conditions if, in this case, the model is extended to describe how a large number of small open economies borrow and lend in perfectly competitive markets for the internationally-traded bond. In any case, when $\rho = r$, what do the optimality conditions you derived in part (a) imply about the behavior of consumption $C(t)$ and the multiplier $\theta(t)$?

c. Still assuming that $\rho = r$, combine the first-order condition for $I(t)$ and the differential equation for $\dot{q}(t)$ that you derived in part (a), so as to eliminate reference to the multiplier $q(t)$ and obtain a differential equation involving $I(t)$ and $K(t)$ alone.

d. Combined with the differential equation for $I(t)$ that you derived in part (c), the differential equation for $\dot{K}(t)$ that you derived in part (a), which simply restates the binding capital accumulation constraint, forms a system of two differential equations describing the optimal paths for investment $I(t)$ and the capital stock $K(t)$ in this small open economy. Use this pair of differential equations to draw a phase diagram that illustrates the following property of the solution to the household's problem: starting from any value $K(0) > 0$ for the initial capital stock, there is a unique value of investment $I(0)$ such that, starting from $I(0)$ and $K(0)$, the optimally-chosen paths for $I(t)$ and $K(t)$ converge to steady state values I^* and K^*. In drawing this phase diagram, it may be helpful to note that while investment $I(t)$ can take on positive or negative values, depending on whether the household is increasing or decreasing the domestic capital stock, the capital stock $K(t)$ itself must always remain positive.

2. Random Walk Consumption and the Marginal Propensity to Consume

This question asks you to use dynamic programming to deduce what, in a famous 1978 article from the *Journal of Political Economy*, Robert Hall called the "stochastic implications of the life-cycle-permanent income hypothesis." The example is similar to the one featuring "saving with multiple random returns" that we studied in class, except that risk is introduced here through random fluctuations in the consumer's labor income instead through random variation in asset returns.

To begin, let A_t denote a consumer's bank account balance at the beginning of each period $t = 0, 1, 2, \ldots$. The consumer takes A_0 as given, but can choose negative values of A_t for any $t = 1, 2, 3, \ldots$, in which case he or she is borrowing from the bank instead of saving.

At the beginning of period t, the consumer receives labor income y_t. Assume that labor income varies randomly over time, according to a Markov process by which the expected value $E_t y_{t+1}$ of y_{t+1} at time t depends only on y_t and not on additional lags $y_{t-1}, y_{t-2}, y_{t-3}, \ldots$. During period t, knowing current income y_t but still taking future income y_{t+j} for $j = 1, 2, 3, \ldots$ as random, the consumer chooses consumption c_t.

As in the example from class, it is convenient in setting up the consumer's dynamic programming problem to define his or her gross savings during period t as

$$s_t = A_t + y_t - c_t \tag{5}$$

Assuming that the interest rate r on savings and borrowing is constant over time, the consumer's bank account balance then evolves according to

$$(1 + r)s_t \geq A_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$.

Now, if the consumer's preferences are described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where the constant discount factor β satisfies $0 < \beta < 1$, the consumer's problem can be described as one of choosing contingency plans for s_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

subject to A_0 given, the constraint in (6) for all $t = 0, 1, 2, \ldots$, and the Markov process generating the random income stream y_t for all $t = 0, 1, 2, \ldots$. The Bellman equation for this problem is

$$v(A_t, y_t) = \max_{s_t} \; u(A_t + y_t - s_t) + \beta E_t\{v[(1 + r)s_t, y_{t+1}]\}.$$

a. Using the Bellman equation from above, write down the first-order condition for s_t and the envelope condition for A_t that help characterize the solution to the consumer's dynamic, stochastic optimization problem.

b. Next, combine your first-order and envelope conditions from part (a) with the help of the constraints shown in (5) and (6) to obtain a single optimality condition (sometimes called the"Euler equation") that links the consumer's intertemporal marginal rate of substitution to the real interest rate.

c. Now assume that the household's discount factor and the constant real interest rate satisfy $\beta(1+r) = 1$. Again, this may seem like a knife-edged restriction, but the relationship emerges as an equilibrium condition in a more complicated model where a large number of individual households borrow and lend in a competitive market for bonds. Assume, as well, that the consumer's single-period utility function is quadratic, with

$$u(c_t) = -(1/2)(c_t - b)^2$$

for some satiation, or "bliss," point b that is large enough so that $c_t - b < 0$ will always hold. Use these assumptions, together with your optimality condition from part (b), to re-derive Hall's most famous result: that consumption should follow a random walk (more precisely, a "martingale"), with

$$c_t = E_t c_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

d. By combining (5) and (6) to obtain

$$A_t + y_t - c_t \geq \frac{A_{t+1}}{1+r},$$

iterating by forward substitution, imposing some finite limit on borrowing to rule out Ponzi schemes, and invoking the transversality condition ruling out an overaccumulation of savings, one can – as we did for a simpler model in class – derive the consumer's present value budget constraint

$$A_t + \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} = \sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j}.$$

Here in this stochastic model, this present value budget constraint must hold for all possible realizations $\{y_{t+j}\}_{j=0}^{\infty}$ of the path for future income. Therefore, the same equality must hold in expected value at time t, so that

$$A_t + \sum_{j=0}^{\infty} \frac{E_t y_{t+j}}{(1+r)^j} = \sum_{j=0}^{\infty} \frac{E_t c_{t+j}}{(1+r)^j}.$$

Using the result from part (c) that optimal consumption follows a martingale, the restriction that $\beta = 1/(1+r)$, and Euclid's formula

$$\sum_{j=0}^{\infty} \beta^j = \frac{1}{1-\beta}$$

for the infinite sum, this version of the present-value budget constraint pins down the optimal level of consumption at each date $t = 0, 1, 2, \ldots$ as

$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty} \beta^j E_t y_{t+j}\right].$$

Suppose now that income follows a first-order autoregressive process, with

$$y_{t+1} = \bar{y} + \rho(y_t - \bar{y}) + \varepsilon_{t+1},$$

where $\bar{y}$ is the long-run average level of income, the parameter ρ, satisfying $0 \leq \rho < 1$, governs the persistence of fluctuations of income above or below its long-run average, and ε_{t+1} is a serially uncorrelated shock with mean zero. This law of motion for income implies that

$$E_t y_{t+j} = \bar{y} + \rho^j (y_t - \bar{y})$$

and hence that

$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty} \beta^j \bar{y} + \sum_{j=0}^{\infty} (\beta\rho)^j (y_t - \bar{y})\right].$$

By applying Euclid's formula to the two infinite sums that remain in this equation, show that Hall's model has two additional implications. Show, first, that the change in c_t brought about by a change in $\bar{y}$ holding $y_t - \bar{y}$ constant (that is, the marginal propensity to consume out of permanent income) equals one. Then show, also, that the change in c_t brought about by a change in $y_t - \bar{y}$ holding $\bar{y}$ constant (that is, the marginal propensity to consume out of deviations of income from its long-run average) is less than one and gets smaller as ρ, measuring the persistence of those deviations, declines.

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2019

Due Thursday, December 19, at 12 noon

1. Consumption, Investment, and Capital Accumulation in a Small Open Economy

The representative household in the small open economy chooses paths for the flow variables $C(t)$ and $I(t)$ for all $t \in [0,\infty)$ and the stock variables $B(t)$ and $K(t)$ for all $t \in (0,\infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t}\ln(C(t))\,\mathrm{d}t, \tag{3}$$

subject to the constraints

$$rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2 \geq \dot{B}(t) \tag{1}$$

and

$$I(t) - \delta K(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0,\infty)$, taking the initial conditions $B(0)$ and $K(0)$ as given.

The maximized current value Hamiltonian for this problem is

$$\begin{aligned} H(B(t),K(t),\theta(t),q(t)) = \max_{C(t),I(t)} \{ &\ln(C(t)) \\ &+ \theta(t)[rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2] \\ &+ q(t)[I(t) - \delta K(t)]\}. \end{aligned} \tag{4}$$

a. From the current-value Hamiltonian, the solution to the dynamic problem must satisfy the first-order conditions

$$\frac{1}{C(t)} - \theta(t) = 0$$

and

$$-\theta(t)[1 + \phi I(t)] + q(t) = 0$$

and the differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_B(B(t),K(t),\theta(t),q(t)) = \rho\theta(t) - \theta(t)r,$$

$$\dot{q}(t) = \rho q(t) - H_K(B(t),K(t),\theta(t),q(t)) = \rho q(t) - \theta(t)\alpha K(t)^{\alpha-1} + q(t)\delta,$$

$$\dot{B}(t) = H_\theta(B(t),K(t),\theta(t),q(t)) = rB(t) + K(t)^\alpha - C(t) - I(t) - (\phi/2)I(t)^2,$$

and

$$\dot{K}(t) = H_q(B(t),K(t),\theta(t),q(t)) = I(t) - \delta K(t).$$

b. If $\rho = r$, the first differential equation from part (a) implies that $\dot{\theta}(t) = 0$, so that $\theta(t)$ is equal to some constant θ for all $t \in [0, \infty)$. The first-order condition for $C(t)$ then implies that consumption is also constant for all $t \in [0, \infty)$. These conditions hold even as investment and capital are still in transition towards their steady-state values. The household borrower or lends internationally to smooth out its consumption over time, consistent with the permanent income hypothesis.

c. Still assuming that $\rho = r$, so that $\theta(t) = \theta$ is constant, the first-order condition for $I(t)$ implies that

$$q(t) = \theta[1 + \phi I(t)]$$

and

$$\dot{q}(t) = \theta\phi\dot{I}(t)$$

for all $t \in [0, \infty)$. Substituting these conditions into the second differential equation from part (a) yields

$$\theta\phi\dot{I}(t) = r\theta[1 + \phi I(t)] - \theta\alpha K(t)^{\alpha-1} + \delta\theta[1 + \phi I(t)].$$

Dividing through by $\theta\phi$ yields the differential equation

$$\dot{I}(t) = (1/\phi)\{(\delta + r)[1 + \phi I(t)] - \alpha K(t)^{\alpha-1}\}$$

involving $I(t)$ and $K(t)$ alone.

d. Combined with the fourth differential equation from part (a),

$$\dot{K}(t) = I(t) - \delta K(t),$$

the differential equation just derived in part (c),

$$\dot{I}(t) = (1/\phi)\{(\delta + r)[1 + \phi I(t)] - \alpha K(t)^{\alpha-1}\}$$

form a system of two differential equations that coincide, exactly, with those that characterize the solution to the "investment with adjustment costs" example from problem set 11. Intuitively, the domestic household's ability to borrow from abroad to smooth consumption also allows it to choose time paths for investment and capital that maximize the present value of output net of investment and adjustment costs, discounted at the world real interest rate. Exactly as in problem set 11, the differential equation for $K(t)$ implies that

$$\dot{K}(t) = 0 \text{ when } I(t) = \delta K(t),$$

$$\dot{K}(t) > 0 \text{ when } I(t) > \delta K(t),$$

and

$$\dot{K}(t) < 0 \text{ when } I(t) < \delta K(t),$$

and differential equation for $I(t)$ implies that

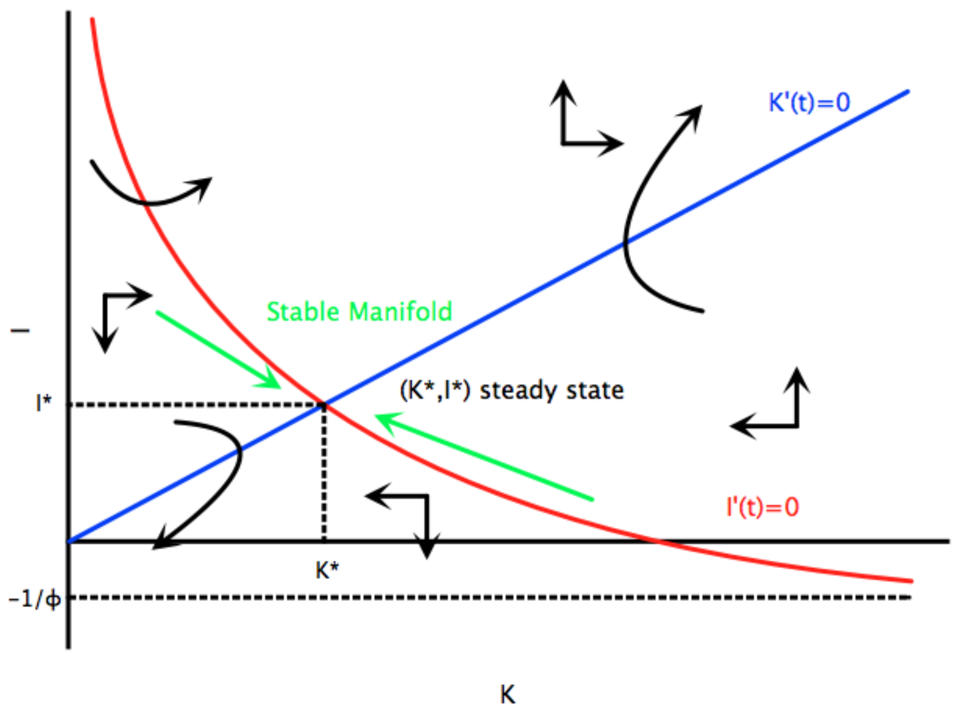
$$\dot{I}(t) = 0 \text{ when } I(t) = \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right],$$

$$\dot{I}(t) > 0 \text{ when } I(t) > \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right],$$

and

$$\dot{I}(t) < 0 \text{ when } I(t) < \frac{1}{\phi}\left[\left(\frac{\alpha}{\delta + r}\right) K(t)^{\alpha-1} - 1\right].$$

The phase diagram shown below illustrates these conditions and also reveals that starting from any value $K(0) > 0$ for the initial capital stock, there is a unique value of investment $I(0)$ such that, starting from $I(0)$ and $K(0)$, the optimally-chosen paths for $I(t)$ and $K(t)$ converge to the steady-state values I^* and K^*.



2. Random Walk Consumption and the Marginal Propensity to Consume

The consumer chooses contingency plans for s_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(A_t + y_t - s_t)$$

subject to A_0 given, the constraint

$$(1 + r)s_t \geq A_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$, and the Markov process generating the random income stream y_t for all $t = 0, 1, 2, \ldots$. The Bellman equation for this problem is

$$v(A_t, y_t) = \max_{s_t} \; u(A_t + y_t - s_t) + \beta E_t\{v[(1 + r)s_t, y_{t+1}]\}.$$

a. Using the Bellman equation from above, the first-order condition for s_t,

$$-u'(A_t + y_t - s_t) + \beta(1+r)E_t\{v_1[(1+r)s_t, y_{t+1}]\} = 0,$$

and the envelope condition for A_t,

$$v_1(A_t, y_t) = u'(A_t + y_t - s_t),$$

b. Using the definition

$$s_t = A_t + y_t - c_t \tag{5}$$

of gross savings s_t and the binding constraint from (6), the first-order and envelope conditions from part (a) can be written more simply as

$$u'(c_t) = \beta(1+r)E_t[v_1(A_{t+1}, y_{t+1})]$$

and

$$v_1(A_t, y_t) = u'(c_t).$$

Since the envelope condition must hold for all $t = 1, 2, 3, \ldots$, it also implies that

$$v_1(A_{t+1}, y_{t+1}) = u'(c_{t+1}),$$

for all $t = 0, 1, 2, \ldots$. Hence, it can be substituted into the first-order condition to obtain the optimality condition

$$u'(c_t) = \beta(1+r)E_t[u'(c_{t+1})],$$

or

$$\frac{1}{1+r} = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right],$$

linking the consumer's intertemporal marginal rate of substitution to the real interest rate.

c. When $\beta(1+r) = 1$ and

$$u(c_t) = -(1/2)(c_t - b)^2$$

the optimality condition from part (b) specializes to

$$-c_t - b = -E_t(c_{t+1} - b),$$

implying that consumption follows a martingale:

$$c_t = E_t c_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

d. By combining (5) and (6) to obtain

$$A_t + y_t - c_t \geq \frac{A_{t+1}}{1+r},$$

iterating by forward substitution, imposing some finite limit on borrowing to rule out Ponzi schemes, and invoking the transversality condition ruling out an overaccumulation of savings, one can derive the consumer's present value budget constraint

$$A_t + \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} = \sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j}.$$

Here in this stochastic model, this present value budget constraint must hold for all possible realizations $\{y_{t+j}\}_{j=0}^{\infty}$ of the path for future income. Therefore, the same equality must hold in expected value at time t, so that

$$A_t + \sum_{j=0}^{\infty} \frac{E_t y_{t+j}}{(1+r)^j} = \sum_{j=0}^{\infty} \frac{E_t c_{t+j}}{(1+r)^j}.$$

Using the result from part (c) that optimal consumption follows a martingale, the restriction that $\beta = 1/(1+r)$, and Euclid's formula

$$\sum_{j=0}^{\infty} \beta^j = \frac{1}{1-\beta}$$

for the infinite sum, this version of the present-value budget constraint pins down the optimal level of consumption at each date $t = 0, 1, 2, \ldots$ as

$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty} \beta^j E_t y_{t+j}\right].$$

Suppose now that income follows a first-order autoregressive process, with

$$y_{t+1} = \bar{y} + \rho(y_t - \bar{y}) + \varepsilon_{t+1},$$

where $\bar{y}$ is the long-run average level of income, the parameter ρ, satisfying $0 \leq \rho < 1$, governs the persistence of fluctuations of income above or below its long-run average, and ε_{t+1} is a serially uncorrelated shock with mean zero. This law of motion for income implies that

$$E_t y_{t+j} = \bar{y} + \rho^j (y_t - \bar{y})$$

and hence that

$$c_t = (1-\beta)\left[A_t + \sum_{j=0}^{\infty} \beta^j \bar{y} + \sum_{j=0}^{\infty} (\beta\rho)^j (y_t - \bar{y})\right].$$

Applying Euclid's formula to the two infinite sums that remain in this equation, yields

$$c_t = (1-\beta)A_t + \bar{y} + \left(\frac{1-\beta}{1-\beta\rho}\right)(y_t - \bar{y}).$$

This last expression reveals that the marginal propensity to consume out of permanent income $\bar{y}$ holding $y_t - \bar{y}$ constant equals one, whereas the marginal propensity to consume out of deviations of income from its long-run average $y_t - \bar{y}$ holding $\bar{y}$ constant is

$$\frac{1-\beta}{1-\beta\rho} < 1$$

and gets smaller as ρ, measuring the persistence of those deviations, declines.

Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2020

Due Tuesday, October 27 at 12noon

This exam has three questions on four pages; before you begin, please check to make sure that your copy has all three questions and all four pages. Each part of each question is worth ten points. Therefore, questions 1 and 2 with three parts are worth 30 points each and question 3 with four parts is worth 40 points, for a total of 100 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. The Kuhn-Tucker Theorem

Consider the constrained optimization problem of choosing values for x and y to maximize the objective function

$$F(x,y) = (1-x-y)(x+2y),$$

subject to the constraints

$$1 \geq x + y,$$

$$x \geq 0,$$

and

$$y \geq 0.$$

a. Define (write down) the Lagrangian for this problem. *Note:* There are a variety of ways to do this. You can choose whatever formulation you find most convenient; just make sure your answers to part (b), below, are consistent with your preferred definition here.

b. Next, write down the first-order conditions, constraints, non-negativity conditions, and complementary slackness conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values x^* and y^* of x and y that solve the problem, together with the corresponding values of the Lagrange multiplier or multipliers, depending on how you chose to handle the nonnegativity constraints on x and y in defining your Lagrangian in part (a).

c. Finally, use your results from part (b) to find the numerical values of x^* and y^* the solve the problem.

2. Consumer Optimization

Consider a consumer who uses his or her income Y to purchase c_1 units of good 1 at the perfectly competitive price of p_1 per unit and c_2 units of good 2 at the perfectly competitive price p_2 to maximize the utility function

$$U(c_1, c_2) = c_1^{1/2} + c_2^{1/2},$$

subject to the budget constraint

$$Y \geq p_1 c_1 + p_2 c_2.$$

a. Write down the first-order conditions for the consumer's optimal choices c_1^* and c_2^* of c_1 and c_2.

b. Use your first-order conditions from part (a), together with the consumer's budget constraint, which will bind at the optimum, to find the solutions for c_1^* and c_2^* in terms of the parameters p_1, p_2, and Y.

c. Use your solutions from part (b) to find expressions for the optimal shares $p_1 c_1/Y$ and $p_2 c_2/Y$ of income spent on the two goods.

3. Symmetry of Marshallian Demands?

Consider two closely-related problems faced by a consumer whose preferences over two goods are described by the utility function $U(c_1, c_2)$, where c_1 is consumption of good 1 and c_2 is consumption of good 2. Both problems assume that markets are perfectly competitive: the consumer can purchase as many or as few units as he or she wishes at the price of p_1 per unit of good 1 and p_2 per unit of good 2.

In the utility-maximization problem, the consumer takes his or her income Y as given, and maximizes utility subject to a budget constraint:

$$\max_{c_1, c_2} U(c_1, c_2) \text{ subject to } Y \geq p_1 c_1 + p_2 c_2.$$

Assume that, for all possible values for the parameters $Y > 0$, $p_1 > 0$, and $p_2 > 0$, this problem has a unique solution. Then, as the parameter values vary, the solutions for consumption of the two goods are described by the Marshallian demand curves $c_1^* = M_1(p_1, p_2, Y)$ and $c_2^* = M_2(p_1, p_2, Y)$ and the maximized value of utility is described by the indirect utility function

$$V(p_1, p_2, Y) = \max_{c_1, c_2} U(c_1, c_2) \text{ subject to } Y \geq p_1 c_1 + p_2 c_2.$$

In the expenditure-minimization problem, the consumer minimizes the cost of achieving a given level of utility $\bar{U}$:

$$\min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U},$$

Assume again that, for all possible values for the parameters $\bar{U}$, $p_1 > 0$, and $p_2 > 0$, this problem has a unique solution. Then, as the parameter values vary, the solutions for consumption of the two goods are described by the Hicksian demand curves $c_1^* = H_1(p_1, p_2, \bar{U})$ and $c_2^* = H_2(p_1, p_2, \bar{U})$ and the minimized cost is described by the expenditure function

$$E(p_1, p_2, \bar{U}) = \min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U},$$

a. Assuming that the expenditure function has continuous second partial derivatives, those derivatives must satisfy the symmetry condition

$$\frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_1 \partial p_2} = \frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_2 \partial p_1}.$$

Use the envelope theorem, applied to the expenditure-minimization problem, to show that under these conditions, the Hicksian demands must satisfy the symmetry condition

$$\frac{\partial H_1(p_1, p_2, \bar{U})}{\partial p_2} = \frac{\partial H_2(p_1, p_2, \bar{U})}{\partial p_1}.$$

The question we'll address next is: under what circumstance do the Marshallian demand curves display a similar symmetry condition? It turns out that this question can be answered with repeated use of the envelope theorem.

b. As a first step, it is useful to re-derive the Slutsky equations linking the Marshallian and Hicksian demands. To begin, note that the Marshallian and Hicksian demands coincide at the point where $Y = E(p_1, p_2, \bar{U})$, so that income in the utility maximization problem equals the minimum cost achieved in the expenditure minimization problem. For all parameter combination $\bar{U}$, $p_1 > 0$, and $p_2 > 0$, therefore, the Marshallian and Hicksian demand curves satisfy

$$H_i(p_1, p_2, \bar{U}) = M_i(p_1, p_2, E(p_1, p_2, \bar{U}))$$

for $i = 1$ and $i = 2$. Use this condition, together with the implications of the envelope theorem applied to the expenditure minimization problem, to show that the Slutsky equation,

$$\frac{\partial M_i(p_1, p_2, Y)}{\partial p_j} = \frac{\partial H_i(p_1, p_2, \bar{U})}{\partial p_j} - \frac{\partial M_i(p_1, p_2, Y)}{\partial Y} M_j(p_1, p_2, Y),$$

must hold for all $i = 1, 2$ and $j = 1, 2$ when $Y = E(p_1, p_2, \bar{U})$.

c. Set $i = 1$ and $j = 2$, so that the Slutsky equation specializes to

$$\frac{\partial M_1(p_1, p_2, Y)}{\partial p_2} = \frac{\partial H_1(p_1, p_2, \bar{U})}{\partial p_2} - \frac{\partial M_1(p_1, p_2, Y)}{\partial Y} M_2(p_1, p_2, Y).$$

Now use the symmetry condition for Hicksian demands to rewrite this expression as

$$\frac{\partial M_1(p_1, p_2, Y)}{\partial p_2} = \frac{\partial H_2(p_1, p_2, \bar{U})}{\partial p_1} - \frac{\partial M_1(p_1, p_2, Y)}{\partial Y} M_2(p_1, p_2, Y),$$

and use the Slutsky equation with $i = 2$ and $j = 1$ to obtain

$$\begin{aligned}\frac{\partial M_1(p_1, p_2, Y)}{\partial p_2} = &\frac{\partial M_2(p_1, p_2, Y)}{\partial p_1} \\ &+ \frac{\partial M_2(p_1, p_2, Y)}{\partial Y} M_1(p_1, p_2, Y) - \frac{\partial M_1(p_1, p_2, Y)}{\partial Y} M_2(p_1, p_2, Y),\end{aligned}$$

d. From this last expression, we can see that the symmetry condition

$$\frac{\partial M_1(p_1, p_2, Y)}{\partial p_2} = \frac{\partial M_2(p_1, p_2, Y)}{\partial p_1}$$

for Marshallian demand curves will not hold in general. By multiplying the second term on the right-hand side by

$$\left[\frac{M_2(p_1, p_2, Y)}{Y}\right] \left[\frac{Y}{M_2(p_1, p_2, Y)}\right] = 1$$

and the third term on the right-hand side by

$$\left[\frac{M_1(p_1, p_2, Y)}{Y}\right] \left[\frac{Y}{M_1(p_1, p_2, Y)}\right] = 1$$

however, you should be able to answer the following question: if the symmetry condition for Marshallian demands *does* hold, what must be true about the income elasticities of demand for goods 1 and 2?

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2020

Due Tuesday, October 27 at 12noon

1. The Kuhn-Tucker Theorem

The problem is to choose x and y to maximize the objective function

$$F(x,y) = (1-x-y)(x+2y),$$

subject to the constraints

$$1 \geq x+y,$$

$$x \geq 0,$$

and

$$y \geq 0.$$

a. One definition of the Lagrangian treats the nonnegativity constraints symmetrically with the constraint on the sum of x and y by assigning a separate multiplier to each:

$$L(x,y,\lambda,\mu_x,\mu_y) = (1-x-y)(x+2y) + \lambda(1-x-y) + \mu_x x + \mu_y y.$$

b. With the Lagrangian defined as above, the Kuhn-Tucker theorem implies that the values x^* and y^* that solve the problem, together with the associated values λ^*, μ_x^*, and μ_y^* of the Lagrange multipliers, must satisfy the first order conditions

$$1 - 2x^* - 3y^* - \lambda^* + \mu_x^* = 0$$

and

$$2 - 3x^* - 4y^* - \lambda^* + \mu_y^* = 0,$$

the constraints

$$1 \geq x^* + y^*,$$

$$x^* \geq 0,$$

and

$$y^* \geq 0,$$

the non-negativity conditions

$$\lambda^* \geq 0,$$

$$\mu_x^* \geq 0,$$

and

$$\mu_y^* \geq 0,$$

and the complementary slackness conditions

$$\lambda^*(1 - x^* - y^*) = 0,$$

$$\mu_x^* x^* = 0,$$

and

$$\mu_y^* y^* = 0.$$

c. In order to find the numerical values of x^* and y^*, consider first the possibility that none of the three constraints bind. In this case, the complementary slackness conditions require all three multipliers to equal zero and the first-order conditions simplify to

$$1 - 2x^* - 3y^* = 0$$

and

$$2 - 3x^* - 4y^* = 0.$$

But these conditions require $x^* = 2$ and $y^* = -1$, violating the non-negativity constraint on y. We now know that at least one of the constraints must bind at the optimum.

Observe next, that the solution cannot be such that all three constraints are binding. For this would require x^* and y^* to both equal zero but also to sum to one. Notice, in fact, that if the constraint

$$1 = x^* + y^*$$

binds at the optimum, the value of the objective function equals zero. Since it is possible to obtain positive values for the objective function by choosing any small but positive values for x^* and y^*, this first constraint cannot bind at the optimum nor, for that matter, can it be the case that x^* and y^* both equal zero, since the objective function will equal zero in that case, too.

We've now narrowed the set of possibilities down to two: either $x^* = 0$ and $\lambda^* = \mu_y^* = 0$ or $y^* = 0$ and $\lambda^* = \mu_x^* = 0$.

Consider, therefore, the case where $x^* = 0$ and $\lambda^* = \mu_y^* = 0$. In this case, the first-order conditions require

$$1 - 3y^* + \mu_x^* = 0$$

and

$$2 - 4y^* = 0,$$

or $y^* = \mu_x^* = 1/2$. As this configuration of values satisfies all of the Kuhn-Tucker conditions, it is a candidate solution.

Consider, next, the case where $y^* = 0$ and $\lambda^* = \mu_x^* = 0$. In this case, the first-order conditions require

$$1 - 2x^* = 0$$

and

$$2 - 3x^* + \mu_y^* = 0,$$

or $x^* = 1/2$ and $\mu_y^* = -1/2$. As the Kuhn-Tucker conditions require μ_y^* to be non-negative, they rule out this case as a possible solution.

Evidently, the solution has $x^* = 0$ and $y^* = 1/2$.

2. Consumer Optimization

The consumer chooses c_1 and c_2 to maximize the utility function

$$U(c_1, c_2) = c_1^{1/2} + c_2^{1/2},$$

subject to the budget constraint

$$Y \geq p_1 c_1 + p_2 c_2.$$

a. With the Lagrangian for the consumer's defined as

$$L(c_1, c_2, \lambda) = c_1^{1/2} + c_2^{1/2} + \lambda(Y - p_1 c_1 - p_2 c_2),$$

the first-order conditions are

$$\frac{1}{2(c_1^*)^{1/2}} - \lambda^* p_1 = 0$$

and

$$\frac{1}{2(c_2^*)^{1/2}} - \lambda^* p_2 = 0.$$

b. Rearrange the first-order conditions so that they read

$$c_1^* = \left(\frac{1}{2\lambda^* p_1}\right)^2$$

and

$$c_2^* = \left(\frac{1}{2\lambda^* p_2}\right)^2,$$

then substitute these expressions into the binding budget constraint to obtain

$$Y = \left(\frac{1}{2\lambda^*}\right)^2 \left(\frac{1}{p_1} + \frac{1}{p_2}\right)$$

or

$$\left(\frac{1}{2\lambda^*}\right)^2 = \left(\frac{p_1 p_2}{p_1 + p_2}\right) Y.$$

Finally, use this last expression to eliminate λ^* from the previous expressions for c_1^* and c_2^*, thereby obtaining the solutions

$$c_1^* = \left(\frac{p_2}{p_1 + p_2}\right)\left(\frac{Y}{p_1}\right)$$

and

$$c_2^* = \left(\frac{p_1}{p_1 + p_2}\right)\left(\frac{Y}{p_2}\right).$$

c. The solutions for c_1^* and c_2^* just derived imply that the shares of income optimally spent are

$$\frac{p_1 c_1^*}{Y} = \frac{p_2}{p_1 + p_2}$$

and

$$\frac{p_2 c_2^*}{Y} = \frac{p_1}{p_1 + p_2}.$$

3. Symmetry of Marshallian Demands?

The Marshallian demand curves $c_1^* = M_1(p_1, p_2, Y)$ and $c_2^* = M_2(p_1, p_2, Y)$ and indirect utility function $V(p_1, p_2, Y)$ describe the solution to the utility-maximization problem

$$V(p_1, p_2, Y) = \max_{c_1, c_2} \; U(c_1, c_2) \text{ subject to } Y \geq p_1 c_1 + p_2 c_2.$$

The Hicksian demand curves $c_1^* = H_1(p_1, p_2, \bar{U})$ and $c_2^* = H_2(p_1, p_2, \bar{U})$ and expenditure function $E(p_1, p_2, \bar{U})$ describe the solution to the cost minimization problem

$$E(p_1, p_2, \bar{U}) = \min_{c_1, c_2} \; p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U},$$

a. The envelope theorem, applied to the cost minimization problem, implies that

$$\frac{\partial E(p_1, p_2, \bar{U})}{\partial p_i} = H_i(p_1, p_2, \bar{U})$$

for $i = 1$ and $i = 2$ and hence that the symmetry conditions for Hicksian demands

$$\frac{\partial H_1(p_1, p_2, \bar{U})}{\partial p_2} = \frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_1 \partial p_2} = \frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_2 \partial p_1} = \frac{\partial H_2(p_1, p_2, \bar{U})}{\partial p_1}$$

must always hold.

b. Since the Marshallian and Hicksian demands coincide when $Y = E(p_1, p_2, \bar{U})$, the Marshallian and Hicksian demand curves satisfy

$$H_i(p_1, p_2, \bar{U}) = M_i(p_1, p_2, E(p_1, p_2, \bar{U}))$$

for $i = 1$ and $i = 2$. Therefore,

$$\begin{aligned}\frac{\partial H_i(p_1, p_2, \bar{U})}{\partial p_j} &= \frac{\partial M_i(p_1, p_2, E(p_1, p_2, \bar{U}))}{\partial p_j} \\ &\quad + \frac{\partial M_i(p_1, p_2, E(p_1, p_2, \bar{U}))}{\partial Y} \frac{\partial E(p_1, p_2, \bar{U})}{\partial p_j}.\end{aligned}$$

The envelope theorem then implies that

$$\frac{\partial H_i(p_1, p_2, \bar{U})}{\partial p_j} = \frac{\partial M_i(p_1, p_2, E(p_1, p_2, \bar{U}))}{\partial p_j} + \frac{\partial M_i(p_1, p_2, E(p_1, p_2, \bar{U}))}{\partial Y} H_j(p_1, p_2, \bar{U})$$

or, more simply,

$$\frac{\partial M_i(p_1,p_2,Y)}{\partial p_j} = \frac{\partial H_i(p_1,p_2,\bar{U})}{\partial p_j} - \frac{\partial M_i(p_1,p_2,Y)}{\partial Y} M_j(p_1,p_2,Y),$$

must hold for all $i = 1,2$ and $j = 1,2$ when $Y = E(p_1,p_2,\bar{U})$.

c. Set $i = 1$ and $j = 2$, so that the Slutsky equation specializes to

$$\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = \frac{\partial H_1(p_1,p_2,\bar{U})}{\partial p_2} - \frac{\partial M_1(p_1,p_2,Y)}{\partial Y} M_2(p_1,p_2,Y).$$

Now use the symmetry condition for Hicksian demands to rewrite this expression as

$$\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = \frac{\partial H_2(p_1,p_2,\bar{U})}{\partial p_1} - \frac{\partial M_1(p_1,p_2,Y)}{\partial Y} M_2(p_1,p_2,Y),$$

and use the Slutsky equation with $i = 2$ and $j = 1$ to obtain

$$\begin{aligned}\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = {} & \frac{\partial M_2(p_1,p_2,Y)}{\partial p_1} \\ & + \frac{\partial M_2(p_1,p_2,Y)}{\partial Y} M_1(p_1,p_2,Y) - \frac{\partial M_1(p_1,p_2,Y)}{\partial Y} M_2(p_1,p_2,Y)\end{aligned}$$

d. This last result implies that

$$\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = \frac{\partial M_2(p_1,p_2,Y)}{\partial p_1} + \frac{M_1(p_1,p_2,Y)M_2(p_1,p_2,Y)}{Y}(\eta_2 - \eta_1),$$

where

$$\eta_i = \frac{Y}{M_i(p_1,p_2,Y)} \frac{\partial M_i(p_1,p_2,Y)}{\partial Y}$$

denotes the income elasticity of demand for good i. From this last expression, we can see that the symmetry condition

$$\frac{\partial M_1(p_1,p_2,Y)}{\partial p_2} = \frac{\partial M_2(p_1,p_2,Y)}{\partial p_1}$$

for Marshallian demand will hold only in the special case where the two goods have equal income elasticities.

Final Exam

ECON 772001 - Math for Economists
Boston College, Department of Economics

Peter Ireland
Fall 2020

Due Thursday, December 10, at 12 noon

This exam has two questions on five pages; please check to make sure that your copy has all five pages. Each question has four parts and each part of each question is worth five points, for a total of $2 \times 4 \times 5 = 40$ points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Consumption and Labor Supply in Continuous Time

This problem asks you to use the maximum principle to solve a consumer's problem with an infinite horizon in continuous time.

The consumer enters each period $t \in [0, \infty)$ with bonds worth $B(t)$ dollars that pay interest at the nominal rate $r(t)$ (in net terms so that $r(t)$ is a number like 0.05). The consumer supplies $h(t)$ units of labor at t, earning $W(t)h(t)$ dollars in labor income, where $W(t)$ in the nominal wage rate. The consumer uses his or her interest and labor income to purchase $c(t)$ units of consumption at total cost of $P(t)c(t)$ dollars, where $P(t)$ is the dollar price of goods (the aggregate nominal price level) at t, according to the budget constraint

$$r(t)B(t) + W(t)h(t) - P(t)c(t) \geq \dot{B}(t). \tag{1}$$

This constraint indicates that if the consumer spends less then he or she earns in interest and labor income during period t, he or she is saving, so that $\dot{B}(t) > 0$. Conversely, if the consumer spends more than he or she earns during period t, he or she is dissaving (either borrowing or running down his or her stock of bonds) so that $\dot{B}(t) < 0$.

Taking his or her initial stock of bonds $B(0)$ as given, the consumer chooses consumption $c(t)$ and labor supply $h(t)$ for all $t \in [0, \infty)$ and bond holdings $B(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1-\sigma} - \frac{\alpha h(t)^{1+\varphi}}{1+\varphi} \right] dt$$

subject to the constraint in (1) for all $t \in [0, \infty)$. In the utility function, $\rho > 0$ is the consumer's rate of time preference, $\sigma > 0$ governs the consumer's willingness to substitute between consumption at different points in time, $\alpha > 0$ is a parameter that governs the weight placed on leisure versus consumption in preferences, and $\varphi > 0$ is a parameter that governs the consumer's elasticity of labor supply. Specifically, higher values of ρ correspond to decreased patience, higher values of σ reduce the intertemporal elasticity of substitution in consumption, higher values of α imply a stronger preference for leisure versus consumption, and higher values of φ correspond to a smaller elasticity of labor supply.

In current-value form, the maximized Hamiltonian for the consumer's problem is

$$H(B(t), \theta(t); t) = \max_{c(t),h(t)} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} - \frac{\alpha h(t)^{1+\varphi}}{1+\varphi} + \theta(t)[r(t)B(t) + W(t)h(t) - P(t)c(t)].$$

Note that, strictly speaking, the maximized current value Hamiltonian depends on t in addition to $B(t)$ and $\theta(t)$ because of time-variation in the interest and wage rates $r(t)$ and $W(t)$ and in the price level $P(t)$.

a. Using the expression for the maximized current-value Hamiltonian given above, write down the first-order conditions for $c(t)$ and $h(t)$ and the pair of differential equations for $\theta(t)$ and $B(t)$ that, according to the maximum principle, help characterize the solution to the consumer's problem.

b. Next, combine two first-order conditions you derived in part (a) to show that the optimizing consumer chooses $c(t)$ and $h(t)$ at each t to equate his or her marginal rate of substitution between consumption and leisure (that is, the ratio of the marginal disutilty of labor supply to the marginal utility of consumption) to the real wage rate $W(t)/P(t)$.

c. Now let

$$\pi(t) = \frac{\dot{P}(t)}{P(t)}$$

denote the rate of price inflation during each period $t \in [0, \infty)$. Use this definition, together with your optimality conditions from part (a), to derive the Euler equation that links the consumer's optimal rate of consumption growth, $\dot{c}(t)/c(t)$, to the parameter σ governing the consumer's intertemporal elasticity of substitution in consumption, the parameter ρ governing the consumer's degree of patience or impatience, and the real (inflation-adjusted) interest rate $r(t) - \pi(t)$.

d. Finally, use your result from part (c) to answer the following question: under what circumstances will the consumer find it optimal to choose a constant level of consumption, with $\dot{c}(t) = 0$ at t?

2. Uncovered Interest Rate Parity?

Consider an investor, residing in the United States, who can trade in bonds issued by the US government and bond issued by the Japanese government. A US government bond costs one dollar today (at time t) and returns R^{usd} next period (at time $t+1$), where R^{usd} is the US dollar interest rate (in gross terms, so R^{usd} is a number like 1.05). A Japanese government bond costs one yen today (at time t) and returns R^{yen} next period (at time $t+1$), where R^{yen} is the Japanese yen interest rate (also in gross terms). The notation reflects an assumption, made here for simplicity, that the US dollar and Japanese yen interest rates remain constant over time.

Since the investor resides in the US, he or she cares about asset prices and payoffs denominated in dollars. Let e_t denote the dollar-per-yen exchange rate today (at time t); that is, e_t is the number of dollars that must be exchanged for one yen today. Likewise, let e_{t+1} denote the dollar-per-yen exchange rate next period (at time $t+1$). With this notation, we can see that purchasing a Japanese government bond requires e_t dollars today; the R^{yen} returned by the bond can then be converted back to $e_{t+1}R^{yen}$ dollars next period.

If we assume that equilibrium in global financial markets requires the expected returns on these two strategies – investing in US government bonds and investing in Japanese government bonds – to be equal, then

$$R^{usd} = R^{yen} E_t\left(\frac{e_{t+1}}{e_t}\right),$$

where E_t denotes investors' expectation at time t. Rearranging this equation yields what international economists call the "uncovered interest parity" or "UIP" condition

$$\frac{R^{usd}}{R^{yen}} = E_t\left(\frac{e_{t+1}}{e_t}\right). \tag{2}$$

This result says that if interest rates on US government bonds are observed to be higher than interest rates on Japanese government bonds, it must be that investors expect the US dollar to depreciate, that is, fall in value so that $E_t(e_{t+1})$ is larger than e_t. Otherwise, investors could earn higher expected returns by selling Japanese bonds and buying US bonds. Conversely, if interest rates on US government bonds are observed to be lower than interest rates on Japanese government bonds, it must be that investors expect the US dollar to appreciate, that is, rise in value so that $E_t(e_{t+1})$ is smaller than e_t.

While intuitively appealing, uncovered interest parity rarely holds in the data. To the contrary, returns on bonds denominated in "high interest rate currencies" tend to have higher average returns than bonds denominated in "low interest rate currencies." With reference to the specific example considered here, the historical experience suggests that, since interest rates are higher in the US than in Japan today, investors in US government bonds will earn higher expected returns than investors in Japanese government bonds, even after taking exchange rate movements into account.

Should we be surprised that the UIP condition is rejected by the data? To find out, this problem asks you to use dynamic programming to solve an asset allocation problem faced by the investor located in the US, who can invest in both US government bonds and Japanese government bonds. In this problem, there are three sources of uncertainty. The investor's labor income W_t, the US price level P_t, and the dollar-per-yen exchange rate e_t all fluctuate randomly, governed by stochastic processes that have the Markov property: from the perspective of time t, the distributions of W_{t+1}, P_{t+1}, and e_{t+1} can depend on the realized values of W_t, P_t, and e_t but not on additional lagged realizations W_{t-j}, P_{t-j}, and e_{t-j} with $j = 1, 2, 3, \ldots$.

The investor enters each period $t = 0, 1, 2, \ldots$ with financial wealth A_t, measured in US dollars. At the beginning of period t, the values of W_t, P_t, and e_t are observed by the investor; however, the values of W_{t+1}, P_{t+1}, and e_{t+1} are still viewed as random. During period t, the investor divides his or her financial wealth A_t, augmented by his or her labor income W_t, into amounts used to purchase consumption goods, US government bonds, and Japanese government bonds.

More specifically, since each consumption good sells for P_t dollars during period t, the investor purchases c_t units of consumption at the total cost of $P_t c_t$ dollars. As above, the interest rates R^{usd} and R^{yen} on US government and Japanese government bonds are assumed to be constant over time. Thus, if B_t^{usd} and B_t^{yen} denote the number of US and Japanese bonds purchased by the investor, the investor spends B_t^{usd} dollars on US bonds and $e_t B_t^{yen}$ dollars on Japanese bonds during period t. These bonds then return $B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}$ dollars to the investor at the beginning of period $t+1$.

Thus, the investor takes initial wealth A_0 as given, and chooses contingency plans for c_t, B_t^{usd}, and B_t^{yen} for $t = 0, 1, 2, \ldots$ and A_t for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

with discount factor β between 0 and 1, subject to the constraints

$$A_t + W_t \geq P_t c_t + B_t^{usd} + e_t B_t^{yen}$$

and

$$B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen} \geq A_{t+1},$$

where each of these two constraints must hold for all $t = 0.1, 2, \ldots.$ and all possible realizations of the random shocks.

As a first step in solving the investor's problem, note that if the utility function u is strictly increasing, both constraints will always hold with equality. Thus, to simplify the Bellman equation, the first constraint can be solved for c_t and substituted into the utility function and the second constraint can be solved for A_{t+1} and substituted into the value function for period $t+1$, yielding

$$\begin{aligned} v(A_t, W_t, P_t, e_t) &= \max_{B_t^{usd}, B_t^{yen}} u\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right) \\ &\quad + \beta E_t v(B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}, W_{t+1}, P_{t+1}, e_{t+1}). \end{aligned}$$

a. Using the Bellman equation from above, derive (write down) the first-order and envelope conditions that characterize the investor's optimal choices of B_t^{usd}, B_t^{yen}, and A_t.

b. Next, use the envelope condition together with the binding constraints from the original formulation of the investor's problem to eliminate reference to the unknown value function in the two first-order conditions for B_t^{usd} and B_t^{yen}.

c. Now let

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

denote the investor's intertemporal marginal rate of substitution (IMRS) between t and $t+1$ and let

$$\pi_{t+1} = \frac{P_{t+1}}{P_t}$$

denote the US inflation rate between t and $t+1$. Rearrange your first-order conditions for B_t^{usd} and B_t^{yen} so that the first links the US dollar interest rate R^{usd} to the conditional expectation

$$E_t\left(\frac{m_{t+1}}{\pi_{t+1}}\right)$$

and the second links the Japanese yen interest rate R^{yen} to the conditional expectation

$$E_t\left[\left(\frac{m_{t+1}}{\pi_{t+1}}\right)\left(\frac{e_{t+1}}{e_t}\right)\right].$$

d. Finally, recall that the definition of covariance implies that

$$E_t\left[\left(\frac{m_{t+1}}{\pi_{t+1}}\right)\left(\frac{e_{t+1}}{e_t}\right)\right] = E_t\left(\frac{m_{t+1}}{\pi_{t+1}}\right)E_t\left(\frac{e_{t+1}}{e_t}\right) + cov_t\left(\frac{m_{t+1}}{\pi_{t+1}}, \frac{e_{t+1}}{e_t}\right).$$

Use this definition, together with your results from part (c), to obtain a more general expression that shows that the UIP condition (2) will only hold if the rate of dollar depreciation e_{t+1}/e_t is uncorrelated with the ratio m_{t+1}/π_{t+1} of the US investor's IMRS to the US inflation rate. This special case will hold if, for example, either the Federal Reserve or Bank of Japan conducts monetary policy in a way that fixes the dollar-per-yen exchange rate over time, so that $e_{t+1}/e_t = 1$ for sure for all $t = 0, 1, 2, \ldots$. But under a regime of floating exchange rates, it seems highly unlikely that the dollar-per-yen exchange rate will be uncorrelated with US inflation and the broader US economic conditions that will affect the investor's intertemporal marginal rate of substitution. Again, the message of this fully dynamic and stochastic model is that we should not be surprised that econometric tests routinely reject the UIP condition (2) – in light of this model, in fact, it would be surprising if, to the contrary, UIP *did* appear to hold in the data from countries with floating exchange rates!

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2020

Due Thursday, December 10, at 12 noon

1. Consumption and Labor Supply in Continuous Time

Taking his or her initial stock of bonds $B(0)$ as given, the consumer chooses consumption $c(t)$ and labor supply $h(t)$ for all $t \in [0, \infty)$ and bond holding $B(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1-\sigma} - \frac{\alpha h(t)^{1+\varphi}}{1+\varphi} \right] dt$$

subject to the constraint constraint

$$r(t)B(t) + W(t)h(t) - P(t)c(t) \geq \dot{B}(t). \tag{1}$$

for all $t \in [0, \infty)$. In current-value form, the maximized Hamiltonian for this problem is

$$H(B(t), \theta(t)) = \max_{c(t), h(t)} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} - \frac{\alpha h(t)^{1+\varphi}}{1+\varphi} + \theta(t)[r(t)B(t) + W(t)h(t) - P(t)c(t)].$$

a. According to the maximum principle, the solution to the consumer's problem is characterized by the first-order conditions

$$c(t)^{-\sigma} - \theta(t)P(t) = 0$$

and

$$-\alpha h(t)^{\varphi} + \theta(t)W(t) = 0$$

for $c(t)$ and $h(t)$ and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - H_B(B(t), \theta(t)) = \rho\theta(t) - \theta(t)r(t)$$

and

$$\dot{B}(t) = H_\theta(B(t), \theta(t)) = r(t)B(t) + W(t)h(t) - P(t)c(t).$$

for $\theta(t)$ and $B(t)$.

b. Rearrange the first-order condition for $c(t)$ to read

$$\theta(t) = \frac{c(t)^{-\sigma}}{P(t)}$$

and substitute this expression into the first-order condition for $h(t)$ to obtain

$$\frac{\alpha h(t)^{\varphi}}{c(t)^{-\sigma}} = \frac{W(t)}{P(t)}.$$

The numerator of the fraction on the left-hand side is the marginal disutility of labor or, equivalently, the marginal utility of leisure. The denominator is the marginal utility of consumption. Hence, the ratio itself is the marginal rate of substitution between consumption and leisure, which the optimizing consumer sets equal to the real wage $W(t)/P(t)$.

c. Rewrite the first-order condition for $c(t)$ as

$$c(t)^{-\sigma} = \theta(t)P(t)$$

and differentiate both sides with respect to t to obtain

$$-\sigma c(t)^{-\sigma-1}\dot{c}(t) = \dot{\theta}(t)P(t) + \theta(t)\dot{P}(t).$$

Next, use the first-order condition for $c(t)$ to replace $c(t)^{-\sigma}$ with $\theta(t)P(t)$ on the left-hand side and the differential equation for $\theta(t)$ to replace $\dot{\theta}(t)$ with $[\rho - r(t)]\theta(t)$ on the right-hand side. Now

$$-\sigma\theta(t)P(t)\left[\frac{\dot{c}(t)}{c(t)}\right] = [\rho - r(t)]\theta(t)P(t) + \theta(t)\dot{P}(t).$$

Divide both sides of this last expression by $-\sigma\theta(t)P(t)$ and use the notation $\pi(t) = \dot{P}(t)/P(t)$ for inflation to obtain

$$\frac{\dot{c}(t)}{c(t)} = (1/\sigma)[r(t) - \pi(t) - \rho],$$

which is the Euler equation for consumption.

d. The Euler equation implies that it will be optimal for the consumer to hold consumption constant, with $\dot{c}(t) = 0$, whenever the real interest rate $r(t) - \pi(t)$ equals the rate of time preference ρ.

2. Uncovered Interest Rate Parity?

The investor takes initial wealth A_0 as given, and chooses contingency plans for c_t, B_t^{usd}, and B_t^{yen} for $t = 0, 1, 2, \ldots$ and A_t for $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the constraints

$$A_t + W_t \geq P_t c_t + B_t^{usd} + e_t B_t^{yen}$$

and

$$B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen} \geq A_{t+1},$$

where each of these two constraints must hold for all $t = 0.1, 2, \dots$. and all possible realizations of the random shocks.

The Bellman equation for this problem is

$$\begin{aligned} v(A_t, W_t, P_t, e_t) &= \max_{B_t^{usd}, B_t^{yen}} u\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right) \\ &\quad + \beta E_t v(B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}, W_{t+1}, P_{t+1}, e_{t+1}). \end{aligned}$$

a. Using the Bellman equation from above, the first-order conditions for B_t^{usd} and B_t^{yen} are

$$\begin{aligned} &\left(\frac{1}{P_t}\right) u'\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right) \\ &= \beta R^{usd} E_t[v_1(B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}, W_{t+1}, P_{t+1}, e_{t+1})] \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{e_t}{P_t}\right) u'\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right) \\ &= \beta R^{yen} E_t[e_{t+1} v_1(B_t^{usd} R^{usd} + e_{t+1} B_t^{yen} R^{yen}, W_{t+1}, P_{t+1}, e_{t+1})], \end{aligned}$$

while the envelope condition for A_t is

$$v_1(A_t, W_t, P_t, e_t) = \left(\frac{1}{P_t}\right) u'\left(\frac{A_t + W_t - B_t^{usd} - e_t B_t^{yen}}{P_t}\right).$$

Note that, in the first-order conditions, the constant interest rates R^{usd} and R^{yen} can be pulled outside of the conditional expectations, but the unknown future exchange rate e_{t+1} must remain inside the expectation.

b. The binding constraints can be used to simplify the first-order and envelope conditions so that they read, more compactly,

$$\frac{u'(c_t)}{P_t} = \beta R^{usd} E_t[v_1(A_{t+1}, W_{t+1}, P_{t+1}, e_{t+1})],$$

$$\frac{e_t u'(c_t)}{P_t} = \beta R^{yen} E_t[e_{t+1} v_1(A_{t+1}, W_{t+1}, P_{t+1}, e_{t+1})],$$

and

$$v_1(A_t, W_t, P_t, e_t) = \frac{u'(c_t)}{P_t}.$$

Since the envelope condition must hold for all $t = 1, 2, 3, \dots$, it also implies that

$$v_1(A_{t+1}, W_{t+1}, P_{t+1}, e_{t+1}) = \frac{u'(c_{t+1})}{P_{t+1}}.$$

Substituting these expressions into the first-order conditions yields

$$\frac{u'(c_t)}{P_t} = \beta R^{usd} E_t \left[\frac{u'(c_{t+1})}{P_{t+1}}. \right]$$

and

$$\frac{e_t u'(c_t)}{P_t} = \beta R^{yen} E_t \left[\frac{e_{t+1} u'(c_{t+1})}{P_{t+1}}. \right].$$

c. Using $m_{t+1} = \beta u'(c_{t+1})/u'(c_t)$ to denote the consumer IMRS and $\pi_{t+1} = P_{t+1}/P_t$ to denote the US inflation rate, the first-order conditions can be written even more compactly as

$$\frac{1}{R^{usd}} = E_t \left(\frac{m_{t+1}}{\pi_{t+1}} \right)$$

and

$$\frac{1}{R^{yen}} = E_t \left[\left(\frac{m_{t+1}}{\pi_{t+1}} \right) \left(\frac{e_{t+1}}{e_t} \right) \right].$$

d. We can now see that the key to understanding observed deviations from the UIP condition (2) is the fact that the expected value of the product of m_{t+1}/π_{t+1} and e_{t+1}/e_t is not the same as the product of the expected values of m_{t+1}/π_{t+1} and e_{t+1}/e_t. In fact, the first-order conditions imply that

$$\begin{aligned} \frac{1}{R^{yen}} &= E_t \left[\left(\frac{m_{t+1}}{\pi_{t+1}} \right) \left(\frac{e_{t+1}}{e_t} \right) \right] \\ &= E_t \left(\frac{m_{t+1}}{\pi_{t+1}} \right) E_t \left(\frac{e_{t+1}}{e_t} \right) + cov_t \left(\frac{m_{t+1}}{\pi_{t+1}}, \frac{e_{t+1}}{e_t} \right) \\ &= \left(\frac{1}{R^{usd}} \right) E_t \left(\frac{e_{t+1}}{e_t} \right) + cov_t \left(\frac{m_{t+1}}{\pi_{t+1}}, \frac{e_{t+1}}{e_t} \right) \end{aligned}$$

so that instead of (2), we have the more general relation

$$\frac{R^{usd}}{R^{yen}} = E_t \left(\frac{e_{t+1}}{e_t} \right) + R^{usd} cov_t \left(\frac{m_{t+1}}{\pi_{t+1}}, \frac{e_{t+1}}{e_t} \right).$$

This relation makes clear that UIP will only hold in the very special case in which m_{t+1}/π_{t+1} and e_{t+1}/e_t are uncorrelated. This special case would arise, most plausibly, under a regime of fixed exchange rates in which $e_{t+1}/e_t = 1$ for sure for all $t = 0, 1, 2, \ldots$. But suppose the US dollar depreciates, so that e_{t+1}/e_t is large, whenever the US economy experiences a period of deflationary recession that is, an episode where m_{t+1} is large because consumption growth is slow and π_{t+1} is small because inflation is low. Now Japanese bonds provide "insurance" against a deflationary recession in the US: the dollar depreciation under those circumstances means that Japanese bonds pay off particularly well, in dollar terms, exactly when US investors need the payoffs most. Under those circumstances, US investors will still be willing to hold Japanese bonds, despite their low yields and even without the expectation that the dollar will depreciate on average.

Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2021

Due Tuesday, November 2

This exam has three questions on five pages; before you begin, please check to make sure that your copy has all three questions and all five pages. Each part of each question is worth ten points. Therefore, question 1 with one part is worth 10 points, question 2 with four parts is worth 40 points, and question 3 with five parts is worth 50 points, for a total of 100 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Utility Maximization

Consider a consumer with preferences over food c_F and all other goods c_O as described by the utility function

$$\alpha \ln(c_F) + (1 - \alpha) \ln(c_O),$$

where the parameter α lies between zero and one, with $0 < \alpha < 1$. If the consumer has income Y and purchases food and all other goods at prices p_F and p_O in perfectly competitive markets, he or she solves

$$\max_{c_F, c_O} \alpha \ln(c_F) + (1 - \alpha) \ln(c_O) \text{ subject to } Y \geq p_F c_F + p_O c_O.$$

Write down the expressions that show how the solutions c_F^* and c_O^* to this problem depend on Y, p_F, p_O, and α.

Note: You can derive these expressions by setting up the Lagrangian, taking the first-order conditions, and using those first-order conditions together with the binding constraint to solve for c_F^* and c_O^*. Or you can just recall that you've solved a version of this problem before, in problem set 1, and simply adapt the solution you derived there to apply to "food" and "all other goods" instead of "good 1" and "good 2."

2. Expenditure Minimization

Next, let's acknowledge that "food" c_F from question 1 is an aggregate of individual food items. Suppose, in particular, that the consumer's total food consumption is determined by his or her consumption c_A of apples and c_B of bananas according to

$$c_F = \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

were $\theta > 0$ is a parameter measuring the elasticity of substitution between apples and bananas in forming or "producing" the food aggregate c_F. While this specification allows for imperfect substitutability in preferences between apples and bananas, it also has the property of being homogeneous of degree one: if the consumer buys twice as many apples and bananas, the food aggregate doubles as well.

The question we want to answer next is: If we know the prices p_A of apples and p_B of bananas, can we also define a corresponding price aggregate to measure p_F to go along with the quantity aggregate c_F that appears in question 1?

To answer this question, suppose that the consumer minimizes the cost of producing at least c_F units of the food aggregate by solving

$$\min_{c_A, c_B} \; p_A c_A + p_B c_B \text{ subject to } \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \geq c_F.$$

a. Define (write down) the Lagrangian for this problem. *Note:* There are a variety of ways to do this. You can choose whatever formulation you find most convenient; just make sure your answers to part (b), below, are consistent with your preferred definition here.

b. Next, write down the two first-order conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values of c_A^* and c_B^* that solve this problem.

c. Together with the binding constraint,

$$c_F = \left({c_A^*}^{\frac{\theta-1}{\theta}} + {c_B^*}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

your two first-order conditions from part (b) form a system of three equations in three unknowns: the values c_A^* and c_B^* that solve the problem and the associated value of the Lagrange multiplier. Use these equations to solve for the optimal choices c_A^* and c_B^* in terms of the parameters: c_F, p_A, p_B, and θ. *Note:* There are a number of ways to do this – use whichever you find most convenient. But perhaps the easiest is to divide one first-order condition by the other to eliminate the multiplier, then substitute the resulting expression into the binding constraint.

d. With the minimum expenditure function defined as

$$E(c_F, p_A, p_B) = \min_{c_A, c_B} \; p_A c_A + p_B c_B \text{ subject to } \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \geq c_F,$$

find an expression for the partial derivative of E with respect to the parameter c_F,

$$\frac{\partial E(c_F, p_A, p_B)}{\partial c_F}$$

in terms of p_A and p_B. *Note:* There are at least two ways to do this – again, use whichever you find most convenient. One is to substitute your solutions for c_A^* and c_B^* from part (c) directly into the expenditure function. The other is to find the solution for the Lagrange multiplier implied by the solutions for c_A^* and c_B^* from part (c), and use the envelope theorem instead.

This partial derivative has an interpretation as the "marginal cost" to the consumer of obtaining an additional unit of the food aggregate c_F. Thus, in in comparing the solution to question 3 below to the solution to question 1 above, it will be our candidate for measuring the price aggregate p_F corresponding to the food aggregate c_F.

3. More Detailed Utility Maximization

Now let's substitute the formula for the food aggregate

$$c_F = \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

from question 2 into the utility function

$$\alpha \ln(c_F) + (1-\alpha)\ln(c_O),$$

from question 1 to see what happens when we consider the optimal choices of individual good items as well as all other goods in a more detailed utility maximization problem. Our general aim is to verify that the solutions to this more detailed problem are consistent with the solutions to questions 1 and 2.

If the consumer has income Y, and purchases apples, bananas, and all other goods at prices p_A, p_B, and p_O in perfectly competitive markets, he or she solves

$$\max_{c_A, c_B, c_O} \left(\frac{\alpha\theta}{\theta-1} \right) \ln\left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right) + (1-\alpha)\ln(c_O) \text{ subject to } Y \geq p_A c_A + p_B c_B + p_O c_O.$$

a. Define (write down) the Lagrangian for this problem. *Note:* Once again, there are a variety of ways to do this. You can choose whatever formulation you find most convenient; just make sure your answers to part (b), below, are consistent with your preferred definition here.

b. Next, write down the first-order conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values of c_A^*, c_B^*, and c_O^* that solve this problem.

c. Together with the binding constraint

$$Y = p_A c_A^* + p_B c_B^* + p_O c_O^*,$$

your three first-order conditions form a system of four equations in four unknowns: the values c_A^*, c_B^*, and c_O^* that solve the problem and the associated value of the Lagrange multiplier. As a first step in solving this system, multiply the first-order condition for c_A by c_A^*, the first-order condition for c_B by c_B^*, and the first-order condition for c_O by c_O^*. Then, after dividing each first-order condition by the Lagrange multiplier, substitute them into the binding budget constraint to find a simple solution for the multiplier in terms of Y. Then, substitute this solution for the multiplier back into the first-order condition for c_O to obtain a solution for c_O^* that coincides exactly with the one you wrote down previously in answering question 1.

d. Next, divide the first-order condition for c_A by the first-order condition for c_B, and use the resulting expression together with the definition of optimal food consumption

$$c_F^* = \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

to find expressions for c_A^* and c_B^* in terms of c_F^*, p_A, p_B, and θ. These expressions should coincide with your solutions from question 2, with c_F^* in place of c_F.

e. It only remains to find c_F^*. This can be done by substituting the expressions for the multiplier, c_A^*, and c_B^* that you just derived in parts (c) and (d) into either the first-order condition for c_A or the first-order condition for c_B. Verify that when the price aggregate p_F for food is defined as suggested by the solution to part (d) of question 2, the solution for c_F^* derived here coincides with the one you wrote down in answering question 1.

Summary and Generalization

Questions 1-3 illustrate that under certain circumstances, it is possible for a consumer to decide first on how much to spend on food versus all other goods and then, only later, how to break spending on food down into components spent on individual food items. Likewise, under the same circumstances, it is possible for an economist who simply wants to study a consumer's total spending on food to solve an aggregated problem like that in question 1, moving on to study more complicated problems like those in questions 2 and 3 only if spending on individual food items is of additional interest.

The general conditions under which this approach is possible are identified by Charles Blackorby, Daniel Primont, and R. Robert Russell in "Separability: A Survey," Chapter 2 in the *Handbook of Utility Theory, Volume 1: Principles*, edited by Salvador Barberà, Peter J. Hammond, and Christian Seidel (Kluwer Academic Publishers, 1998). First, preferences over individual food items must be weakly separable from those over all other goods, meaning that they can be described by a utility function that takes the form

$$\mathcal{U}(c_A, c_B, c_O) = U(f(c_A, c_B), c_O)$$

for some functions U and f. In this general specification as in the example above,

$$c_F = f(c_A, c_B)$$

defines the quantity aggregate for food. Second, the aggregator function f must be homothetic; this assumption guarantees that there exists a corresponding price aggregate,

$$p_F = g(p_A, p_B),$$

defining p_F in terms of p_A and p_B. Together, these assumptions hold if and only if the utility function over the three goods takes the form

$$\mathcal{U}(c_A, c_B, c_O) = \tilde{U}(\tilde{f}(c_A, c_B), c_O),$$

where $\tilde{f}$ is homogeneous of degree one.

The CES (constant elasticity of substitution) aggregator introduced here in question 2 can be a bit tedious to work with, but usefully allows for different degrees of substitutability between components of the aggregate and is still more tractable than most alternatives. Therefore, it gets used a lot in macroeconomics, international trade, and international finance.

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2021

Due Tuesday, November 2

1. Utility Maximization

The consumer solves

$$\min_{c_F, c_O} \; \alpha \ln(c_F) + (1-\alpha)\ln(c_O) \text{ subject to } Y \geq p_F c_F + p_0 c_0.$$

The Lagrangian for this problem can be written as

$$L(c_F, c_O, \lambda) = \alpha \ln(c_F) + (1-\alpha)\ln(c_O) + \lambda(Y - p_F c_F - p_0 c_0).$$

The first-order conditions for c_F^* and c_O^* are

$$\frac{\alpha}{c_F^*} - \lambda^* p_F = 0$$

and

$$\frac{1-\alpha}{c_O^*} - \lambda^* p_O = 0,$$

and can be rearranged to read

$$c_F^* = \frac{\alpha}{\lambda^* p_F}$$

and

$$c_O^* = \frac{1-\alpha}{\lambda^* p_O}.$$

Substituting these last expressions into the binding constraint

$$Y = p_F c_F^* + p_O c_O^*$$

provides the solution

$$\lambda^* = \frac{1}{Y},$$

which can then be substituted back into the previous expressions to obtain

$$c_F^* = \frac{\alpha Y}{p_F}$$

and

$$c_O^* = \frac{(1-\alpha)Y}{p_O}.$$

2. Expenditure Minimization

The consumer solves

$$\min_{c_A, c_B} \; p_A c_A + p_B c_B \text{ subject to } \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \geq c_F.$$

a. Letting λ denote the nonnegative multiplier on the constraint, the Lagrangian for this problem can be defined as

$$L(c_A, c_B, \lambda) = p_A c_A + p_B c_B - \lambda \left[\left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} - c_F \right]$$

b. With the Lagrangian defined as above, the Kuhn-Tucker theorem implies that the values c_A^* and c_B^* that solve the problem, together with the associated value λ^* of the Lagrange multiplier, must satisfy the first-order conditions

$$p_A - \lambda^* \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} c_A^{*-\frac{1}{\theta}} = 0$$

and

$$p_B - \lambda^* \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} c_B^{*-\frac{1}{\theta}} = 0$$

c. Together with the binding constraint

$$c_F = \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

the first-order conditions form a system of three equations in the three unknowns c_A^*, c_B^*, and λ^*. Although there are many ways of solving this system, perhaps the easiest is to start by dividing the first-order condition for c_A by the first-order condition for c_B to obtain

$$\frac{p_A}{p_B} = \left(\frac{c_B^*}{c_A^*} \right)^{\frac{1}{\theta}}$$

or

$$c_B^* = \left(\frac{p_A}{p_B} \right)^{\theta} c_A^*,$$

and then substitute this expression for c_B^* into the binding constraint to obtain the solution

$$c_A^* = c_F \left(\frac{p_A^{1-\theta} + p_B^{1-\theta}}{p_A^{1-\theta}} \right)^{\frac{\theta}{1-\theta}}$$

Substituting this solution for c_A^* back into the previous expression for c_B^* then yields the solution

$$c_B^* = c_F \left(\frac{p_A^{1-\theta} + p_B^{1-\theta}}{p_B^{1-\theta}} \right)^{\frac{\theta}{1-\theta}}.$$

Although this approach allows us to find c_A^* and c_B^* without having to solve for λ^*, for future reference it is helpful to note that, by substituting these solutions into either of the two first-order conditions, one can find

$$\lambda^* = \left(p_A^{1-\theta} + p_B^{1-\theta} \right)^{\frac{1}{1-\theta}}.$$

d. With the minimum expenditure function defined as

$$E(c_F, p_A, p_B) = \min_{c_A, c_B} \; p_A c_A + p_B c_B \text{ subject to } \left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \geq c_F,$$

it is possible to find the partial derivative of E with respect to c_F in either of two ways. The first is to use the solutions for c_A^* and c_B^* to find

$$E(c_F, p_A, p_B) = p_A c_A^* + p_B c_B^* = c_F \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}},$$

then differentiate to obtain

$$\frac{\partial E(c_F, p_A, p_B)}{\partial c_F} = \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

The second is to use the envelope theorem and the solution for λ^* to obtain

$$\frac{\partial E(c_F, p_A, p_B)}{\partial c_F} = \lambda^* = \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

Since this partial derivative can be interpreted as the marginal cost of producing the quantity aggregate, it suggests defining the price aggregate for food as

$$p_F = \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

3. More Detailed Utility Maximization

The consumer solves

$$\max_{c_A, c_B, c_O} \left(\frac{\alpha\theta}{\theta-1}\right) \ln\left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right) + (1-\alpha)\ln(c_O) \text{ subject to } Y \geq p_A c_A + p_B c_B + p_O c_O.$$

a. Letting λ denote the nonnegative multiplier on the constraint, the Lagrangian for this problem can be defined as

$$L(c_A, c_B, c_O, \lambda) = \left(\frac{\alpha\theta}{\theta-1}\right) \ln\left(c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}\right) + (1-\alpha)\ln(c_O) + \lambda(Y - p_A c_A - p_B c_B - p_O c_O).$$

b. With the Lagrangian defined as above, the Kuhn-Tucker theorem implies that the values c_A^*, c_B^*, and c_O^* that solve this problem, together with the associated value λ^* of the Lagrange multiplier, must satisfy the first-order conditions

$$\frac{\alpha c_A^{*-\frac{1}{\theta}}}{c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}} - \lambda^* p_A = 0,$$

$$\frac{\alpha c_B^{*-\frac{1}{\theta}}}{c_A^{\frac{\theta-1}{\theta}} + c_B^{\frac{\theta-1}{\theta}}} - \lambda^* p_B = 0,$$

and

$$\frac{1-\alpha}{c_O^*} - \lambda^* p_O = 0.$$

c. Together with the binding constraint

$$Y = p_0 c_O^* + p_A c_A^* + p_B c_B^*,$$

the first-order conditions for a system of four equations in the four unknowns c_A^*, c_B^*, c_O^*, and λ^*. As a first step in solving this system, multiply the first-order condition for c_A by c_A^*, the first-order condition for c_B by c_B^*, and the first-order condition for c_O by c_O^*. Then, after dividing each first-order condition by λ^*, substituting them all into the binding budget constraint provides the solutions

$$\lambda^* = \frac{1}{Y}$$

and hence

$$c_0^* = \frac{(1-\alpha)Y}{p_O},$$

just as in question 1.

d. Next, divide the first-order condition for c_A by the first-order condition for c_B to obtain

$$\left(\frac{c_B^*}{c_A^*}\right)^{\frac{1}{\theta}} = \frac{p_A}{p_B}.$$

or

$$c_B^* = \left(\frac{p_A}{p_B}\right)^{\theta} c_A^*,$$

and substitute this expression into

$$c_F^* = \left(c_A^{*\frac{\theta-1}{\theta}} + c_B^{*\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

to obtain

$$c_A^* = c_F^* \left(\frac{p_A^{1-\theta} + p_B^{1-\theta}}{p_A^{1-\theta}}\right)^{\frac{\theta}{1-\theta}}$$

Substituting this expression for c_A^* back into the previous expression for c_B^* then yields

$$c_B^* = c_F^* \left(\frac{p_A^{1-\theta} + p_B^{1-\theta}}{p_B^{1-\theta}}\right)^{\frac{\theta}{1-\theta}}.$$

Both of these solutions coincide with those from question 2, with c_F^* in place of c_F.

e. It only remains to find c_F^*. This can be done by substituting the expressions for λ^*, c_A^*, and c_B^* back into the first-order condition for c_A to obtain

$$c_F^* = \frac{\alpha Y}{\left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.},$$

which coincides with the solution to question 1 if we define the price aggregate for food as

$$p_F = \left(p_A^{1-\theta} + p_B^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2021

Due Tuesday, December 21 at 12noon

This exam has three questions on seven pages; before you begin, please check to make sure that your copy has all three questions and all seven pages. Each part of each question is worth ten points. Therefore, questions 1 with one part is worth 10 points, question 2 with four parts is worth 40 points, and question 3 with five parts is worth 50 points, for a total of 100 points overall.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, the papers referenced, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. The Tobin Effect

How does inflation affect the process of economic growth? James Tobin proposed one answer to this question in his article, "Money and Economic Growth" (*Econometrica* 33 (1965): 671-684). In Tobin's framework, consumers hold wealth in the form of physical capital and money. When inflation rises, they substitute out of money and into capital. Hence, via this "Tobin effect," output rises as well.

One would not want to take this argument too far. No economist or central banker would argue that hyperinflation is optimal monetary policy because it causes consumers to hold almost all of their wealth in the form of physical capital and almost none as money. But, maybe the Tobin effect operates at modest rates of inflation, explaining why periods when inflation runs low – such as the Great Depression of the 1930s or the more recent period following the financial crisis and Great Recession of 2008-9 – are also periods when output is disappointingly low as well.

Tobin's argument can be formalized using a monetary version of the Solow ("A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics* 70 (1956): 65-94) growth model. Consider an economy in which, at each time $t \in [0, \infty)$, output $Y(t)$ is produced with $K(t)$ units of capital and $N(t)$ workers according to the Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} N(t)^{1-\alpha},$$

where $0 < \alpha < 1$. Suppose, also, that the supply of money – "dollars" – is $D(t)$ at time $t \in [0, \infty)$. Suppose that the government increases the money supply over time by making transfers $X(t) = \mu D(t)$ to the public at time t. Then the nominal money supply grows at the constant rate μ over time.

Let $P(t)$ denote the aggregate nominal price level at time t, so that

$$M(t) = D(t)/P(t)$$

denotes real money balances held by the public at the same date. If δ denotes the rate of physical capital depreciation and $\pi(t) = \dot{P}(t)/P(t)$ denotes the inflation rate (here, as always, $\dot{P}(t)$ denotes the derivative of $P(t)$ with respect to time t), then the public's disposable income at time t is

$$Y^d(t) = K(t)^\alpha N(t)^{1-\alpha} + \frac{X(t)}{P(t)} - \delta K(t) - \pi(t)M(t).$$

The first term on the right-hand side of this expression is real output and the second is the real value of the government transfer. The third term subtracts off the value of depreciated capital, and the fourth corrects for the erosion due to inflation of the real value of money held by the public. Remembering that $X(t) = \mu D(t)$ and $M(t) = D(t)/P(t)$, this last expression can be written more simply as

$$Y^d(t) = K(t)^\alpha N(t)^{1-\alpha} - \delta K(t) + [\mu - \pi(t)]M(t).$$

Now, following Solow (1956), assume that the savings rate out of disposable income is constant and equal to s between zero and one: $0 < s < 1$. Extending Solow's model as suggested by Tobin (1965), however, total saving is used here to increase both the stock of physical capital and real money balances. Therefore,

$$\dot{K}(t) + \dot{M}(t) = sY^d(t) = s\{K(t)^\alpha N(t)^{1-\alpha} - \delta K(t) + [\mu - \pi(t)]M(t)\}, \tag{1}$$

where, again as always, $\dot{K}(t)$ and $\dot{M}(t)$ denote the derivatives of $K(t)$ and $M(t)$ with respect to time t.

It is helpful to rewrite (1) in per-capita terms. Let $k(t) = K(t)/N(t)$ and $m(t) = M(t)/N(t)$ denote capital and real money balances per worker. And assume, again following Solow (1956), that the population/labor force expands at the constant rate $n > 0$. Then

$$\dot{k}(t) + \dot{m}(t) = s\{k(t)^\alpha - \delta k(t) + [\mu - \pi(t)]m(t)\} - nk(t) - nm(t). \tag{2}$$

In a steady state where the stocks of physical capital and real balances per capita are constant, with $\dot{k}(t) = \dot{m}(t) = 0$, $k(t) = k$, and $m(t) = m$. The steady-state condition $\dot{m}(t) = 0$ also determines the constant inflation rate $\pi(t) = \pi$ as

$$\pi = \mu - n.$$

Hence, in steady state, (2) requires

$$0 = s(k^\alpha - \delta k) - nk - (1 - s)nm. \tag{3}$$

To complete our analysis, we need a second steady-state condition, describing how the public divides its financial wealth between real money and physical capital. A simple specification that captures the essence of Tobin's (1965) argument assumes that the steady-state ratio m/k is inversely related to the inflation rate π, so that

$$m = \phi(\pi)k, \tag{4}$$

where the function ϕ satisfies $\phi(\pi) > 0$ and $\phi'(\pi) < 0$ for all steady-state inflation rates π.

Combining (3) and (4) yields

$$0 = s(k^{\alpha} - \delta k) - nk - (1-s)n\phi(\pi)k. \tag{5}$$

a. Use the total differential of (5) to compute $dk/d\pi$. Assuming that the stability condition

$$s(\alpha k^{\alpha-1} - \delta) - n < 0$$

from the original Solow (1956) model holds here as well, you should be able to use your result to show that

$$\frac{dk}{d\pi} > 0$$

as conjectured by Tobin (1965).

2. An Optimizing Model of Saving and Asset Allocation

The model from question 1 might be criticized for simply assuming that the savings rate is constant and that the steady-state real money-to-capital ratio is a simple function of the inflation rate without explicitly deriving those behavioral patterns from a utility-maximization problem. The constant savings rate assumption, in particular, might be playing an important role in shaping the results: if the savings rate falls when inflation rises, for example, an increase in inflation could decrease physical capital as well as real money balances, offsetting or even reversing the Tobin effect.

What happens when we extend our analysis from question 1 by formulating and solving the dynamic optimization problem faced by a household that saves to accumulate both physical capital and real money balances over time? Miguel Sidrauski answered this question in his paper "Rational Choice and Patterns of Growth in a Monetary Economy" (*American Economic Review* 57 (1967): 534-544). Here, we'll use his framework to see for ourselves.

Consider an infinitely-lived representative household consisting of $N(t)$ members, with preferences described by

$$\int_0^{\infty} e^{-\rho t} N(t) U\left(\frac{C(t)}{N(t)}, \frac{D(t)}{P(t)N(t)}\right) \mathrm{d}t,$$

where $\rho > 0$ is the discount rate, $C(t)$ is total consumption, and as before, $D(t)$ are the total number of dollars held and $P(t)$ is the aggregate nominal price level. Hence, according to this specification, the household enjoys consumption of goods and also benefits from holding real money balances, perhaps because doing so saves time in transacting with other households.

Assume, again, that the population grows at the constant rate $n > 0$, and for simplicity, normalize the population at $t = 0$ to equal one: $N(0) = 1$. Let $c(t) = C(t)/N(t)$ denote per-capita consumption, and let $M(t) = D(t)/P(t)$ and $m(t) = M(t)/N(t)$ denote total and per-capital real money balances. Then the utility function can be written more simply as

$$\int_0^\infty e^{-(\rho-n)t} U(c(t), m(t)) \mathrm{d}t. \tag{6}$$

During each period t, the household divides its real wealth $A(t)$ up into an amount $K(t)$ to be held as physical capital and an amount $M(t)$ to be held as real money balances. With $a(t) = A(t)/N(t)$ and $k(t) = K(t)/N(t)$, this asset-allocation constraint can be written in per-capita terms as

$$a(t) \geq k(t) + m(t) \tag{7}$$

for all $t \in [0, \infty)$. In (7), the constraint appears as an inequality to allow for the possibility of free disposal. If the single-period utility function U is increasing in both its arguments, however, (7) will always bind when the household is optimizing.

As before, as sources of funds in each period t, the household has output, produced according to a Cobb-Douglas production function with capital's share $0 < \alpha < 1$, and a government transfer of newly-printed dollars $X(t)$ having real value $X(t)/P(t)$. Also as before, the household's capital stock depreciates at the constant rate δ and the real value of its money holdings is eroded by $\pi(t)M(t)$ because of inflation $\pi(t)$. And as an additional use of funds, not explicitly considered before, the household has its consumption $C(t)$. The household, therefore, accumulates real assets subject to the constraint

$$K(t)^\alpha N(t)^{1-\alpha} + X(t)/P(t) - \delta K(t) - \pi(t)M(t) - C(t) \geq \dot{A}(t),$$

or, in per-capita terms,

$$k(t)^\alpha + x(t) - \delta k(t) - \pi(t)m(t) - c(t) - na(t) \geq \dot{a}(t), \tag{8}$$

for all $t \in [0, \infty)$, where $x(t) = X(t)/P(t)$ denotes per-capita real money transfers received during period t.

The household's optimization problem can now be stated as: taking initial wealth $a(0)$ as given, choose $c(t)$, $k(t)$, and $m(t)$ for all $t \in [0, \infty)$ and $a(t)$ for all $t \in (0, \infty)$ to maximize utility in (6) subject to the constraints (7) and (8) for all $t \in [0, \infty)$. Note that in this problem, $a(t)$ appears as a stock variable and $c(t)$, $k(t)$, and $m(t)$ appear as flows. If treating capital and money as flow variables seems counterintuitive, think instead of $k(t)$ and $m(t)$ as measuring service flows from physical capital and real money balances received by the household using its stock of financial assets $a(t)$ according to the constraint in (7).

a. As a first step in characterizing the household's optimizing behavior using the maximum principle, write down an expression for the maximized Hamiltonian for its problem. *Note:* For this problem, it may be easiest to work with the Hamiltonian in current value form. But of course you'll reach the same conclusions starting from the Hamiltonian in present value form. So use whichever formulation you prefer; just make sure the optimality conditions you list in part (b), below, are consistent with your chosen definition of the Hamiltonian.

b. Next, write down the first-order conditions for $c(t)$, $k(t)$, and $m(t)$ and the pair of differential equations for $a(t)$ and the associated multiplier from the Hamiltonian that, according to the maximum principle, help characterize the solution to the household's problem.

c. In any steady-state, all per-capita variables will be constant, with $c(t) = c$, $k(t) = k$, $m(t) = m$, and $a(t) = a$ for all t. And if, as in question 1, the government sets the transfer $x(t) = \mu m(t)$ to provide for constant money growth at rate μ, the steady-state condition $\dot{m}(t) = 0$ once again determines the constant inflation rate $\pi(t) = \pi$ as

$$\pi = \mu - n.$$

Use these steady-state conditions, together with your optimality conditions from part (b), to obtain a solution for the steady-state capital stock k in terms of the model's parameters: ρ, n, α, δ, and π. *Note:* You should be able to solve for k using the steady-state and optimality conditions without necessarily having to find the steady-state values of other variables: this was one of the surprising results derived by Sidrauski (1965). In addition, the value of k will not depend on all of the parameters, just a subset of them.

d. Finally, use the solution you just derived for k to answer the question posed at outset: does the Tobin effect of inflation on the capital stock still appear when savings and asset allocation decisions are derived from optimizing behavior as modeled here and by Sidrauski (1967)?

3. Precautionary Saving

This question asks you to work through a version of a model developed by Ricardo Caballero in his article "Consumption Puzzles and Precautionary Savings" (*Journal of Monetary Economics* 25 (1990): 113-136), in which an optimizing consumer saves to insure against adverse shocks to income. With a set of restrictions imposed on Caballero's more general framework, it is possible to solve explicitly for the value function that enters into the dynamic programming formulation of the consumer's problem via the guess-and-verify method.

The consumer in this problem seeks to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = E_0 \sum_{t=0}^{\infty} \beta^t [-\frac{1}{\theta} \exp(-\theta c_t)]. \tag{9}$$

where the discount factor lies between zero and one, $0 < \beta < 1$, and the positive parameter $\theta = -u''(c_t)/u'(c_t) > 0$ measures the consumer's constant coefficient of absolute risk aversion.

The consumer's bank account balance at the beginning of each period t is A_t. During the period, the consumer decides how much to withdraw from the bank in order to finance his or her consumption c_t. Unspent balances earn interest at the constant, gross rate $R > 1$, which satisfies $\beta R = 1$.

At the very start of the following period $t+1$, the consumer learns how much income Y_{t+1} he or she will earn during the following period $t+1$. Thus, the consumer must decide on spending c_t and saving $A_t - c_t$ before knowing Y_{t+1}. Cabellero allows Y_{t+1} to follow a more general stochastic process; here, instead, we will assume that Y_{t+1} is independently and identically distributed and, in particular, normally distributed with mean $\bar{Y}$ and variance σ^2:

$$Y_{t+1} \sim N(\bar{Y}, \sigma^2)$$

for all $t = 0, 1, 2, \ldots$. Our goal is to find out how the consumer's spending and saving decisions depend on the coefficient of absolute risk aversion θ and the variance σ^2 of income.

The consumer's problem can be stated as: given A_0, choose contingency plans for c_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$ in order to maximize expected utility in (9) subject to the constraints

$$R(A_t - c_t) + Y_{t+1} \geq A_{t+1} \tag{10}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of Y_{t+1}.

a. As a first step in characterizing the consumer's optimal spending and savings decisions via dynamic programming, write down the Bellman equation for this problem. *Note:* In doing this, it is helpful to observe that since income Y_{t+1} is iid and since c_t must be chosen before the value of Y_{t+1} is known, the stock variable A_t completely summarizes the state of the world as it appears to the consumer at the beginning of period t. Thus, in the dynamic programming formulation, c_t can serve as the flow or control variable, A_t as the stock or state variable, and the value function $v(A_t)$ for period t will depend only on A_t.

b. Now guess that the value function takes the specific form

$$v(A_t) = -\frac{1}{\theta F} \exp[-\theta(FA_t + G)],$$

where F and G are constant coefficients to be determined. Rewrite the Bellman equation using this guess, and use the Bellman equation to derive the the first-order condition for c_t and the envelope condition for A_t that help characterize the solution to the consumer's problem. In doing this, it is helpful to note that since the constraint (10) will always bind at the optimum, the guess for the value function also implies that

$$\begin{aligned}
v(A_{t+1}) &= -\frac{1}{\theta F} \exp[-\theta(FA_{t+1} + G)] \\
&= -\frac{1}{\theta F} \exp[-\theta FR(A_t - c_t) - \theta FY_{t+1} - \theta G] \\
&= -\frac{1}{\theta F} \exp(-\theta G) \exp[-\theta FR(A_t - c_t)] \exp(-\theta FY_{t+1})
\end{aligned}$$

Moreover, because A_t and c_t are both known at time t,

$$E_t v(A_{t+1}) = -\frac{1}{\theta F}\exp(-\theta G)\exp[-\theta F R(A_t - c_t)]E_t[\exp(-\theta F Y_{t+1})].$$

Finally, since $Y_{t+1} \sim N(\bar{Y}, \sigma^2)$,

$$-\theta F Y_{t+1} \sim N(-\theta F \bar{Y}, \theta^2 F^2 \sigma^2)$$

and therefore $\exp(-\theta F Y_{t+1})$ is log-normally distributed with

$$E_t[\exp(-\theta F Y_{t+1})] = \exp\left(-\theta F \bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right).$$

It follows that the second term on the right-hand side of the Bellman equation can be rewritten as

$$\beta E_t v(A_{t+1}) = -\frac{\beta}{\theta F}\exp(-\theta G)\exp\left(-\theta F \bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp[-\theta F R(A_t - c_t)]$$

before deriving the first-order and envelope conditions.

c. Next, combine the first-order and envelope conditions to show that the optimal choice for c_t must be given by

$$c_t = F A_t + G.$$

d. It still remains to determine how the unknown coefficients F and G depend on the parameters of the consumer's optimization problem. To do this, substitute $c_t = F A_t + G$ back into the envelope condition. Then, find a solution for F so that the terms involving A_t on the left-hand side are equal to the terms involving A_t on the right-hand side. This solution should indicate that F depends only on R, the gross interest rate on savings. After finding the solution for F, find the solution for G so that the remaining terms on the left-hand side of the envelope condition match the remaining terms on the right-hand side of the envelope condition. This solution should indicate that G depends on the parameters $\bar{Y}$, R, θ, and σ^2.

e. Finally, to complete our analysis, substitute your solutions for F and G back into the expression $c_t = F A_t + G$ to see how consumption c_t and saving $A_t - c_t$ depend on the coefficient of absolute risk aversion θ and the variance of income σ^2. These expressions should indicate that the consumer saves more to guard against adverse income shocks when he or she is more risk averse (that is, when θ is larger) or when the volatility of income goes up (that is, when σ^2 is larger).

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2021

Due Tuesday, December 21 at 12noon

1. The Tobin Effect

The condition linking the steady-state value k of physical capital to the inflation rate π is

$$0 = s(k^\alpha - \delta k) - nk - (1-s)n\phi(\pi)k, \tag{5}$$

where $\phi(\pi) > 0$ and $\phi'(\pi) < 0$ for all π.

a. The total differential of (5) is

$$0 = [s(\alpha k^{\alpha-1} - \delta) - n - (1-s)n\phi(\pi)]dk - (1-s)n\phi'(\pi)kd\pi,$$

which can be rearranged to obtain the expression

$$\frac{dk}{d\pi} = \frac{(1-s)n\phi'(\pi)k}{s(\alpha k^{\alpha-1} - \delta) - n - (1-s)n\phi(\pi)}.$$

Since $\phi'(\pi) < 0$, the numerator is negative. And since $\phi(\pi) > 0$, the denominator is also negative if the stability condition from the original Solow (1956) model is assumed to hold. Therefore, the derivative is positive, confirming Tobin's conjecture.

2. An Optimizing Model of Saving and Asset Allocation

The household takes initial wealth $a(0)$ as given and chooses $c(t)$, $k(t)$, and $m(t)$ for all $t \in [0, \infty)$ and $a(t)$ for all $t \in (0, \infty)$ to maximize utility

$$\int_0^\infty e^{-(\rho-n)t} U(c(t), m(t))\mathrm{d}t. \tag{6}$$

subject to the constraints

$$a(t) \geq k(t) + m(t) \tag{7}$$

and

$$k(t)^\alpha + x(t) - \delta k(t) - \pi(t)m(t) - c(t) - na(t) \geq \dot{a}(t) \tag{8}$$

for all $t \in [0, \infty)$.

a. In current value form, the maximized Hamiltonian for the household's problem is

$$\begin{aligned} H(a(t),\theta(t);t) = \max_{c(t),k(t),m(t)} & \; U(c(t),m(t)) \\ & + \theta(t)[k(t)^\alpha + x(t) - \delta k(t) - \pi(t)m(t) - c(t) - na(t)] \\ & \text{subject to } a(t) \geq k(t) + m(t). \end{aligned}$$

Note that the maximized Hamiltonian depends on t as well as $a(t)$ and $\theta(t)$ because inflation $\pi(t)$ may in general depend on t.

b. According to the maximum principle, the solution to the household's problem is characterized by the first-order condition for $c(t)$,

$$U_1(c(t),m(t)) - \theta(t) = 0,$$

the first-order condition for $k(t)$,

$$\theta(t)[\alpha k(t)^{\alpha-1} - \delta] - \lambda(t) = 0,$$

and the first-order condition for $m(t)$,

$$U_2(c(t),m(t)) - \theta(t)\pi(t) - \lambda(t) = 0,$$

along with the pair of differential equations

$$\dot{\theta}(t) = (\rho - n)\theta(t) + \theta(t)n - \lambda(t) = \rho\theta(t) - \lambda(t).$$

and

$$\dot{a}(t) = k(t)^\alpha + x(t) - \delta k(t) - \pi(t)m(t) - c(t) - na(t).$$

In these optimality conditions, U_1 and U_2 denote the partial derivatives of the utility function U with respect to consumption and real money balances per capita and $\lambda(t)$ is the Lagrange multiplier on the constraint from the static problem defining the maximized Hamiltonian.

c. In any steady-state, all per-capita variables will be constant, including $c(t) = c$, $k(t) = k$, and $m(t) = m$. Moreover, the inflation rate $\pi(t) = \pi$ is constant as well. Thus, the first-order condition for $c(t)$ implies that $\theta(t) = \theta$ is constant, and then the first-order condition for $m(t)$ implies that $\lambda(t) = \lambda$ is also constant. Since $\dot{\theta}(t) = 0$, the first differential equation requires that

$$\lambda = \rho\theta.$$

Substituting this result into the first-order condition for $k(t)$ shows that

$$\alpha k^{\alpha-1} - \delta - \rho = 0.$$

Interestingly, this expression for the steady-state capital is the same as the one from the non-monetary Ramsey model. It implies that the steady-state capital stock is

$$k = \left(\frac{\delta+\rho}{\alpha}\right)^{1/(\alpha-1)}.$$

d. Since the expression of k just derived does not depend on the inflation rate π, in this model the Tobin effect disappears. Instead, money is "superneutral" in the sense that changes in the inflation have no effect on steady-state capital or output.

It should be noted that while inflation has no effect on the capital stock in the steady state of Sidrauski's (1967) model, it will affect capital accumulation along the transition path from an arbitrary value of $a(0)$ to the steady state. This was shown by Stanley Fischer in his article "Capital Accumulation on the Transition Path in a Monetary Optimizing Model" (*Econometrica* 47 (1979): 1433-1439). It should be noted, as well, that further modifications of or extensions to Sidrauski's (1967) model can re-introduce effects of steady-state inflation on the capital stock, with some model variants implying that inflation increases capital and others implying that inflation decreases capital. For a partial survey, see Athanasios Orphanides and Robert M. Solow's "Money, Inflation and Growth" (Chapter 6 in Volume 1 of the *Handbook of Monetary Economics*, edited by B.M. Friedman and F.H. Hahn, Elsevier, 1990).

3. Precautionary Saving

The consumer takes A_0 as given, and chooses contingency plans for c_t, $t = 0, 1, 2, \ldots$, and A_t, $t = 1, 2, 3, \ldots$ in order to maximize expected utility

$$E_0 \sum_{t=0}^{\infty} \beta^t [-\frac{1}{\theta} \exp(-\theta c_t)] \tag{9}$$

subject to the constraints

$$R(A_t - c_t) + Y_{t+1} \geq A_{t+1} \tag{10}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of $Y_{t+1} \sim N(\bar{Y}, \sigma^2)$.

a. The Bellman equation for the consumer's problem is

$$v(A_t) = \max_{c_t} \; -\frac{1}{\theta} \exp(-\theta c_t) + \beta E_t v[R(A_t - c_t) + Y_{t+1}].$$

b. Using the conjecture that the value function takes the specific form

$$v(A_t) = -\frac{1}{\theta F} \exp[-\theta(F A_t + G)],$$

along with the algebraic manipulations shown previously, the Bellman equation specializes to

$$\begin{aligned} &-\frac{1}{\theta F} \exp[-\theta(F A_t + G)] \\ = \max_{c_t} \; &-\frac{1}{\theta} \exp(-\theta c_t) - \frac{\beta}{\theta F} \exp(-\theta G) \exp\left(-\theta F \bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right) \exp[-\theta F R(A_t - c_t)]. \end{aligned}$$

The first-order condition for c_t is then

$$\exp(-\theta c_t) - \exp(-\theta G)\exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp[-\theta F R(A_t - c_t)] = 0$$

and the envelope condition is

$$\exp[-\theta(FA_t + G)] = \exp(-\theta G)\exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp[-\theta F R(A_t - c_t)],$$

where, in both cases, use has been made of the assumption that $\beta R = 1$.

c. Use the envelope condition to replace the second term in the first-order condition and thereby obtain the much simpler expression

$$\exp(-\theta c_t) - \exp[-\theta(FA_t + G)] = 0,$$

which implies that

$$c_t = FA_t + G$$

and hence

$$A_t - c_t = (I - F)A_t - G.$$

d. Substitute the expression for saving $A_t - c_t$ just derived into the right-hand side of the envelope condition to obtain

$$\exp[-\theta(FA_t + G)] = \exp(-\theta G)\exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp\{-\theta F R[(I - F)A_t - G]\},$$

or

$$\exp(-\theta F A_t) = \exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp[-\theta F R(I - F)A_t]\exp(\theta F R G).$$

If the term on the left involving A_t is to equal the term on the right involving A_t, it must be that

$$1 = R(1 - F).$$

This expression leads to the solution

$$F = \frac{R - 1}{R}.$$

With this solution substituted in, the envelope condition simplifies to

$$1 = \exp\left(-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2}\right)\exp(\theta F R G),$$

which requires that

$$-\theta F\bar{Y} + \frac{\theta^2 F^2 \sigma^2}{2} + \theta F R G = 0.$$

This expression leads to the solution

$$G = \frac{\bar{Y}}{R} - \frac{\theta F \sigma^2}{2R} = \frac{\bar{Y}}{R} - \frac{\theta (R - 1)\sigma^2}{2R^2}.$$

e. Substituting the solutions for F and G into the expressions for consumption and saving derived previously yields

$$c_t = \left(\frac{R-1}{R}\right) A_t + \frac{\bar{Y}}{R} - \frac{\theta(R-1)\sigma^2}{2R^2}$$

and

$$A_t - c_t = \left(\frac{1}{R}\right) A_t - \frac{\bar{Y}}{R} + \frac{\theta(R-1)\sigma^2}{2R^2}.$$

From the last term in this last expression, we can confirm that the consumer saves more when he or she is more risk averse and when the volatility of income goes up.

Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2022

Due Tuesday, November 1

This exam has three questions, each with four parts, on six pages. Before you begin, please check to make sure that your copy has all three questions and all six pages. Each question will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Properties of Hicksian Demands

Consider a consumer with preferences over two goods described by the utility function $U(c_1, c_2)$, where c_1 and c_2 are the amounts consumed of goods 1 and 2. Suppose this consumer purchases the goods in perfectly competitive markets at prices p_1 and p_2 in order to minimize the total expenditure needed to reach utility level $\bar{U}$, solving the constrained minimization problem:

$$\min_{c_1,c_2}\ p_1c_1 + p_2c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

a. Set up (define) the Lagrangian for this problem. Then, write down the first-order conditions for the consumer's optimal choices c_1^* and c_2^*. *Note:* As discussed in class, there are a number of ways to define the Lagrangian for any given constrained optimization problem. Use whichever form you prefer, but make sure your first-order conditions are consistent with your chosen definition of the Lagrangian.

b. Your first-order conditions from part (a) should indicate that so long as the goods prices p_1 and p_2 are strictly positive and the utility function U is strictly increasing in both its arguments, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices c_1^* and c_2^* and the corresponding value of the Lagrange multiplier. Assuming that the utility function is also such that the consumer's expenditure minimization problem has a unique solution, these three equations define the Hicksian demand functions $c_1^* = h_1(p_1, p_2, \bar{U})$ and $c_2^* = h_2(p_1, p_2, \bar{U})$. In particular, these Hicksian demands must satisfy the efficiency condition

$$\frac{p_1}{p_2} = \frac{U_1[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})]}{U_2[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})]}, \tag{1}$$

where U_i, $i = 1, 2$, denote the partial derivatives of the utility function with respect to each of its two arguments, and the binding constraint

$$U[h_1(p_1, p_2, \bar{U}), h_2(p_1, p_2, \bar{U})] = \bar{U} \tag{2}$$

for all values of $(p_1, p_2, \bar{U})$. Differentiate (2) with respect to p_1, then use the resulting expression together with (1) to obtain a restriction, implied by the consumer's optimizing behavior, that links the partial derivative

$$\frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_1}$$

to the prices p_1 and p_2 and the partial derivative

$$\frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_1}.$$

Similarly, differentiate (2) with respect to p_2 and use (1) to obtain a restriction that links

$$\frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_2}$$

to the prices p_1 and p_2 and

$$\frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_2}.$$

c. Now define the minimum expenditure function E as

$$E(p_1, p_2, \bar{U}) = \min_{c_1, c_2} \ p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}.$$

Use the envelope theorem, together with the symmetry condition

$$\frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_1 \partial p_2} = \frac{\partial^2 E(p_1, p_2, \bar{U})}{\partial p_2 \partial p_1}$$

to obtain yet another restriction implied by consumer optimality that links

$$\frac{\partial h_1(p_1, p_2, \bar{U})}{\partial p_2}.$$

to

$$\frac{\partial h_2(p_1, p_2, \bar{U})}{\partial p_1}.$$

d. Finally, consider the matrix

$$\begin{bmatrix} \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1} & \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} \\ \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} & \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Use the restrictions you derived in parts (b) and (c) to show how each of the values b, c, and d in this matrix depends on the value of a and the prices p_1 and p_2. These restrictions imply that, in this two-good case, knowing one partial derivative of one of the Hicksian demand curves allows you to infer both of the partial derivatives of both of the Hicksian demand curves.

2. Cost Minimization and the Production Function

Consider a firm that produces output y with capital k and labor l according to the production function

$$y = f(k, l).$$

Suppose this firm rents capital at the rental rate r and hires labor at the wage rate w in perfectly competitive markets in order to minimize the costs of producing at least $\bar{y}$ units of output, solving the constrained minimization problem

$$\min_{k,l} \; rk + wl \text{ subject to } f(k, l) \geq \bar{y}.$$

a. Set up (define) the Lagrangian for this problem. Then, write down the first-order conditions for the firm's optimal choices k^* and l^*. *Note:* As discussed in class, there are a number of ways to define the Lagrangian for any given constrained optimization problem. Use whichever form you prefer, but make sure your first-order conditions are consistent with your chosen definition of the Lagrangian.

b. Your first-order conditions from part (a) should indicate that so long as the factor prices r and w are strictly positive and the production function is strictly increasing in both of its arguments, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices k^* and l^* and the corresponding value of the Lagrange multiplier. Assuming that the production function is also such that the firm's cost minimization problem has a unique solution, these three equations define the conditional factor demand curves $k^* = k(r, w, \bar{y})$ and $l^* = l(w, r, \bar{y})$. In particular, these conditional factor demands must satisfy the efficiency condition

$$\frac{r}{w} = \frac{f_1[k(r, w, \bar{y}), l(r, w, \bar{y})]}{f_2[k(r, w, \bar{y}), l(r, w, \bar{y})]}, \tag{3}$$

where f_i, $i = 1, 2$, denote the partial derivatives of the production function with respect to each of its two arguments, and the binding constraint

$$f[k(r, w, \bar{y}), l(r, w, \bar{y})] = \bar{y} \tag{4}$$

for all values of $(r, w, \bar{y})$. Note that the factor prices r and w enter directly into these optimality conditions only through their ratio r/w. What does this fact tell you about the conditional factor demand curves: what, in particular, will happen to the optimal choices k^* and l^* when r and w both double, with the output requirement $\bar{y}$ left unchanged? Now define the minimum cost function by

$$C(r, w, \bar{y}) = \min_{k,l} \; rk + wl \text{ subject to } f(k, l) \geq \bar{y}.$$

What happens to the value of $C(r, w, \bar{y})$ when r and w both double with the output requirement $\bar{y}$ left unchanged?

c. Your answers to the questions in part (b) should imply that the minimum cost function is homogenous of degree one in the factor prices r and w. Suppose that, after recognizing this property of the cost function, we hypothesize further that the cost function takes the specific form

$$C(r, w, \bar{y}) = \bar{y}^b r^a w^{1-a}, \tag{5}$$

where the exponents on the factor prices sum to one. Can we determine what the assumed functional form (5) for the cost function implies for the form of the firm's production function $f(k, l)$? The answer is yes! To see how, start by applying the envelope theorem (in this case, Shephard's lemma) to the firm's cost minimization problem, to obtain two equations that link the conditional factor demands $k(r, w, \bar{y})$ and $l(r, w, \bar{y})$ to the partial derivatives of the cost function.

d. Your answers from part (b), together with the assumed functional form from (5), should indicate that

$$k(r, w, \bar{y}) = \frac{\partial C(r, w, \bar{y})}{\partial r} = a\bar{y}^b r^{a-1} w^{1-a}$$

and

$$l(r, w, \bar{y}) = \frac{\partial C(r, w, \bar{y})}{\partial w} = (1-a)\bar{y}^b r^a w^{-a}.$$

Rearranging and then combining these two results shows that when $k = k(r, w, \bar{y})$ and $l = l(r, w, \bar{y})$ are chosen by the cost-minimizing firm,

$$\left(\frac{a\bar{y}^b}{k}\right)^{1/(1-a)} = \frac{r}{w} = \left[\frac{l}{(1-a)\bar{y}^b}\right]^{1/a}$$

or, perhaps more simply,

$$\left(\frac{a\bar{y}^b}{k}\right)^a = \left[\frac{l}{(1-a)\bar{y}^b}\right]^{1-a}.$$

To complete the derivation, use this last equality to solve for $\bar{y}$ in terms of k, l, a, and b. This solution should reveal the form of the production function $f(k, l)$!

3. Discretion versus Commitment in Optimal Monetary Policymaking

At the Federal Reserve in the United States and at other central banks around the world, monetary policymakers prefer to maintain a large degree of discretion, making their policy decisions meeting-by-meeting instead of committing to a pre-announced plan ahead of time. But does this approach help or hurt, when it comes to achieving their stabilization goals for unemployment and inflation? Finn E. Kydland and Edward C. Prescott ("Rules Rather than Discretion: The Inconsistency of Optimal Plans," *Journal of Political Economy*, June 1977) and Robert J. Barro and David B. Gordon ("A Positive Theory of Monetary Policy in a Natural Rate Model," *Journal of Political Economy*, August 1983) present examples to show that better outcomes can be achieved under commitment.

This question asks you to work through a version of their model that highlights three key assumptions needed to make their argument work. First, a strong form of the "natural rate hypothesis" must hold: inflationary monetary policy can only lower the unemployment rate by surprising private agents. Second, private agents have "rational expectations:" they know the central bank's objectives and can therefore anticipate how monetary policy decisions will be made. Third, the central bank must wish to push unemployment below the "natural rate," that is, the rate prevailing when there are no monetary surprises.

Setting up the example requires two equations. The first is a Phillips curve

$$U = U^n - \alpha(\pi - \pi^e), \tag{6}$$

with $\alpha > 0$, that reflects the natural rate hypothesis: the central bank can push unemployment U below the natural rate $U^n > 0$ only by creating inflation π that exceeds the rate π^e expected by private agents. The second is an objective function for the central bank

$$-(1/2)(U - kU^n)^2 - (b/2)\pi^2, \tag{7}$$

with $0 < k < 1$ and $b > 0$, that penalizes deviations of unemployment U from a target kU^n that, in light of the assumption that $0 < k < 1$, lies below the natural rate U^n and deviations of inflation π from a target of zero. In (7), $b > 0$ is the weight that the central bank places on its objective for inflation relative to unemployment.

a. Suppose first that the central bank operates with "discretion," meaning that it chooses the rates of unemployment U and inflation π after private agents have formed their expectation π^e. In this case, taking U^n and π^e as given, the central bank maximizes the objective function (7) subject to the constraint in (6). You can treat this as constrained optimization problem and set up the Lagrangian before taking first-order conditions for U and π, or you can just substitute the constraint (6) into the objective function (7) and take the first-order condition for π. Either way, use your results to find an expression that shows how the central bank's choice of π depends on U^n, π^e, and the parameters α, b, and k.

b. Assume next that private agents have rational expectations and set their expected rate of inflation π^e equal to the value of π implied by the optimality condition you just derived in part (a). Note that this rational expectation assumption implies, through (6), that despite the central bank's efforts to use surprise inflation to lower the unemployment rate, U will always equal U^n in equilibrium. Use your equation from part (a) to show that, nevertheless, the discretionary central bank will choose a positive rate of inflation π so long as $U^n > 0$, $\alpha > 0$, $b > 0$, and $0 < k < 1$.

c. Now suppose instead that the central bank commits to a choice of the inflation rate π before private agents have fixed their expectation π^e. Suppose also that the central bank anticipates that, once it announces its choice of π, rational expectations in the private sector will imply that $\pi^e = \pi$. In light of (1), the central bank under commitment recognizes that it is futile to try creating surprise inflation and realizes that $U = U^n$ will prevail no matter what. In this case with commitment, therefore, the central bank solves the simpler problem

$$\max_{\pi} \; -(1/2)[(1-k)U^n]^2 - (b/2)\pi^2.$$

What is the central bank's optimal choice of π in this case?

d. Finally, use your results from parts (b) and (c) to confirm that, in this example, the central bank achieves a higher value of its objective function by choosing commitment instead of discretion.

Solutions to Midterm Exam

ECON 772001 - Math for Economists
Boston College, Department of Economics

Peter Ireland
Fall 2022

Due Tuesday, November 1

1. Properties of Hicksian Demands

The consumer solves

$$\min_{c_1,c_2} \; p_1c_1 + p_2c_2 \text{ subject to } U(c_1,c_2) \geq \bar{U}.$$

a. With the Lagrangian defined as

$$L(c_1,c_2,\lambda) = p_1c_1 + p_2c_2 - \lambda[U(c_1,c_2) - \bar{U}],$$

the first-order conditions for the consumer's problem are

$$p_1 - \lambda^* U_1(c_1^*, c_2^*) = 0$$

and

$$p_2 - \lambda^* U_2(c_1^*, c_2^*) = 0$$

b. The first-order conditions from part (a) indicate that so long as the goods prices p_1 and p_2 are strictly positive and the utility function U is strictly increasing in both its arguments, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices c_1^* and c_2^* and the corresponding value of the Lagrange multiplier. Assuming that the utility function is also such that the consumer's expenditure minimization problem has a unique solution, these three equations define the Hicksian demand functions $c_1^* = h_1(p_1,p_2,\bar{U})$ and $c_2^* = h_2(p_1,p_2,\bar{U})$. In particular, these Hicksian demands must satisfy the efficiency condition

$$\frac{p_1}{p_2} = \frac{U_1[h_1(p_1,p_2,\bar{U}), h_2(p_1,p_2,\bar{U})]}{U_2[h_1(p_1,p_2,\bar{U}), h_2(p_1,p_2,\bar{U})]} \tag{1}$$

and the binding constraint

$$U[h_1(p_1,p_2,\bar{U}), h_2(p_1,p_2,\bar{U})] = \bar{U} \tag{2}$$

for all values of $(p_1,p_2,\bar{U})$. Differentiating (2) with respect to p_1 yields

$$U_1[h_1(p_1,p_2,\bar{U}), h_2(p_1,p_2,\bar{U})]\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1} + U_2[h_1(p_1,p_2,\bar{U}), h_2(p_1,p_2,\bar{U})]\frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} = 0.$$

Moving the first term from the left to the right, dividing through by U_2, and using (1) then shows that

$$\frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} = -\left(\frac{p_1}{p_2}\right)\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1}.$$

Similarly, differentiating (2) with respect to p_2 yields

$$\begin{aligned} &U_1[h_1(p_1,p_2,\bar{U}),h_2(p_1,p_2,\bar{U})]\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} \\ &+U_2[h_1(p_1,p_2,\bar{U}),h_2(p_1,p_2,\bar{U})]\frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} = 0. \end{aligned}$$

Moving the first term from the left to the right, dividing through by U_2, and using (1) then shows that

$$\frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} = -\left(\frac{p_1}{p_2}\right)\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2}.$$

c. Now define the minimum expenditure function E as

$$E(p_1,p_2,\bar{U}) = \min_{c_1,c_2}\ p_1c_1 + p_2c_2 \text{ subject to } U(c_1,c_2) \geq \bar{U}.$$

The envelope theorem implies that

$$\frac{\partial E(p_1,p_2,\bar{U})}{\partial p_1} = h_1(p_1,p_2,\bar{U}),$$

$$\frac{\partial E(p_1,p_2,\bar{U})}{\partial p_2} = h_2(p_1,p_2,\bar{U}),$$

and hence

$$\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} = \frac{\partial^2 E(p_1,p_2,\bar{U})}{\partial p_1 \partial p_2} = \frac{\partial^2 E(p_1,p_2,\bar{U})}{\partial p_2 \partial p_1} = \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1}.$$

d. Finally, consider the matrix

$$\begin{bmatrix} \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1} & \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} \\ \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} & \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Given a value for

$$a = \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1},$$

the results from (b) and (c) imply that

$$c = \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} = -\left(\frac{p_1}{p_2}\right)\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_1} = -a\left(\frac{p_1}{p_2}\right),$$

$$b = \frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} = \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_1} = -a\left(\frac{p_1}{p_2}\right),$$

and

$$d = \frac{\partial h_2(p_1,p_2,\bar{U})}{\partial p_2} = -\left(\frac{p_1}{p_2}\right)\frac{\partial h_1(p_1,p_2,\bar{U})}{\partial p_2} = a\left(\frac{p_1}{p_2}\right)^2.$$

2. Cost Minimization and the Production Function

The firm solves

$$\min_{k,l} \ rk + wl \text{ subject to } f(k,l) \geq \bar{y}.$$

a. With the Lagrangian defined as

$$L(k,l,\lambda) = rk + wl - \lambda[f(k,l) - \bar{y}],$$

the first-order conditions for the firm's problem are

$$r - \lambda^* f_1(k^*, l^*) = 0$$

and

$$w - \lambda^* f_2(k^*, l^*) = 0.$$

b. The first-order conditions from part (a) indicate that so long as the factor prices r and w are strictly positive and the production function is strictly increasing in both of its arguments, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices k^* and l^* and the corresponding value of the Lagrange multiplier. Assuming that the production function is also such that the firm's cost minimization problem as a unique solution, these three equations define, in particular, the conditional factor demand curves $k^* = k(r,w,\bar{y})$ and $l^* = l(w,r,\bar{y})$. In particular, these conditional factor demands must satisfy the efficiency condition

$$\frac{r}{w} = \frac{f_1[k(r,w,\bar{y}), l(r,w,\bar{y})]}{f_2[k(r,w,\bar{y}), l(r,w,\bar{y})]} \tag{3}$$

and the binding constraint

$$f[k(r,w,\bar{y}), l(r,w,\bar{y})] = \bar{y} \tag{4}$$

for all values of $(r,w,\bar{y})$. Note that the factor prices r and w enter directly into these optimality conditions only through their ratio r/w. This tells us that the conditional factor demand curves will be homogenous of degree zero in the factor prices: the optimal choices k^* and l^* will not change when r and w both double, with the output requirement $\bar{y}$ left unchanged. It also tells us that the minimum cost function, defined by

$$C(r,w,\bar{y}) = \min_{k,l} \ rk + wl \text{ subject to } f(k,l) \geq \bar{y},$$

will be homogeneous of degree one in the factor prices: the value of $C(r,w,\bar{y})$ will double when r and w both double with the output requirement $\bar{y}$ left unchanged.

c. Suppose that, after recognizing this property of the cost function, we hypothesize further that the cost function takes the specific form

$$C(r,w,\bar{y}) = \bar{y}^b r^a w^{1-a}, \tag{5}$$

where the exponents on the factor prices sum to one. Can we determine what the assumed functional form (5) for the cost function implies for the form of the firm's production function $f(k,l)$? The answer is yes! To see how, start by applying the envelope theorem (in this case, Shephard's lemma) to the firm's cost minimization problem, to obtain two equations that link the conditional factor demands $k(r,w,\bar{y})$ and $l(r,w,\bar{y})$ to the partial derivatives of the cost function. Now we know that

$$k(r,w,\bar{y}) = \frac{\partial C(r,w,\bar{y})}{\partial r} = a\bar{y}^b r^{a-1} w^{1-a}$$

and

$$l(r,w,\bar{y}) = \frac{\partial C(r,w,\bar{y})}{\partial w} = (1-a)\bar{y}^b r^a w^{-a}.$$

d. Rearranging and then combining these two results shows that when $k = k(r,w,\bar{y})$ and $l = l(r,w,\bar{y})$ are chosen by the cost-minimizing firm,

$$\left(\frac{a\bar{y}^b}{k}\right)^{1/(1-a)} = \frac{r}{w} = \left[\frac{l}{(1-a)\bar{y}^b}\right]^{1/a}$$

or, perhaps more simply,

$$\left(\frac{a\bar{y}^b}{k}\right)^{a} = \left[\frac{l}{(1-a)\bar{y}^b}\right]^{1-a},$$

or, even more simply,

$$\bar{y} = Ak^{a/b}l^{(1-a)/b},$$

where

$$A = \left[\frac{1}{a^a(1-a)^{1-a}}\right]^{1/b}.$$

Evidently, the firm's production function is

$$f(k,l) = Ak^{a/b}l^{(1-a)/b}.$$

3. Discretion versus Commitment in Optimal Monetary Policymaking

The central bank chooses U and π to maximize

$$-(1/2)(U - kU^n)^2 - (b/2)\pi^2, \tag{7}$$

subject to the constraint

$$U = U^n - \alpha(\pi - \pi^e). \tag{6}$$

Two cases below are distinguished by whether the central bank takes π^e as given when it makes its decisions under discretion or whether it recognizes that π^e will equal π under commitment.

a. Under discretion, the central bank solves

$$\max_{\pi} = -(1/2)[(1-k)U^n - \alpha(\pi - \pi^e)]^2 - (b/2)\pi^2.$$

The first-order condition

$$\alpha[(1-k)U^n - \alpha(\pi^* - \pi^e)] - b\pi^* = 0$$

implies that

$$\pi^* = \left(\frac{\alpha}{b+\alpha^2}\right)[(1-k)U^n + \alpha\pi^e].$$

b. In equilibrium, private agents with rational expectations will set $\pi^e = \pi^*$. Substituting this condition into the expression for the central bank's optimal choice of π yields

$$\pi^* = \left[\frac{\alpha(1-k)}{b}\right] U^n > 0.$$

Under discretion, the central bank chooses positive inflation in a futile attempt to exploit the Phillips curve.

c. With commitment, the central bank recognizes that it cannot fool the private sector and solves

$$\max_{\pi} = -(1/2)[(1-k)U^n]^2 - (b/2)\pi^2$$

by setting inflation equal to zero

$$\pi^* = 0.$$

d. Under both discretion and commitment, $U = U^n$ holds in equilibrium. Under discretion, however, the central bank ends up choosing a positive inflation rate, whereas with commitment it chooses inflation equal to its target of zero. Thus, in this example, the central bank achieves a higher value of its objective function by choosing commitment instead of discretion.

Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2022

Due Tuesday, December 20

This exam has two questions on four pages; before you begin, please check to make sure that your copy has both questions and all four pages. The two questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, my notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class. The answers you submit must be yours and yours alone.

1. Optimal Pollution Abatement

Consider a variant of the neoclassical growth model in which more production inevitably creates more pollution, but also where the representative consumer can allocate resources to pollution abatement as well as consumption and investment. The infinitely-lived consumer has preferences described by the utility function

$$\int_0^\infty e^{-\rho t}[\ln(c(t)) + \beta \ln(E(t))]dt,$$

where $c(t)$ and $E(t)$ are consumption and environmental quality at each time $t \in [0, \infty)$, $\rho > 0$ is the discount rate, and $\beta > 0$ is the weight that the consumer places on environmental quality relative to consumption.

At each time $t \in [0, \infty)$, the consumer uses the capital stock $k(t)$ to produce $k(t)^\alpha$ units of output, where $0 < \alpha < 1$. The capital stock depreciates in use at rate $\delta > 0$. In addition to consumption $c(t)$, the consumer allocates $a(t)$ units of output to pollution reduction (abatement), and invests in new capital. The capital stock evolves according to

$$k(t)^\alpha - \delta k(t) - c(t) - a(t) \geq \dot{k}(t)$$

for all $t \in [0, \infty)$.

As noted above, production in this economy inevitably leads to pollution, which degrades environmental quality. The consumer's spending on abatement, however, helps offset this by improving environmental quality. Specifically, environmental quality evolves according to

$$\gamma a(t) - k(t)^\alpha \geq \dot{E}(t)$$

for all $t \in [0, \infty)$, where the parameter $\gamma > 1$ measures the productivity of resources allocated to pollution abatement (the restriction that γ is greater than one is needed to ensure that the economy has a well-defined steady state).

The consumer's problem is therefore: given $k(0)$ and $E(0)$, choose $c(t)$ and $a(t)$ for all $t \in [0, \infty)$ and $k(t)$ and $E(t)$ for all $t \in (0, \infty)$ to maximize the utility function subject to the constraints determining $\dot{k}(t)$ and $\dot{E}(t)$.

a. Define (write down) the maximized Hamiltonian for the consumer's problem, using $k(t)$ and $E(t)$ as stock variables, $c(t)$ and $a(t)$ as flow variables, and remembering to introduce a separate multiplier for each of the two constraints, as illustrated more generally in section 6 of the notes on the maximum principle and more specifically in examples from question 1 on the 2017 final exam and question 1 on the 2019 final exam. For this problem, you can use whichever form of the Hamiltonian – present or current value – you find most convenient, although it might be easiest to use the current-value formulation. Either way, however, just make sure the optimality conditions you'll write down next, for part (b), are consistent with your choice.

b. Now write down the first-order conditions for $c(t)$ and $a(t)$ and the two pairs of differential equations for the multipliers and stock variables that, according to the maximum principle, characterize the solution to the consumer's problem.

c. Now consider a steady state, in which $c(t) = c$, $a(t) = a$, $k(t) = k$, and $E(t) = E$ are all constant. Use your optimality conditions from part (b) to find an expression that shows how the steady-state capital stock k depends on the parameters ρ, δ, α, γ, and β (*Note:* k may depend on some, but not all of these parameters).

d. How does the steady-state capital stock in this model of pollution and abatement compare to the steady-state capital stock in the original (without pollution and abatement) neoclassical (Ramsey) model with the same values for ρ, δ, and α? How does the steady-state capital stock in this model change as the parameter γ increases? What happens to the steady-state capital stock in this model as γ becomes arbitrarily large?

2. Investment Under Uncertainty

Consider a firm that produces output Y_t with capital K_t according to the linear technology

$$Y_t = A_t K_t,$$

where productivity A_t fluctuates randomly around a long-run mean A according to

$$A_{t+1} - A = \rho(A_t - A) + \varepsilon_{t+1},$$

with persistence governed by the parameter ρ, satisfying $0 < \rho < 1$, and where ε_{t+1} is serially uncorrelated with mean zero and constant variance σ^2.

During each period $t = 0, 1, 2, \ldots$, the firm chooses its optimal level of investment I_t after observing the realized value of ε_t and hence knowing the value of A_t. It earns profits

$$A_t K_t - I_t - \left(\frac{\phi}{2}\right) I_t^2,$$

during period t, where $\phi > 0$ governs the magnitude of adjustment costs for capital. It then carries

$$K_{t+1} = (1 - \delta)K_t + I_t$$

units of capital unto period $t+1$, where the depreciation rate δ satisfies $0 < \delta < 1$. The firm discounts future profits at the constant rate r.

The firm's problem is therefore: given K_0 and A_0, choose contingency plans for I_t, $t = 0, 1, 2, \ldots$, and K_{t+1}, $t = 1, 2, 3, \ldots$, to maximize

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right) \left[A_t K_t - I_t - \left(\frac{\phi}{2}\right) I_t^2\right]$$

subject to the constraints

$$(1 - \delta)K_t + I_t \geq K_{t+1}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of A_{t+1}.

Although there are a number of ways to solve this problem via dynamic programming, perhaps the easiest starts by writing the Bellman equation as

$$v(K_t, A_t) = \max_{I_t, K_{t+1}} A_t K_t - I_t - \left(\frac{\phi}{2}\right) I_t^2 + \left(\frac{1}{1+r}\right) E_t v(K_{t+1}, A_{t+1})$$
$$\text{subject to } (1 - \delta)K_t + I_t \geq K_{t+1}.$$

Note that in the static, constrained optimization problem on the right-hand side of this Bellman equation, I_t and K_{t+1} appear as choice variables while K_t appears as a parameter.

a. Using λ_t to denote the Lagrange multiplier on the constraint from the problem on the right-hand side of the Bellman equation, derive (write down) the first-order conditions for I_t and K_{t+1} and the envelope condition for K_t that characterize the solution to the firm's problem.

b. As the next step in characterizing the firm's optimal investment, combine the first-order condition for K_{t+1} with the envelope condition for K_t so as to eliminate reference in the optimality conditions to the unknown value functions $v(K_t, A_t)$ and $v(K_{t+1}, A_{t+1})$.

c. Your result from part (b) should take the form of a difference equation that links λ_t to the expected future values $E_t A_{t+1}$ and $E_t \lambda_{t+1}$. Use recursive forward substitution to obtain an expression that shows how λ_t depends on the expected future values of productivity $E_t A_{t+j}$ for all $j = 1, 2, 3, \ldots$ stretching out into the infinite future. To derive this result, you'll need to use the law of iterated expectations, which implies that

$$E_t(E_{t+j-1} A_{t+j}) = E_t A_{t+j}$$

and

$$E_t(E_{t+j-1} \lambda_{t+j}) = E_t \lambda_{t+j}$$

for all $j = 1, 2, 3, \ldots$. You can also assume that the transversality condition for the firm's infinite-horizon problem implies that

$$\lim_{j \to \infty} \left(\frac{1-\delta}{1+r} \right)^j E_t \lambda_{t+j} = 0$$

so that the infinite sum in your expression for λ_t converges.

d. Finally, notice that the autoregressive law of motion for productivity implies that

$$E_t A_{t+j} = A + \rho^j (A_t - A)$$

for all $j = 1, 2, 3, \ldots$. Use this forecasting equation, together with your solution for λ_t from part (c) and your first-order condition for I_t from part (a), to show how the firm's optimal investment I_t depends on the current level of productivity A_t, the average level of productivity A, and the parameters ρ, δ, r and ϕ.

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2022

Due Tuesday, December 20

1. Optimal Pollution Abatement

The consumer's problem is: given $k(0)$ and $E(0)$, choose $c(t)$ and $a(t)$ for all $t \in [0, \infty)$ and $k(t)$ and $E(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t}[\ln(c(t)) + \beta \ln(E(t))]dt$$

subject to the constraints

$$k(t)^\alpha - \delta k(t) - c(t) - a(t) \geq \dot{k}(t)$$

and

$$\gamma a(t) - k(t)^\alpha \geq \dot{E}(t)$$

for all $t \in [0, \infty)$.

a. The maximized current-value Hamiltonian for the consumer's problem is

$$\begin{aligned} H(k(t), E(t), \theta(t), \varphi(t)) = \max_{c(t), a(t)} \; & \ln(c(t)) + \beta \ln(E(t)) \\ & + \theta(t)[k(t)^\alpha - \delta k(t) - c(t) - a(t)] \\ & + \varphi(t)[\gamma a(t) - k(t)^\alpha] \end{aligned}$$

b. With the maximized Hamiltonian as defined above, the maximum principle implies that the solution to the consumer's problem is characterized by the first order conditions

$$\frac{1}{c(t)} - \theta(t) = 0$$

and

$$-\theta(t) + \gamma \varphi(t) = 0$$

for all $t \in [0, \infty)$, the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - \theta(t)[\alpha k(t)^{\alpha-1} - \delta] + \varphi(t)\alpha k(t)^{\alpha-1}$$

and

$$\dot{\varphi}(t) = \rho\varphi(t) - \frac{\beta}{E(t)}$$

for all $t \in [0, \infty)$ and the binding constraints

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t) - a(t)$$

and

$$\dot{E}(t) = \gamma a(t) - k(t)^\alpha$$

for all $t \in [0, \infty)$.

c. In a steady-state, $c(t) = c$, $a(t) = a$, $k(t) = k$, $E(t) = E$, $\theta(t) = \theta$, and $\varphi(t) = \varphi$ are all constant. Since the first-order condition for $a(t)$ implies

$$\varphi = (1/\gamma)\theta,$$

the differential equation for $\dot{\theta}(t)$ requires

$$0 = \rho\theta - \theta(\alpha k^{\alpha-1} - \delta) + (1/\gamma)\theta\alpha k^{\alpha-1},$$

which simplifies to

$$0 = \rho - \alpha k^{\alpha-1} + \delta + (1/\gamma)\alpha k^{\alpha-1}$$

and leads to the desired expression for the steady-state capital stock:

$$k = \left[\frac{\rho+\delta}{\alpha\left(1-\frac{1}{\gamma}\right)}\right]^{1/(\alpha-1)}.$$

Although this expression for k is the only one needed to answer the questions in part (d), we can for the sake of completeness we can use the remaining optimality conditions to find steady-state values for the other variables. In particular, the constraint for $\dot{E}(t)$ determines

$$a = (1/\gamma)k^\alpha,$$

which shows why the restriction $\gamma > 1$ is needed: to make spending on pollution abatement a fraction instead of a multiple of output. The constraint for $\dot{k}(t)$ then implies

$$c = \left(1 - \frac{1}{\gamma}\right)k^\alpha - \delta k.$$

The first-order condition for $c(t)$ determines

$$\theta = 1/c,$$

and the differential equation for $\dot{\varphi}(t)$ determines

$$E = \frac{\beta\gamma}{\rho\theta}.$$

Thus, once the value of k is found, the values for c, a, E, θ, and φ can all be computed as well.

d. Returning to the expression for steady-state capital,

$$k = \left[\frac{\rho + \delta}{\alpha \left(1 - \frac{1}{\gamma}\right)} \right]^{1/(\alpha - 1)},$$

the restriction $\gamma > 1$ implies that this value is smaller than corresponding value

$$k = \left(\frac{\rho + \delta}{\alpha} \right)^{1/(\alpha - 1)}$$

from the neoclassical model without pollution and abatement. Intuitively, the optimal steady-state capital stock falls when production is accompanied by pollution. The steady-state capital stock here increases when the productivity parameter γ increases and converges to the steady-state capital stock from model without pollution as γ becomes arbitrarily large. Again, this is intuitive: we can think of the original neoclassical model as one in which, even if pollution occurs, abatement is costless. For extensions of and elaborations on the relatively simple model of pollution abatement considered here, see Emmett Keeler, Michael Spence, and Richard Zeckhauser ("The Optimal Control of Pollution," *Jorurnal of Economic Theory* 1971).

2. Investment Under Uncertainty

The Bellman equation for the firm's problem is

$$v(K_t, A_t) = \max_{I_t, K_{t+1}} A_t K_t - I_t - \left(\frac{\phi}{2}\right) I_t^2 + \left(\frac{1}{1+r}\right) E_t v(K_{t+1}, A_{t+1})$$
$$\text{subject to } (1 - \delta)K_t + I_t \geq K_{t+1}.$$

a. Use λ_t to denote the Lagrange multiplier on the constraint from the static problem on the right-hand side of the Bellman equation. The first-order conditions for I_t and K_{t+1} are obtained by differentiating the Lagrangian for the the static problem on the right-hand side of the Bellman equation partially with respect to I_t

$$-1 - \phi I_t + \lambda_t = 0$$

and with respect to K_{t+1}.

$$\left(\frac{1}{1+r}\right) E_t v_1(K_{t+1}, A_{t+1}) - \lambda_t = 0.$$

The envelope condition for K_t is found by differentiating both sides of the Bellman equation with respect to K_t, once again after forming the Lagrangian for the the static problem on the right-hand side:

$$v_1(K_t, A_t) = A_t + \lambda_t(1 - \delta).$$

b. As the envelope condition must hold for all periods $t = 1, 2, 3, \ldots$, it implies that

$$v_1(K_{t+1}, A_{t+1}) = A_{t+1} + \lambda_{t+1}(1-\delta)$$

for all $t = 0, 1, 2, \ldots$. Substituting this expression into the first-order condition for K_{t+1} yields

$$\lambda_t = \left(\frac{1}{1+r}\right) E_t A_{t+1} + \left(\frac{1-\delta}{1+r}\right) E_t \lambda_{t+1}$$

for all $t = 0, 1, 2, \ldots$.

c. Note that this last result also implies

$$\lambda_{t+1} = \left(\frac{1}{1+r}\right) E_{t+1} A_{t+2} + \left(\frac{1-\delta}{1+r}\right) E_{t+1} \lambda_{t+2}$$

which can be substituted into the right-hand side of the original expression for λ_t to get

$$\lambda_t = \left(\frac{1}{1+r}\right) E_t A_{t+1} + \left(\frac{1}{1+r}\right)\left(\frac{1-\delta}{1+r}\right) E_t E_{t+1} A_{t+2} + \left(\frac{1-\delta}{1+r}\right)^2 E_t E_{t+1} \lambda_{t+2}$$

or, using the law of iterated expectations to simplify,

$$\lambda_t = \left(\frac{1}{1+r}\right) E_t A_{t+1} + \left(\frac{1}{1+r}\right)\left(\frac{1-\delta}{1+r}\right) E_t A_{t+2} + \left(\frac{1-\delta}{1+r}\right)^2 E_t \lambda_{t+2}$$

Since, likewise,

$$\lambda_{t+2} = \left(\frac{1}{1+r}\right) E_{t+2} A_{t+3} + \left(\frac{1-\delta}{1+r}\right) E_{t+2} \lambda_{t+3},$$

we can continue the forward recursive substitution to get

$$\begin{aligned}\lambda_t = &\left(\frac{1}{1+r}\right) E_t A_{t+1} + \left(\frac{1}{1+r}\right)\left(\frac{1-\delta}{1+r}\right) E_t A_{t+2} \\ &+ \left(\frac{1}{1+r}\right)\left(\frac{1-\delta}{1+r}\right)^2 E_t A_{t+2} + \left(\frac{1-\delta}{1+r}\right)^3 E_t \lambda_{t+3}\end{aligned}$$

Repeating this process and using the transversality condition

$$\lim_{j\to\infty} \left(\frac{1-\delta}{1+r}\right)^j E_t \lambda_{t+j} = 0$$

then implies

$$\lambda_t = \left(\frac{1}{1+r}\right) \sum_{j=1}^{\infty} \left(\frac{1-\delta}{1+r}\right)^{j-1} E_t A_{t+j}.$$

d. Since the autoregressive law of motion for productivity implies that

$$E_t A_{t+j} = A + \rho^j (A_t - A)$$

for all $j = 1, 2, 3, \ldots$, the solution for λ_t implies

$$\begin{aligned}
\lambda_t &= \left(\frac{1}{1+r}\right) \sum_{j=1}^{\infty} \left(\frac{1-\delta}{1+r}\right)^{j-1} A + \left(\frac{\rho}{1+r}\right) \sum_{j=1}^{\infty} \left[\frac{\rho(1-\delta)}{1+r}\right]^{j-1} (A_t - A) \\
&= \left(\frac{1}{r+\delta}\right) A + \left[\frac{\rho}{1+r-\rho(1-\delta)}\right] (A_t - A).
\end{aligned}$$

The first-order condition for I_t then implies that

$$I_t = \frac{1}{\phi} \left\{ \left(\frac{1}{r+\delta}\right) A + \left[\frac{\rho}{1+r-\rho(1-\delta)}\right] (A_t - A) - 1 \right\}$$

for all $t = 0, 1, 2, \ldots$. Assuming that $r > 0$ and $1 > \rho \geq 0$, this expression implies that optimal investment I_t depends positively on average productivity A. Assuming also that $\rho > 0$, the expression implies that investment I_t is larger during periods when realized productivity A_t is larger. When $\rho = 0$, however, I_t is constant and, in particular, independent of A_t. This is because investment is forward-looking: it depends on expected future productivity instead of today's productivity. And with $\rho = 0$, future productivity is unforecastable. In general, the value of A_t today matters only to the extent that is helps forecast future values of productivity A_{t+j}, $j = 1, 2, 3, \ldots$. That the solution for optimal investment depends importantly on the persistence parameter ρ from the law of motion for productivity provides an example of the "cross-equation restrictions" that Lars Hansen and Thomas Sargent ("Formulating and Estimating Dynamic Linear Rational Expectations Models," *Journal of Economic Dynamics and Control* 1980) called the "hallmark of rational expectations models."

Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2023

Due Thursday, November 2

This exam has five questions on seven pages. Before you begin, please check to make sure that your copy has all five questions and all seven pages. Each question will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Hicksian Demands

We've already seen that when a consumer's preferences over apples and bananas are described by the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b),$$

the consumer's Marshallian demand curves for apples and bananas are

$$c_a^* = c_a(p_a, p_b, Y) = \frac{\alpha Y}{p_a}$$

and

$$c_b^* = c_b(p_a, p_b, Y) = \frac{(1-\alpha)Y}{p_b},$$

where Y is income and p_a and p_b are the prices of apples and bananas. Substituting these Marshallian demand curves back into the utility function yields the indirect utility function

$$V(p_a, p_b, Y) = \alpha \ln(\alpha) + (1-\alpha)\ln(1-\alpha) - \alpha \ln(p_a) - (1-\alpha)\ln(p_b) + \ln(Y).$$

This problem asks you to derive, instead, the Hicksian demand curves from the expenditure minimization problem

$$\min_{c_a, c_b} \; p_a c_a + p_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}.$$

a. Set up (define) the Lagrangian for expenditure minimization problem. Then, write down the first-order conditions for the consumer's optimal choices c_a^* and c_b^*. *Note:* As discussed in class, there are a number of ways to define the Lagrangian for any given constrained optimization problem. Use whichever form you prefer, but make sure your first-order conditions are consistent with your chosen definition of the Lagrangian.

b. Your first-order conditions from part (a) should indicate that so long both of the goods prices p_a or p_b are strictly positive, the constraint from the problem will bind at the optimum. In this case, the two first-order conditions and the binding constraint form a system of three equations in three unknowns: the optimal choices c_a^* and c_b^* and the corresponding value of the Lagrange multiplier. Use these three equations to find the Hicksian demand functions $c_a^* = h_a(p_a, p_b, \bar{U})$ and $c_b^* = h_b(p_a, p_b, \bar{U})$.

c. Finally, to check your answers from part (b), show that when $\bar{U} = V(p_a, p_b, Y)$, the Hicksian demands satisfy

$$h_a[p_a, p_b, V(p_a, p_b, Y)] = \frac{\alpha Y}{p_a} = c_a(p_a, p_b, Y)$$

and

$$h_b[p_a, p_b, V(p_a, p_b, Y)] = \frac{(1-\alpha)Y}{p_b} = c_b(p_a, p_b, Y).$$

2. The Cost of Living

If the prices of all goods and services always increased at exactly the same rate, it would be easy to measure the rising cost of living by their common rate of change. But in the more general case where different prices change at different rates, how should those differential rates of increase be aggregated to summarize the rising cost of living? This question asks you to consider and compare three alternatives.

To begin by fixing notation, let p_a and p_b denote the prices of apples and bananas in a base year, labeled $t = 0$, and let q_a and q_b denote the prices of apples and bananas in a subsequent year, labeled $t = 1$. Assume throughout that the consumer's preferences are described by the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b).$$

a. The first index of the cost of living is grounded most firmly in microeconomic theory, and will therefore be called the "true" price index P. Define the minimum cost function with reference to the expenditure minimization problem you studied above, in question 1, so that for any fixed utility level $\bar{U}$,

$$C(p_a, p_b, \bar{U}) = \min_{c_a, c_b} \; p_a c_a + p_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}$$

and

$$C(q_a, q_b, \bar{U}) = \min_{c_a, c_b} \; q_a c_a + q_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}.$$

Now consider using the increase in the cost of achieving the given utility level $\bar{U}$, computed as

$$P(q_a, q_b, p_a, p_b, \bar{U}) = \frac{C(q_a, q_b, \bar{U})}{C(p_a, p_b, \bar{U})},$$

to measure the rise in the cost of living between $t = 0$ and $t = 1$. In general, a drawback to this measure is that it will depend on the utility level $\bar{U}$. Use the Hicksian demand curves that you derived in answering question 1 to show that in this case, however, the aggregate price index depends only on prices and the preference parameter α and to find the specific functional form for

$$P(q_a, q_b, p_a, p_b, \bar{U}) = P(q_a, q_b, p_a, p_b).$$

b. The second index of the cost of living, used more commonly in practice, simply compares the cost of purchasing a fixed basket of c_a^* apples and c_b^* bananas when prices are q_a and q_b to the cost of purchasing the same basket of apples and bananas when prices are p_a and p_b. This is how, for example, the consumer price index (CPI) is computed in the United States. Suppose c_a^* apples and c_b^* are taken to be the quantities consumed in the base year, $t = 0$. Then the Marshallian demand curves imply that

$$c_a^* = c_a(p_a, p_b, Y) = \frac{\alpha Y}{p_a}$$

and

$$c_b^* = c_b(p_a, p_b, Y) = \frac{(1-\alpha)Y}{p_b}.$$

In this case, this second measure of the cost of living,

$$L(q_a, q_b, p_a, p_b, Y) = \frac{q_a c_a(p_a, p_b, Y) + q_b c_b(p_a, p_b, Y)}{p_a c_a(p_a, p_b, Y) + p_b c_b(p_a, p_b, Y)},$$

corresponds to the Laspeyres price index. Use of the Laspeyres price index can be justified using a first-order approximation to the true minimum cost function and an appeal to the envelope theorem, as follows:

$$\begin{aligned}
C(q_a, q_b, \bar{U}) &\approx C(p_a, p_b, \bar{U}) + \frac{\partial C(p_a, p_b, \bar{U})}{\partial p_a}(q_a - p_a) + \frac{\partial C(p_a, p_b, \bar{U})}{\partial p_b}(q_b - p_b) \\
&= C(p_a, p_b, \bar{U}) + h_a(p_a, p_b, \bar{U})(q_a - p_a) + h_b(p_a, p_b, \bar{U})(q_b - p_b) \\
&= h_a(p_a, p_b, \bar{U})q_a + h_b(p_a, p_b, \bar{U})q_b,
\end{aligned}$$

where the first equality follows from the envelope theorem and the second from the definitions of the cost function $C(p_a, p_b, \bar{U})$ and the Hicksian demands $h_a(p_a, p_b, \bar{U})$ and $h_b(p_a, p_b, \bar{U})$. Therefore

$$P(q_a, q_b, p_a, p_b, \bar{U}) = \frac{C(q_a, q_b, \bar{U})}{C(p_a, p_b, \bar{U})} \approx \frac{h_a(p_a, p_b, \bar{U})q_a + h_b(p_a, p_b, \bar{U})q_b}{h_a(p_a, p_b, \bar{U})p_a + h_b(p_a, p_b, \bar{U})p_b}.$$

Finally, setting $\bar{U} = V(p_a, p_b, Y)$ and using

$$h_a[p_a, p_b, V(p_a, p_b, Y)] = c_a(p_a, p_b, Y) = c_a^*$$

and

$$h_b[p_a, p_b, V(p_a, p_b, Y)] = c_b(p_a, p_b, Y) = c_b^*,$$

it follows that

$$P(q_a, q_b, p_a, p_b, V(p_a, p_b, Y)) \approx \frac{c_a(p_a, p_b, Y)q_a + c_b(p_a, p_b, Y)q_b}{c_a(p_a, p_b, Y)p_a + c_b(p_a, p_b Y)p_b} = L(q_a, q_b, p_a, p_b, Y).$$

In general, a drawback to the Laspeyres price index if that it will depend on the level of income Y. Use the Marshallian demand curves $c_a(p_a, p_b, Y)$ and $c_b(p_a, p_b, Y)$ to show that in this case, however, the Laspeyres price index depends only on prices and the preference parameter α and to find the specific functional form for

$$L(q_a, q_b, p_a, p_b, Y) = L(q_a, q_b, p_a, p_b).$$

c. Although the argument outlined above shows that while the Laspeyres index provides a first-order approximation to the true microeconomic cost of living index, the functional form for $L(q_a, q_b, p_a, p_b)$ does not coincide exactly with the functional form for $P(q_a, q_b, p_a, p_b)$. As a preferable alternative to the Laspeyres index, W.E. Diewert ("Exact and Suplerative Index Numbers," *Journal of Econometrics*, Vol.4, 1976, pp.115-145) suggests the Törnqvist-Theil index, defined by

$$T(q_a, q_b, p_a, p_b, Y) = \left(\frac{q_a}{p_a}\right)^{\bar{s}_a(q_a, q_b, p_a, p_b, Y)} \left(\frac{q_b}{p_b}\right)^{\bar{s}_b(q_a, q_b, p_a, p_b, Y)},$$

where

$$\bar{s}_a(q_a, q_b, p_a, p_b, Y) = \frac{1}{2}\left[\frac{p_a c_a^*(p_a, p_b, Y)}{Y} + \frac{q_a c_a^*(q_a, q_b, Y)}{Y}\right]$$

and

$$\bar{s}_b(q_a, q_b, p_a, p_b, Y) = \frac{1}{2}\left[\frac{p_b c_b^*(p_a, p_b, Y)}{Y} + \frac{q_b c_b^*(q_a, q_b, Y)}{Y}\right]$$

are the average budget shares of apples and bananas from $t = 0$ and $t = 1$. Recalling from problem set 1 that these budget shares are constant, so that

$$\bar{s}_a(q_a, q_b, p_a, p_b, Y) = \alpha$$

and

$$\bar{s}_a(q_a, q_b, p_a, p_b, Y) = 1 - \alpha,$$

show that when preferences are described by the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1 - \alpha)\ln(c_b),$$

the Törnqvist-Theil index depends only on prices and the preference parameter α and that the functional form of

$$T(q_a, q_b, p_a, p_b, Y) = T(q_a, q_b, p_a, p_b)$$

coincides exactly with the form of the true microeconomic price index $P(q_a, q_b, p_a, p_b)$ that you derived in part (a).

3. Portfolio Allocation: Part 1

Consider an individual investor who must allocate the savings in his or her retirement plan either to a perfectly safe asset that pays a known rate of return r_f or to a risky stock mutual fund with random return having expected value μ_r and standard deviation σ_r. Then if w is the fraction of total savings allocated to the risky mutual fund and $1 - w$ the fraction allocated to the risk-free alternative, the investor's portfolio will have random return with expected value

$$\mu_p = w\mu_r + (1 - w)r_f$$

and standard deviation

$$\sigma_p = w\sigma_r.$$

Suppose the investor prefers to earn a higher expected return on his or her portfolio but dislikes volatility in the portfolio's random return, as described by a utility function $U(\mu_p, \sigma_p)$ that is strictly increasing in its first argument but strictly decreasing in its second argument. Then the investor can be depicted as solving the constrained optimization problem

$$\max_{w,\mu_p,\sigma_p} \; U(\mu_p, \sigma_p) \text{ subject to } w\mu_r + (1 - w)r_f \geq \mu_p \text{ and } \sigma_p \geq w\sigma_r.$$

a. Write down the first-order conditions for the individual investor's problem. *Note:* For this problem and the two that follow, there are a number of ways of deriving the first-order conditions. In particular, you can substitute some or all of the constraints into the objective function or set up the Lagrangian for the constrained optimization problem. Use whichever approach you find most convenient.

b. Use the first-order conditions that you derived in part (a), together with the binding constraints, to show that the optimizing investor sets the marginal rate of substitution between risk and return,

$$-\frac{U_2(\mu_p^*, \sigma_p^*)}{U_1(\mu_p^*, \sigma_p^*)}$$

equal to the "Sharpe ratio" of the risky mutual fund, defined as

$$\frac{\mu_r - r_f}{\sigma_r}.$$

4. Portfolio Management

Now consider a professional funds manager, who makes investment decisions for the risky mutual fund offered to the individual investor from question 3. To retain analytic tractability while, at the same time, highlighting the basic economic ideas, suppose that the professional manager is able to allocate funds across two individual stocks, with random returns that have expected values μ_1 and μ_2, standard deviations σ_1 and σ_2, and covariance σ_{12}. Then if v is the fraction of total funds allocated to stock 1 and $1 - v$ is the fraction of total funds allocated to stock 2, the mutual fund will have a random return with expected value

$$\mu_r = v\mu_1 + (1 - v)\mu_2$$

and standard deviation

$$\sigma_r = [v^2\sigma_1^2 + (1 - v)^2\sigma_2^2 + 2v(1 - v)\sigma_{12}]^{1/2}.$$

Suppose that the manager is instructed to maximize the Sharpe ratio offered by the mutual fund, as a measure of "expected excess return" (as measured by $\mu_r - r_f$ in the numerator) "per unit of risk" (as measure by σ_r in the denominator). Then the manager can be depicted as solving the constrained optimization problem

$$\max_{v,\mu_r,\sigma_r} \frac{\mu_r - r_f}{\sigma_r} \text{ subject to } v\mu_1 + (1 - v)\mu_2 \geq \mu_r$$
$$\text{and } \sigma_r \geq [v^2\sigma_1^2 + (1 - v)^2\sigma_2^2 + 2v(1 - v)\sigma_{12}]^{1/2}.$$

a. Write down the first-order conditions for the manager's problem.

b. Use the first-order conditions that you derived in part (a), together with the binding constraints, to show that the manager's optimal choice of v^* must satisfy

$$\mu_1 - \mu_2 = \frac{[v^*\mu_1 + (1 - v^*)\mu_2 - r_f][v^*\sigma_1^2 - (1 - v^*)\sigma_2^2 + (1 - 2v^*)\sigma_{12}]}{v^{*2}\sigma_1^2 + (1 - v^*)^2\sigma_2^2 + 2v^*(1 - v^*)\sigma_{12}}.$$

5. Portfolio Allocation: Part 2

Suppose that the individual investor from question 3, worried perhaps that the professional funds manager's incentives do not align perfectly with his or her own preferences over risk versus return, decides to allocate funds directly to the two individual stocks as well as the risk-free alternative. Although there are numerous ways of setting up the investor's expanded problem, perhaps the easiest is to think of the individual investor as deciding, first, on the shares of total savings w and $1-w$ allocated to the stock market as a whole versus the risk-free alternative and deciding, second, on the shares of funds allocated to the stock market as a whole v and $1-v$ allocated to individual stocks 1 and 2. Now the investor can be depicted as solving the constrained optimization problem

$$\max_{w,v,\mu_r,\sigma_r,\mu_p,\sigma_p} U(\mu_p,\sigma_p) \text{ subject to } v\mu_1+(1-v)\mu_2 \geq \mu_r,$$
$$\sigma_r \geq [v^2\sigma_1^2+(1-v)^2\sigma_2^2+2v(1-v)\sigma_{12}]^{1/2},$$
$$w\mu_r+(1-w)r_f \geq \mu_p,$$
$$\text{and } \sigma_p \geq w\sigma_r.$$

a. Write down the first-order conditions for the individual investor's expanded problem.

b. Use the first-order conditions that you derived in part (a) to show that the optimizing investor still sets the marginal rate of substitution between risk and return equal to the Sharpe ratio of his or her optimally-chosen risky portfolio,

$$-\frac{U_2(\mu_p^*,\sigma_p^*)}{U_1(\mu_p^*,\sigma_p^*)} = \frac{\mu_r^*-r_f}{\sigma_r^*},$$

and that his or her optimally-chosen v^* satisfies exactly the same condition

$$\mu_1-\mu_2 = \frac{[v^*\mu_1+(1-v^*)\mu_2-r_f][v^*\sigma_1^2-(1-v^*)\sigma_2^2+(1-2v^*)\sigma_{12}]}{v^{*2}\sigma_1^2+(1-v^*)^2\sigma_2^2+2v^*(1-v^*)\sigma_{12}}$$

that dictates the professional funds manager's choice of v^*. This comparison illustrates the famous "separation" or "two-fund" theorem from financial economics. This theorem says that individual investors with preferences that can be described by "mean-variance" utility functions don't need to worry about choosing individual stocks. Instead, they can focus on allocating their savings across the professionally-managed stock mutual fund with the highest feasible Sharpe ratio and a risk-free alternative.

Solutions to Midterm Exam

ECON 772001 - Math for Economists
Boston College, Department of Economics

Peter Ireland
Fall 2023

Due Thursday, November 2

1. Hicksian Demands

The Hicksian demand curves can be derived from the expenditure minimization problem

$$\min_{c_a, c_b} \; p_a c_a + p_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}.$$

a. With the Lagrangian defined as

$$L(c_a, c_b, \lambda) = p_a c_a + p_b c_b - \lambda[\alpha \ln(c_a) + (1-\alpha)\ln(c_b) - \bar{U}],$$

the first-order conditions for the consumer's optimal choices c_a^* and c_b^* are

$$p_a - \frac{\lambda^* \alpha}{c_a^*} = 0$$

and

$$p_a - \frac{\lambda^*(1-\alpha)}{c_b^*} = 0.$$

b. Rewrite the first-order conditions from part (a) as

$$c_a^* = \frac{\lambda^* \alpha}{p_a}$$

and

$$c_b^* = \frac{\lambda^*(1-\alpha)}{p_b},$$

then substitute these expressions into the binding constraint to obtain

$$\ln(\lambda^*) + \alpha \ln(\alpha) + (1-\alpha)\ln(1-\alpha) - \alpha \ln(p_a) - (1-\alpha)\ln(p_b) = \bar{U}$$

or

$$\lambda^* = \exp\{\bar{U} - \alpha \ln(\alpha) - (1-\alpha)\ln(1-\alpha) + \alpha \ln(p_a) + (1-\alpha)\ln(p_b)\} = \frac{\exp(\bar{U}) p_a^\alpha p_b^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}}.$$

Finally, substitute this solution for λ^* back into the first-order conditions to find the Hicksian demand functions

$$c_a^* = h_a(p_a, p_b, \bar{U}) = \exp(\bar{U}) \left[\frac{\alpha p_b}{(1-\alpha)p_a}\right]^{1-\alpha}$$

and

$$c_b^* = h_b(p_a, p_b, \bar{U}) = \exp(\bar{U}) \left[\frac{(1-\alpha)p_a}{\alpha p_b}\right]^{\alpha}.$$

c. To check the answers from part (b), note that when

$$\bar{U} = V(p_a, p_b, Y) = \alpha \ln(\alpha) + (1-\alpha)\ln(1-\alpha) - \alpha \ln(p_a) - (1-\alpha)\ln(p_b) + \ln(Y)$$

and therefore

$$\exp(\bar{U}) = \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} Y}{p_a^\alpha p_b^{1-\alpha}},$$

the Hicksian demands satisfy

$$h_a[p_a, p_b, V(p_a, p_b, Y)] = \left[\frac{\alpha^\alpha (1-\alpha)^{1-\alpha} Y}{p_a^\alpha p_b^{1-\alpha}}\right]\left[\frac{\alpha p_b}{(1-\alpha)p_a}\right]^{1-\alpha} = \frac{\alpha Y}{p_a} = c_a(p_a, p_b, Y)$$

and

$$h_b[p_a, p_b, V(p_a, p_b, Y)]) = \left[\frac{\alpha^\alpha (1-\alpha)^{1-\alpha} Y}{p_a^\alpha p_b^{1-\alpha}}\right]\left[\frac{(1-\alpha)p_a}{\alpha p_b}\right]^{\alpha} = \frac{(1-\alpha)Y}{p_b} = c_b(p_a, p_b, Y).$$

2. The Cost of Living

Let p_a and p_b denote the prices of apples and bananas in a base year, labeled $t = 0$, and let q_a and q_b denote the prices of apples and bananas in a subsequent year, labeled $t = 1$. Suppose the consumer's preferences are once again described by the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b).$$

a. Define the minimum cost function with reference to the expenditure minimization problem studied above, in question 1, so that for any fixed utility level $\bar{U}$,

$$C(p_a, p_b, \bar{U}) = \min_{c_a, c_b} \; p_a c_a + p_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}$$

and

$$C(q_a, q_b, \bar{U}) = \min_{c_a, c_b} \; q_a c_a + q_b c_b \text{ subject to } \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \geq \bar{U}.$$

The Hicksian demand curves derived in answering question 1 then imply that

$$\begin{aligned} C(p_a, p_b, \bar{U}) &= p_a \exp(\bar{U}) \left[\frac{\alpha p_b}{(1-\alpha)p_a}\right]^{1-\alpha} + p_b \exp(\bar{U}) \left[\frac{(1-\alpha)p_a}{\alpha p_b}\right]^{\alpha} \\ &= \exp(\bar{U}) p_a^\alpha p_b^{1-\alpha} \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right] \end{aligned}$$

and similarly

$$C(q_a, q_b, \bar{U}) = \exp(\bar{U}) q_a^\alpha q_b^{1-\alpha} \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right].$$

It follows from these expressions that

$$P(q_a, q_b, p_a, p_b, \bar{U}) = \frac{C(q_a, q_b, \bar{U})}{C(p_a, p_b, \bar{U})} = \left(\frac{q_a}{p_a}\right)^{\alpha} \left(\frac{q_b}{p_b}\right)^{1-\alpha}.$$

In particular, because the utility function is homothetic, this "true" cost of living index does not depend on the utility level $\bar{U}$.

b. Using the Marshallian demands curves

$$c_a(p_a, p_b, Y) = \frac{\alpha Y}{p_a}$$

and

$$c_b(p_a, p_b, Y) = \frac{(1-\alpha)Y}{p_b},$$

the Laspeyres price index is

$$L(q_a, q_b, p_a, p_b, Y)] = \frac{c_a(p_a, p_b, Y)q_a + c_b(p_a, p_b, Y)q_b}{c_a(p_a, p_b, Y)p_a + c_b(p_a, p_b, Y)p_b} = \alpha\left(\frac{q_a}{p_a}\right) + (1-\alpha)\left(\frac{q_b}{p_b}\right).$$

Again because the utility function is homothetic, the Laspeyres price index does not depend on the level of income Y. But it does not coincide exactly with the true cost of living index from part (a).

c. Since the utility function

$$U(c_a, c_b) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b)$$

implies that the budget shares of apples and bananas are constant and equal to α and $1-\alpha$, the Törnqvist-Thiel price index

$$T(q_a, q_b, p_a, p_b, Y) = \left(\frac{q_a}{p_a}\right)^{\alpha} \left(\frac{q_b}{p_b}\right)^{1-\alpha}.$$

coincides with exactly the true price index from part (a).

3. Portfolio Allocation: Part 1

The individual investor solves the constrained optimization problem

$$\max_{w,\mu_p,\sigma_p} U(\mu_p, \sigma_p) \text{ subject to } w\mu_r + (1-w)r_f \geq \mu_p \text{ and } \sigma_p \geq w\sigma_r.$$

a. With the Lagrangian for the investor's problem defined as

$$L(w, \mu_p, \sigma_p, \lambda_1, \lambda_2) = U(\mu_p, \sigma_p) + \lambda_1[w\mu_r + (1-w)r_f - \mu_p] + \lambda_2(\sigma_p - w\sigma_r),$$

the first-order conditions can be written as

$$\lambda_1^*(\mu_r - r_f) - \lambda_2^* \sigma_r,$$

$$U_1(\mu_p^*, \sigma_p^*) - \lambda_1^* = 0,$$

and

$$U_2(\mu_p^*, \sigma_p)^* + \lambda_2^* = 0.$$

b. Use the first-order conditions for μ_p^* and σ_p^* to solve for λ_1^* and λ_2^*, and substitute these solutions into the first-order condition for w^* to get the desired result that the investor sets the marginal rate of substitution between risk and return equal to the "Sharpe ratio" of the risky mutual fund:

$$-\frac{U_2(\mu_p^*, \sigma_p^*)}{U_1(\mu_p^*, \sigma_p^*)} = \frac{\mu_r - r_f}{\sigma_r}.$$

4. Portfolio Management

The professional funds manager solves the constrained optimization problem

$$\max_{v,\mu_r,\sigma_r} \frac{\mu_r - r_f}{\sigma_r} \text{ subject to } v\mu_1 + (1-v)\mu_2 \geq \mu_r$$
$$\text{and } \sigma_r \geq [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2}.$$

a. With the Lagrangian for the investor's problem defined as

$$L(v, \mu_r, \sigma_r, \lambda_1, \lambda_2) = \frac{\mu_r - r_f}{\sigma_r} + \lambda_1[v\mu_1 + (1-v)\mu_2 - \mu_r]$$
$$+ \lambda_2\{\sigma_r - [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2}\},$$

the first-order conditions can be written as

$$\lambda_1^*(\mu_1 - \mu_2) - \lambda_2^* \left\{ \frac{v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}}{[v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}]^{1/2}} \right\} = 0$$

$$\frac{1}{\sigma_r^*} - \lambda_1^* = 0,$$

and

$$-\frac{\mu_r^* - r_f}{\sigma_r^{*2}} + \lambda_2^* = 0.$$

b. Use the first-order conditions for μ_r^* and σ_r^* to solve for λ_1^* and λ_2^*, and substitute these solutions together with the binding constrains for μ_r^* and σ_r^* into the first-order condition for v^* to get the desired result that v^* must satisfy

$$\mu_1 - \mu_2 = \frac{[v^*\mu_1 + (1-v^*)\mu_2 - r_f][v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}]}{v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}}.$$

5. Portfolio Allocation: Part 2

When the individual investor chooses between individual stocks, he or she solves the constrained optimization problem

$$\max_{w,v,\mu_r,\sigma_r,\mu_p,\sigma_p} U(\mu_p,\sigma_p) \text{ subject to } v\mu_1 + (1-v)\mu_2 \geq \mu_r,$$

$$\sigma_r \geq [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2},$$

$$w\mu_r + (1-w)r_f \geq \mu_p,$$

$$\text{and } \sigma_p \geq w\sigma_r.$$

a. With the Lagrangian for the investor's problem defined as

$$\begin{aligned} L(w,v,\mu_r,\sigma_r,\mu_p,\sigma_p,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = {} & U(\mu_p,\sigma_p) \\ & + \lambda_1[v\mu_1 + (1-v)\mu_2 - \mu_r] \\ & + \lambda_2\{\sigma_r - [v^2\sigma_1^2 + (1-v)^2\sigma_2^2 + 2v(1-v)\sigma_{12}]^{1/2}\}, \\ & + \lambda_3[w\mu_r + (1-w)r_f - \mu_p] \\ & + \lambda_4(\sigma_p - w\sigma_r), \end{aligned}$$

the first-order conditions can be written as

$$\lambda_3^*(\mu_r^* - r_f) - \lambda_4^*\sigma_r^* = 0,$$

$$\lambda_1^*(\mu_1-\mu_2) - \lambda_2^*\left\{\frac{v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}}{[v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}]^{1/2}}\right\} = 0,$$

$$-\lambda_1^* + \lambda_3^*w^* = 0,$$

$$\lambda_2^* - \lambda_4^*w^* = 0,$$

$$U_1(\mu_p^*,\sigma_p^*) - \lambda_3^* = 0,$$

and

$$U_2(\mu_p^*,\sigma_p)^* + \lambda_4^* = 0.$$

b. Use the first-order conditions for μ_p^* and σ_p^* to solve for λ_3^* and λ_4^*, and substitute these solutions into the first-order condition for w^* to get the same result as in question 3:

$$-\frac{U_2(\mu_p^*,\sigma_p^*)}{U_1(\mu_p^*,\sigma_p^*)} = \frac{\mu_r^* - r_f}{\sigma_r^*}.$$

Then use the first-order conditions for w^*, μ_r^*, and σ_r^* to rewrite the first-order condition for v^* as

$$\mu_1 - \mu_2 = \frac{[v^*\mu_1 + (1-v^*)\mu_2 - r_f][v^*\sigma_1^2 - (1-v^*)\sigma_2^2 + (1-2v^*)\sigma_{12}]}{v^{*2}\sigma_1^2 + (1-v^*)^2\sigma_2^2 + 2v^*(1-v^*)\sigma_{12}},$$

which replicates the condition from question 4 that dictates the professional funds manager's choice of v^*.

Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2023

Due Tuesday, December 20

This exam has four questions on five pages; before you begin, please check to make sure that your copy has all four questions and all five pages. The four questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, my notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class. The answers you submit must be yours and yours alone.

1. Endogenous Growth: Continuous Time

As we've seen, the Ramsey (neoclassical growth) model implies that, in the absence of exogenous technological change, the economy converges to a steady state in which consumption and the capital stock remain constant. A straightforward modification of that model, suggested by Sergio Rebelo ("Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy*, Vol.99, June 1991) dispenses with the traditional assumption of diminishing returns to capital accumulation to obtain a variant that is consistent with continuing long-run growth. In his paper (footnote 7, p.507), Rebelo explains that his assumption of constant returns is consistent with a broader interpretation of the capital stock that includes human and well as physical capital and therefore implies that in effect there are no fixed factors of production.

Following Rebelo, assume that the representative consumer's preferences are described by the utility function

$$\int_0^\infty e^{-\rho t}\left(\frac{1}{1-\sigma}\right) C(t)^{1-\sigma}\ \mathrm{d}t, \tag{1}$$

where $C(t)$ is consumption at time $t \in [0,\infty)$ and the discount rate and curvature parameters $\rho > 0$ and $\sigma > 0$ are both strictly positive. *Note:* In the special case where $\sigma = 1$, the instantaneous utility function in (1) should be replaced by the natural log $\ln(C(t))$. For the purposes of solving this problem, however, you can simply assume that $\sigma \neq 1$ to avoid having to the consider the limiting case separately.

Output at time t is a linear function $AK(t)$ of the capital stock, where $A > 0$. The capital stock depreciates at the constant rate $\delta > 0$. Capital accumulation is therefore governed by

$$(A-\delta)K(t) - C(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0,\infty)$.

Like the Ramsey model, Rebelo's describes an economic environment in which the two welfare theorems apply. Equilibrium allocations can therefore be characterized by solving a social planner's problem: Given the initial capital stock $K(0)$, choose $C(t)$ for all $t \in [0, \infty)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function (1) subject to the constraint (2) for $t \in [0, \infty)$.

a. Define (write down) the maximized Hamiltonian for the social planner's problem, and use the maximum principle to derive the set of optimality conditions (the first-order conditions and pair of differential equations) that describe the solution to this problem. *Note:* For this problem, you can use whichever form of the Hamiltonian – present or current value – you find most convenient. Either way, however, just make sure the optimality conditions you write down are consistent with your choice.

b. Interestingly, this endogenous growth model has no transition dynamics. It implies, in particular, that consumption always grows at a constant rate, with

$$\dot{C}(t)/C(t) = g_c$$

for all $t \in [0, \infty)$. Use the optimality conditions you derived in part (a) to solve for the constant growth rate g_c in terms of the model's parameters: ρ, σ, A, and δ.

2. Endogenous Growth with Comparison Utility: Continuous Time

Christoper Carroll, Jody Overland, and David Weil ("Comparison Utility in a Growth Model," *Journal of Economic Growth*, Vol.2, December 1997) develop an interesting variant of Rebelo's endogenous growth model in which the representative consumer's utility depends on his or her own consumption relative to an economy-wide average. They trace this idea back to James Duesenberry's PhD dissertation (published as *Income, Saving, and the Theory of Consumer Behavior*, Harvard University Press, 1949).

In Carroll, Overland, and Weil's model, capital accumulation is still governed by the constraint in (2), but the representative consumer's preferences are described by

$$\int_0^\infty e^{-\rho t} \left(\frac{1}{1-\sigma}\right) \left[\frac{C(t)}{Z(t)^\gamma}\right]^{1-\sigma} \mathrm{d}t, \tag{3}$$

where $Z(t)$ is an index of past aggregate consumption that the representative consumer takes as given. The preference parameters in (3) are now assumed to satisfy $\rho > 0$, $\sigma > 1$, and $\gamma \in [0, 1)$. Capital accumulation continues to be governed by the constraint in (2).

Because this model features a kind of negative externality, through which a rise in aggregate consumption impacts adversely on the representative consumer's utility, the welfare theorems of economics no longer hold. Here, we'll focus on equilibrium allocations by solving the individual consumer's problem: Taking the initial capital stock $K(0)$ and the trajectory for $Z(t)$ as given, choose $C(t)$ for all $t \in [0, \infty)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function (3) subject to the constraint (2) for $t \in [0, \infty)$.

a. Define (write down) the maximized Hamiltonian for the consumer's problem, and use the maximum principle to derive the set of optimality conditions (the first-order conditions and pair of differential equations) that describe the solution to this problem. *Note:* Again, you can use whichever form of the Hamiltonian – present or current value – you find most convenient. Either way, however, just make sure the optimality conditions you write down are consistent with your choice.

b. In deriving the implications of their model for long-run growth rates, Carroll, Overland, and Weil go on to assume that the aggregate index of consumption $Z(t)$ evolves according to

$$\dot{Z}(t) = \alpha[\bar{C}(t) - Z(t)],$$

where $\bar{C}(t)$ is average consumption per capita and the speed-of-adjustment parameter $\alpha > 0$ is positive. If all consumers in the economy are identical, then $\bar{C}(t) = C(t)$ in equilibrium, and this adjustment equation becomes

$$\dot{Z}(t) = \alpha[C(t) - Z(t)]. \tag{4}$$

Equation (4) reveals that this model is similar to one in which there is habit formation in preferences although, here, each consumer neglects the effects that his or her consumption choice today has on the future habit stock $Z(t)$. Carroll, Overland, and Weil show that when the dynamics of $Z(t)$ are governed by (4), the economy converges to a steady-state growth path in which consumption and the habit stock $Z(t)$ grow at the same constant rate, with

$$\dot{C}(t)/C(t) = \dot{Z}(t)/Z(t) = g_c$$

for all t. Use the optimality conditions you derived in part (a) to solve for the constant growth rate g_c in terms of the model's parameters: ρ, σ, γ, A, and δ. Your answer should show that under the assumption that $\sigma > 1$, the steady-state growth rate of consumption g_c increases as the parameter γ measuring the importance of the habit stock $Z(t)$ in utility becomes larger.

3. Endogenous Growth: Discrete Time

In a discrete-time version of Rebelo's model, the representative consumer's preferences are described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right) C_t^{1-\sigma}, \tag{5}$$

where $\beta \in (0,1)$ and $\sigma > 0$. Again, when $\sigma = 1$ the single-period utility function in (5) can be replaced by the natural log $\ln(C_t)$ but here, for simplicity, you can just assume that $\sigma \neq 1$ to avoid having to consider the limiting case separately.

Capital accumulation takes place according to the constraint

$$AK_t + (1-\delta)K_t - C_t \geq K_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$, where $A > 0$ and $\delta > 0$. The social planner takes K_0 as given and chooses C_t for $t = 0, 1, 2, \ldots$ and K_t for $t = 1, 2, 3, \ldots$ to maximize the utility function (5) subject to the constraint (6) for all $t = 0, 1, 2, \ldots$.

a. Define (write down) the Bellman equation for the social planner's problem, and use dynamic programming to derive the set of optimality conditions (the first-order and envelope conditions, plus the binding constraint) that describe the solution to this problem. *Note:* There are several ways to set up the Bellman equation; feel free to use whichever you find most convenient. Perhaps the easiest approach, though, is to start by substituting the binding constraint from (6) into the utility function in (5), and then to use K_t as the state variable and K_{t+1} as the control variable.

b. Once again, this model implies that consumption growth will be constant, with

$$C_{t+1}/C_t = G_c$$

for all $t = 0, 1, 2, \ldots$. Use your optimality conditions from part (a) to solve for the constant growth rate G_c in terms of the model's parameters β, σ, A, and δ.

4. Endogenous Growth with Comparison Utility: Discrete Time

In a discrete-time version of Carroll, Overland, and Weil's model, the representative consumer's preferences are described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right)\left(\frac{C_t}{Z_t^\gamma}\right)^{1-\sigma}, \tag{7}$$

where $\beta \in (0, 1)$, $\sigma > 1$, and $\gamma \in [0, 1)$. Capital accumulation continues to be governed by the constraint in (6).

The consumer takes the initial capital stock K_0 and the trajectory for Z_t as given, and chooses C_t for $t = 0, 1, 2, \ldots$ and K_t for $t = 1, 2, 3, \ldots$ to maximize the utility function (7) subject to the constraint (6) for all $t = 0, 1, 2, \ldots$.

a. Define (write down) the Bellman equation for the consumer's problem, and use dynamic programming to derive the set of optimality conditions (the first-order and envelope conditions, plus the binding constraint) that describe the solution to this problem. *Note:* Once again, there are several ways to set up the Bellman equation; feel free to use whichever you find most convenient. Perhaps the easiest approach, though, is to start by substituting the binding constraint from (6) into the utility function in (7), and then to use K_t as the state variable and K_{t+1} as the control variable.

b. When the dynamics of the aggregate consumption index are governed by

$$Z_{t+1} = \alpha \bar{C}_t + (1-\alpha) Z_t$$

with $\alpha \in (0,1)$ and when average per capita consumption $\bar{C}_t$ equals the representative consumer's consumption C_t for all $t = 0, 1, 2, \ldots$, the discrete-time version of (4)

$$Z_{t+1} = \alpha C_t + (1-\alpha) Z_t$$

will hold in equilibrium and the economy will once again converge to a steady-state growth path in which consumption C_t and the habit stock Z_t grow at the same constant rate, with

$$C_{t+1}/C_t = Z_{t+1}/Z_t = G_c$$

for all $t = 0, 1, 2, \ldots$. Use your optimality conditions from part (a) to solve for the constant growth rate G_c in terms of the model's parameters: β, σ, γ, A, and δ.

Solutuions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2023

Due Tuesday, December 19

1. Endogenous Growth: Continuous Time

Given the initial capital stock $K(0)$, the social planner chooses $C(t)$ for all $t \in [0, \infty)$ and $K(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left(\frac{1}{1-\sigma} \right) C(t)^{1-\sigma} \, \mathrm{d}t, \tag{1}$$

subject to the constraint

$$(A-\delta)K(t) - C(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0, \infty)$.

a. With the maximized present-value Hamiltonian for the social planner's problem defined as

$$H(K(t), \pi(t); t) = \max_{C(t)} \; e^{-\rho t} \left(\frac{1}{1-\sigma} \right) C(t)^{1-\sigma} + \pi(t)[(A-\delta)K(t) - C(t)],$$

the maximum principle implies that the solution to the problem must satisfy the first-order condition

$$e^{-\rho t} C(t)^{-\sigma} - \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_K(K(t), \pi_t; t) = -\pi(t)(A-\delta)$$

and

$$\dot{K}(t) = (A-\delta)K(t) - C(t)$$

for all $t \in [0, \infty)$.

b. Rewrite the first-order condition as

$$\pi(t) = e^{-\rho t} C(t)^{-\sigma}$$

and differentiate both sides with respect to t to obtain

$$\dot{\pi}(t) = -\rho e^{-\rho t} C(t)^{-\sigma} - \sigma e^{-\rho t} C(t)^{-\sigma-1} \dot{C}(t).$$

Now substitute these two expressions into the differential equation for $\pi(t)$ to obtain

$$-\rho e^{-\rho t} C(t)^{-\sigma} - \sigma e^{-\rho t} C(t)^{-\sigma-1} \dot{C}(t) = -(A-\delta) e^{-\rho t} C(t)^{-\sigma}$$

or, more simply

$$\frac{\dot{C}(t)}{C(t)} = g_c = \frac{A-\delta-\rho}{\sigma}.$$

2. Endogenous Growth with Comparison Utility: Continuous Time

Taking the initial capital stock $K(0)$ and the trajectory for $Z(t)$ as given, the consumer chooses $C(t)$ for all $t \in [0,\infty)$ and $K(t)$ for all $t \in (0,\infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t} \left(\frac{1}{1-\sigma}\right) \left[\frac{C(t)}{Z(t)^\gamma}\right]^{1-\sigma} \text{d}t, \tag{3}$$

subject to the constraint

$$(A-\delta)K(t) - C(t) \geq \dot{K}(t) \tag{2}$$

for all $t \in [0,\infty)$.

a. With the maximized present-value Hamiltonian for the social planner's problem defined as

$$H(K(t),\pi(t);t) = \max_{C(t)} \; e^{-\rho t}\left(\frac{1}{1-\sigma}\right) C(t)^{1-\sigma} Z(t)^{-\gamma(1-\sigma)} + \pi(t)[(A-\delta)K(t) - C(t)],$$

the maximum principle implies that the solution to the problem must satisfy the first-order condition

$$e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)} - \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi}(t) = -H_K(K(t),\pi_t;t) = -\pi(t)(A-\delta)$$

and

$$\dot{K}(t) = (A-\delta)K(t) - C(t)$$

for all $t \in [0,\infty)$.

b. Rewrite the first-order condition as

$$\pi(t) = e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)}$$

and differentiate both sides with respect to t to obtain

$$\begin{aligned}\dot{\pi}(t) = &-\rho e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)} \\ &- \sigma e^{-\rho t} C(t)^{-\sigma-1} Z(t)^{-\gamma(1-\sigma)} \dot{C}(t) - \gamma(1-\sigma) e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)-1} \dot{Z}(t).\end{aligned}$$

Now substitute these two expressions into the differential equation for $\pi(t)$ to obtain

$$\begin{aligned}(A-\delta)e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)} = &\; \rho e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)} \\ &+ \sigma e^{-\rho t} C(t)^{-\sigma-1} Z(t)^{-\gamma(1-\sigma)} \dot{C}(t) \\ &+ \gamma(1-\sigma) e^{-\rho t} C(t)^{-\sigma} Z(t)^{-\gamma(1-\sigma)-1} \dot{Z}(t).\end{aligned}$$

or, more simply,

$$A - \delta - \rho = \sigma \left[\frac{\dot{C}(t)}{C(t)}\right] + \gamma(1-\sigma)\left[\frac{\dot{Z}(t)}{Z(t)}\right].$$

Hence, along a steady-state growth path with

$$\dot{C}(t)/C(t) = \dot{Z}(t)/Z(t) = g_c$$

it must be that

$$A - \delta - \rho = \sigma g_c + \gamma(1-\sigma)g_c$$

or

$$g_c = \frac{A - \delta - \rho}{\sigma + \gamma(1-\sigma)}.$$

3. Endogenous Growth: Discrete Time

Given the initial capital stock K_0, the social planner chooses C_t for $t = 0, 1, 2, \ldots$ and K_t for $t = 1, 2, 3, \ldots$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right) C_t^{1-\sigma}, \tag{5}$$

subject to the constraint

$$AK_t + (1-\delta)K_t - C_t \geq K_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$.

a. With the Bellman equation written as

$$v(K_t) = \max_{K_{t+1}} \left(\frac{1}{1-\sigma}\right) [(A + 1 - \delta)K_t - K_{t+1}]^{1-\sigma} + \beta v(K_{t+1}),$$

dynamic programming arguments imply that the solution to the planner's problem must satisfy the first-order condition

$$-[(A + 1 - \delta)K_t - K_{t+1}]^{-\sigma} + \beta v'(K_{t+1}) = 0$$

for all $t = 0, 1, 2, \ldots$, the envelope condition

$$v'(K_t) = (A + 1 - \delta)[(A + 1 - \delta)K_t - K_{t+1}]^{-\sigma},$$

for all $t = 1, 2, 3, \ldots$, and the binding constraint

$$K_{t+1} = (A + 1 - \delta)K_t - C_t$$

for all $t = 0, 1, 2, \ldots$.

b. Use the binding constraint to rewrite the first-order and envelope conditions more simply as

$$C_t^{-\sigma} = \beta v'(K_{t+1})$$

and

$$v'(K_t) = (A + 1 - \delta)C_t^{-\sigma}.$$

Since this last equation must hold for all $t = 1, 2, 3, \ldots$, it also implies that

$$v'(K_{t+1}) = (A + 1 - \delta)C_{t+1}^{-\sigma}$$

for all $t = 0, 1, 2, \ldots$. Substituting this expression into the first-order condition yields

$$C_t^{-\sigma} = \beta(A + 1 - \delta)C_{t+1}^{-\sigma}$$

or

$$\frac{C_{t+1}}{C_t} = G_c = [\beta(A + 1 - \delta)]^{1/\sigma}.$$

4. Endogenous Growth with Comparison Utility: Discrete Time

Taking K_0 and the trajectory for Z_t as given, the consumer chooses C_t for $t = 0, 1, 2, \ldots$ and K_t for $t = 1, 2, 3, \ldots$ to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right)\left(\frac{C_t}{Z_t^\gamma}\right)^{1-\sigma}, \tag{7}$$

subject to the constraint

$$AK_t + (1 - \delta)K_t - C_t \geq K_{t+1} \tag{6}$$

for all $t = 0, 1, 2, \ldots$.

a. With the Bellman equation written as

$$v(K_t) = \max_{K_{t+1}} \left(\frac{1}{1-\sigma}\right)[(A + 1 - \delta)K_t - K_{t+1}]^{1-\sigma} Z_t^{-\gamma(1-\sigma)} + \beta v(K_{t+1}),$$

dynamic programming arguments imply that the solution to the planner's problem must satisfy the first-order condition

$$-[(A + 1 - \delta)K_t - K_{t+1}]^{-\sigma} Z_t^{-\gamma(1-\sigma)} + \beta v'(K_{t+1}) = 0$$

for all $t = 0, 1, 2, \ldots$, the envelope condition

$$v'(K_t) = (A + 1 - \delta)[(A + 1 - \delta)K_t - K_{t+1}]^{-\sigma} Z_t^{-\gamma(1-\sigma)},$$

for all $t = 1, 2, 3, \ldots$, and the binding constraint

$$K_{t+1} = (A + 1 - \delta)K_t - C_t$$

for all $t = 0, 1, 2, \ldots$.

b. Use the binding constraint to rewrite the first-order and envelope conditions more simply as

$$C_t^{-\sigma} Z_t^{-\gamma(1-\sigma)} = \beta v'(K_{t+1})$$

and

$$v'(K_t) = (A + 1 - \delta) C_t^{-\sigma} Z_t^{-\gamma(1-\sigma)}.$$

Since this last equation must hold for all $t = 1, 2, 3, \ldots$, it also implies that

$$v'(K_{t+1}) = (A + 1 - \delta) C_{t+1}^{-\sigma} Z_{t+1}^{-\gamma(1-\sigma)}$$

for all $t = 0, 1, 2, \ldots$. Substituting this expression into the first-order condition yields

$$C_t^{-\sigma} Z_t^{-\gamma(1-\sigma)} = \beta(A + 1 - \delta) C_{t+1}^{-\sigma} Z_{t+1}^{-\gamma(1-\sigma)}$$

or, more simply,

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} \left(\frac{Z_{t+1}}{Z_t}\right)^{\gamma(1-\sigma)} = \beta(A + 1 - \delta)$$

Hence, along a steady-state growth path with

$$C_{t+1}/C_t = Z_{t+1}/Z_t = G_c$$

it must be that

$$G_c^{\sigma} G_c^{\gamma(1-\sigma)} = \beta(A + 1 - \delta)$$

or

$$G_c = [\beta(A + 1 - \delta)]^{1/[\sigma+\gamma(1-\sigma)]}.$$

Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2024

Due Thursday, November 7

This exam has three questions on five pages. Please check to make sure that your copy has all three questions and all five pages. Each question will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Profit Maximization

Consider a firm that hires n workers to produce y units of output according to the production function

$$y = f(n) = 10n - n^2.$$

Normalize the price of output to equal one, and let $w > 0$ denote the real wage. Then the profit-maximizing firm solves

$$\max_n 10n - n^2 - wn \text{ subject to } n \geq 0.$$

There are a variety of ways to find the solution to this problem; use whichever you find most convenient to answer the questions below.

a. Find the value of $\bar{w}$ such that, for all $w \geq \bar{w}$, the firm hires no workers, choosing $n^* = 0$.

b. Assuming instead that $0 < w < \bar{w}$, find the firm's labor demand curve, that is, the function $n(w)$ linking the firm's optimal choice of $n^* \geq 0$ of n to the wage rate w.

c. For any value of $w > 0$, define the firm's profit function π as its maximized profit for that value of w, that is:

$$\pi(w) = \max_n 10n - n^2 - wn \text{ subject to } n \geq 0.$$

Using your results from parts (a) and (b), above, find the expressions that show how $\pi(w)$ depends on w for the two cases, where $w \geq \bar{w}$ and $0 < w < \bar{w}$.

d. Finally, find an expression for $\pi'(w)$, the derivative of the profit function with respect to the wage, assuming that $0 < w < \bar{w}$.

2. Optimal Risk Sharing

Consider an economy consisting of a finite number of consumers, indexed by $j = 1, 2, \ldots, J$. There are two periods: a planning period ($t = 0$) and a production and consumption period ($t = 1$). Looking ahead from the planning period $t = 0$, there are a finite number of possible states of the world in the production-consumption period $t = 1$, indexed by $s = 1, 2, \ldots, S$; each occurs with probability $\pi(s)$, with $0 < \pi(s) < 1$ for all $s = 1, 2, \ldots, S$ and, of course,

$$\sum_{s=1}^{S} \pi(s) = 1.$$

Consumers have identical preferences, as described by the expected utility function

$$\sum_{s=1}^{S} \pi(s) u[c_j(s)],$$

with

$$u[c_j(s)] = -\frac{1}{\sigma} \exp[-\sigma c_j(s)]$$

and where $\sigma > 0$ and $c_j(s)$ is consumer j's consumption in state s at $t = 1$. This utility function implies that the coefficient of absolute risk aversion $-u''(c)/u'(c)$ is constant and equal to σ.

In each state $s = 1, 2, \ldots, S$ during period $t = 1$, consumer j supplies one unit of labor inelastically to produce $y_j(s)$ units of output. Hence, looking ahead from $t = 0$ the fundamental source of uncertainty is over each worker's productivity at $t = 1$. In the analysis to follow, it will be helpful to let

$$y_a(s) = \frac{1}{J} \sum_{j=1}^{J} y_j(s)$$

denote average output per worker in each state $s = 1, 2, \ldots, S$ at $t = 1$.

Suppose now that at $t = 0$, a social planner takes outputs $y_j(s)$, $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$ as given, and chooses consumptions $c_j(s)$, $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$, for all consumers in all states of the world to maximize an equally-weighted sum of their utilities

$$\sum_{j=1}^{J} \sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\},$$

subject to the aggregate resource constraints

$$\sum_{j=1}^{J} y_j(s) \geq \sum_{j=1}^{J} c_j(s)$$

for all $s = 1, 2, \ldots, S$. This is a constrained maximization problem with a finite number $J \times S$ of choice variables and a finite number S of constraints. Hence, the Kuhn-Tucker theorem can be applied by defining the Lagrangian

$$\sum_{j=1}^{J}\sum_{s=1}^{S}\pi(s)\left\{-\frac{1}{\sigma}\exp[-\sigma c_j(s)]\right\} + \sum_{s=1}^{S}\lambda(s)\left[\sum_{j=1}^{J}y_j(s) - \sum_{j=1}^{J}c_j(s)\right],$$

where for each $s = 1, 2, \ldots, S$, $\lambda(s)$ is the Lagrange multiplier on the aggregate resource constraint for state s.

a. Write down the first-order conditions for the social planner's problem. *Note:* Given the symmetry of this problem, it is possible to derive all of these first-order conditions at once by fixing a particular value of j and a particular value of s, differentiating the Lagrangian by $c_j(s)$ for those values of j, and s, and setting the result equal to zero. The same first-order condition will then have to hold for all $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$.

b. Next, use the first-order conditions you derived above to solve for the optimal $c_j^*(s)$ in terms of the coefficient of absolute risk aversion σ, the probability $\pi(s)$, and the associated value of the Lagrange multiplier $\lambda^*(s)$.

c. Now substitute your solutions for $c_j^*(s)$ into the binding constraint for state s,

$$Jy_a(s) = \sum_{j=1}^{J}y_j(s) = \sum_{j=1}^{J}c_j^*(s),$$

to find a solution for the multiplier $\lambda^*(s)$ in terms of the coefficient of absolute risk aversion σ, the probability $\pi(s)$, and the average output per worker $y_a(s)$.

d. Finally, substitute your solution for $\lambda^*(s)$ back into the expression for $c_j^*(s)$ you derived in part (b) to find the solution for $c_j^*(s)$ in terms of $y_a(s)$. Note that this solution implies that the social planner optimally chooses to fully insure each individual consumer against "idiosyncratic risk," since each consumer's consumption depends only on aggregate output and not on his or her own level of productivity. At the same time, however, the solution indicates that the social planner cannot insure consumers against "aggregate risk," since their consumptions must rise or fall as the average output per worker $y_a(s)$ rises or falls across states.

3. Equilibrium Risk Sharing

The second welfare theorem of economics implies that the Pareto optimal allocation that solves the social planner's problem in question 2 can be supported in a competitive equilibrium. But how?

Suppose that the initial planning period $t = 0$ in the economy from question 2 is replaced by an initial trading period $t = 0$, in which consumers buy and sell claims for state-contingent delivery of goods during the production and consumption period $t = 1$. A consumer who sells a contingent claim for any state $s = 1, 2, \ldots, S$ receives $p(s)$ dollars at $t = 0$, but is required to deliver one unit of consumption in that state s at $t = 1$. A consumer who buys a contingent claim for any state $s = 1, 2, \ldots, S$ must pay $p(s)$ dollars at $t = 0$, but will receive one unit of consumption in that state s at $t = 1$.

Now, in the trading period $t = 0$, each consumer j takes current prices $p(s)$, $s = 1, 2, \ldots, S$, and future outputs $y_j(s)$, $s = 1, 2, \ldots, S$ as given, and chooses future consumptions $c_j(s)$, $s = 1, 2, \ldots, S$, to maximize his or her own utility

$$\sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\},$$

subject to the budget constraint

$$\sum_{s=1}^{S} p(s) y_j(s) \geq \sum_{s=1}^{S} p(s) c_j(s).$$

This is a constrained maximization problem with a finite number S of choice variables and a single constraint. Hence, the Kuhn-Tucker theorem can be applied by defining the Lagrangian

$$\sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\} + \lambda_j \left[\sum_{s=1}^{S} p(s) y_j(s) - \sum_{s=1}^{S} p(s) c_j(s) \right],$$

where λ_j is the single Lagrange multiplier on consumer j's budget constraint.

a. Write down the first-order conditions for the consumer's problem. *Note:* Once again, given the symmetry of this problem, it is possible to derive all of these first-order conditions at once by fixing a particular value of s, differentiating the Lagrangian by $c_j(s)$ for that value of s, and setting the result equal to zero. The same first-order condition will then have to hold for all $s = 1, 2, \ldots, S$.

b. Next, use the first-order conditions you derived above to solve for the optimal $c_j^*(s)$ in terms of the coefficient of absolute risk aversion σ, the probability $\pi(s)$, the contingent claim price $p(s)$, and the associated value of the Lagrange multiplier λ_j^*.

c. Now let's guess that the equilibrium prices that support the optimal allocation from question 2 are

$$p(s) = \pi(s) \exp[-\sigma y_a(s)]$$

for all $s = 1, 2, \ldots, S$. Where does this conjecture come from? To see, take a look back at your solution for $\lambda^*(s)$ from question 2, part (c), and recall the interpretation of Lagrange multipliers as "shadow prices." Recall, also, that the social planner from question 2 weighted all individual agent's utility equally when solving for the optimal allocations. Thus, it makes sense to guess that the prices from above will prevail in a competitive equilibrium when

$$\sum_{s=1}^{S} p(s) y_j(s) = \sum_{s=1}^{S} p(s) y_a(s)$$

for all $j = 1, 2, \ldots, J$. Note that this symmetry condition does not require agents to have equal productivities state-by-state; it simply requires that agents have equal endowments of wealth when valued in the time $t = 0$ trading session. Using this symmetry condition for wealth, substitute your solutions for $c_j^*(s)$, together with the conjectured equilibrium prices, into consumer j's binding budget constraint,

$$\sum_{s=1}^{S} p(s) y_a(s) = \sum_{s=1}^{S} p(s) y_j(s) = \sum_{s=1}^{S} p(s) c_j^*(s),$$

to find a solution for the multiplier λ_j^*.

d. Finally, substitute the conjectured prices $p(s)$ and your solution for λ_j^* back into the expression for $c_j^*(s)$ you derived in part (b) to find the solution for $c_j^*(s)$ in terms of $y_a(s)$. To verify that these are equilibrium allocations, it only remains to show that the market clearing conditions,

$$\sum_{j=1}^{J} c_j^*(s) = \sum_{j=1}^{J} y_j(s) = J y_a(s)$$

hold for all $s = 1, 2, \ldots, S$, but this is easy. Your solutions for $c_j^*(s)$, $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$, here should coincide those that you derived from the social planner's problem in question 2. This confirms that the welfare theorems hold and, more specifically, that the time $t = 0$ markets for contingent claims allow each consumer to insure against idiosyncratic but not aggregate risk.

Solutions to Midterm Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2024

Due Thursday, November 7

1. Profit Maximization

The profit-maximizing firm solves

$$\max_n 10n - n^2 - wn \text{ subject to } n \geq 0.$$

a. With the Lagrangian defined as

$$L(n, \lambda) = 10n - n^2 - wn + \lambda n,$$

the Kuhn-Tucker conditions describing the solution to the firm's problem are the first-order condition

$$10 - 2n^* - w + \lambda^* = 0.$$

the constraint

$$n^* \geq 0,$$

the nonnegativity conditiion

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^* n^* = 0.$$

Suppose first that $n^* = 0$. Then the first-order condition requires

$$\lambda^* = w - 10,$$

which is greater than or equal to zero as required by the nonnegativity condition only when $w \geq 10$. Suppose on the other hand that $n^* > 0$. Then the complementary slackness condition requires $\lambda^* = 0$ and the first-order conditions implies that

$$n^* = \frac{10 - w}{2},$$

which is greater than or equal to zero as required by the constraint only when $0 < w \leq 10$. Combining these observations shows that $n^* = 0$ if $w \geq \bar{w}$, where $\bar{w} = 10$.

b. The analysis from above also shows that when $0 < w < \bar{w} = 10$, the firm's labor demand curve is

$$n(w) = \frac{10 - w}{2}.$$

c. With the profit function defined as

$$\pi(w) = \max_n 10n - n^2 - wn \text{ subject to } n \geq 0,$$

the results derived above imply that

$$\pi(w) = 0$$

for $w \geq \bar{w}$ and

$$\pi(w) = (10 - w)\left(\frac{10 - w}{2}\right) - \left(\frac{10 - w}{2}\right)^2 = \frac{1}{4}(10 - w)^2,$$

for $0 < w < \bar{w}$.

d. Differentiating the expression for $\pi(w)$ derived above with respect to w yields

$$\pi'(w) = -\frac{10 - w}{2}.$$

for all $w < \bar{w} = 10$. Note that this same result follows from the envelope theorem:

$$\pi'(w) = -n(w) = -\frac{10 - w}{2}.$$

2. Optimal Risk Sharing

At $t = 0$, the social planner chooses consumptions $c_j(s)$, $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$, for all consumers in all states of the world to maximize

$$\sum_{j=1}^{J} \sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\},$$

subject to the aggregate resource constraints

$$\sum_{j=1}^{J} y_j(s) \geq \sum_{j=1}^{J} c_j(s)$$

for all $s = 1, 2, \ldots, S$. The Lagrangian for this problem can be written as

$$\sum_{j=1}^{J} \sum_{s=1}^{S} \pi(s) \left\{ -\frac{1}{\sigma} \exp[-\sigma c_j(s)] \right\} + \sum_{s=1}^{S} \lambda(s) \left[\sum_{j=1}^{J} y_j(s) - \sum_{j=1}^{J} c_j(s) \right].$$

a. Fixing values of j and s, differentiating the Lagrangian by $c_j(s)$ for those values of j, and s, and setting the result equal to zero to yields the first-order conditions

$$\pi(s) \exp[-\sigma c_j^*(s)] - \lambda^*(s) = 0$$

for all $j = 1, 2, \ldots, J$ and $s = 1, 2, \ldots, S$.

b. Rearranging each first-order condition yields

$$c_j^*(s) = \frac{\ln[\pi(s)] - \ln[\lambda^*(s)]}{\sigma}$$

for all $j = 1, 2, \dots, J$ and $s = 1, 2, \dots, S$.

c. Substituting these solutions for $c_j^*(s)$ into the binding constraint for state s yields

$$Jy_a(s) = \sum_{j=1}^{J} y_j(s) = \sum_{j=1}^{J} c_j^*(s) = J\left\{\frac{\ln[\pi(s)] - \ln[\lambda^*(s)]}{\sigma}\right\},$$

which implies

$$\lambda^*(s) = \pi(s)\exp[-\sigma y_a(s)]$$

for all $s = 1, 2, \dots, S$.

d. Substituting this solution for $\lambda^*(s)$ back into the previous expression for $c_j^*(s)$ provides the solutions

$$c_j^*(s) = y_a(s)$$

for all $j = 1, 2, \dots, J$ and $s = 1, 2, \dots, S$. This optimal allocation insures each consumer against the shock to his or her own individual level of productivity, but requires that all consumers reduce their consumption when average productivity falls.

3. Equilibrium Risk Sharing

At $t = 0$, each consumer chooses consumptions $c_j(s)$, $s = 1, 2, \dots, S$, to maximize

$$\sum_{s=1}^{S} \pi(s)\left\{-\frac{1}{\sigma}\exp[-\sigma c_j(s)]\right\},$$

subject to the budget constraint

$$\sum_{s=1}^{S} p(s)y_j(s) \geq \sum_{s=1}^{S} p(s)c_j(s).$$

The Lagrangian for this problem can be written as

$$\sum_{s=1}^{S} \pi(s)\left\{-\frac{1}{\sigma}\exp[-\sigma c_j(s)]\right\} + \lambda_j\left[\sum_{s=1}^{S} p(s)y_j(s) - \sum_{s=1}^{S} p(s)c_j(s)\right].$$

a. Fixing a value of s, differentiating the Lagrangian by $c_j(s)$ for that value of s, and setting the result equal to zero yields the first-order conditions

$$\pi(s)\exp[-\sigma c_j^*(s)] - \lambda_j^* p(s) = 0$$

for all $s = 1, 2, \dots, S$.

b. Rearranging each first-order condition yields

$$c_j^*(s) = \frac{\ln[\pi(s)] - \ln(\lambda_j^*) - \ln[p(s)]}{\sigma}$$

for all $s = 1, 2, \ldots, S$.

c. If, as conjectured,

$$p(s) = \pi(s)\exp[-\sigma y_a(s)],$$

for all $s = 1, 2, \ldots, S$, then the expression for $c_j^*(s)$ derived above simplifies to

$$c_j^*(s) = y_a(s) - \frac{\ln(\lambda_j^*)}{\sigma}.$$

Using this expression for $c_j^*(s)$, as well as the symmetry condition for initial wealth,

$$\sum_{s=1}^{S} p(s)y_j(s) = \sum_{s=1}^{S} p(s)y_a(s)$$

for all $j = 1, 2, \ldots, J$, consumer j's binding budget constraint implies

$$\sum_{s=1}^{S} p(s)y_a(s) = \sum_{s=1}^{S} p(s)y_j(s) = \sum_{s=1}^{S} p(s)c_j^*(s) = \sum_{s=1}^{S} p(s)y_a(s) - \sum_{s=1}^{S} p(s)\left[\frac{\ln(\lambda_j^*)}{\sigma}\right]$$

or, more simply,

$$0 = \left[\frac{\ln(\lambda_j^*)}{\sigma}\right]\sum_{s=1}^{S} p(s).$$

Since $\sigma > 0$ and $p(s) > 0$ for all $s = 1, 2, \ldots, S$, this condition only holds if $\ln(\lambda_j^*) = 0$ or $\lambda_j^* = 1$.

d. Substituting this solution for λ_j^* back into the expression for $c_j^*(s)$ confirms that

$$c_j^*(s) = y_a(s)$$

for all $s = 1, 2, \ldots, S$. In equilibrium, as under the optimal allocation, each consumer is able fully insure against the shock to his or her own individual level of productivity but not against aggregate fluctuations in reflected in average productivity.

Final Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2024

Due Tuesday, December 17

This exam has two questions on five pages; before you begin, please check to make sure that your copy has both questions and all five pages. The two questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, my notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class. The answers you submit must be yours and yours alone.

1. Technical Knowledge Spillovers and Long-Run Economic Growth

As we've seen, the Ramsey (neoclassical growth) model implies that, in the absence of exogenous technological change, the economy converges to a steady state in which consumption and the capital stock remain constant. An interesting and influential modification to that model, suggested by Paul Romer ("Increasing Returns and Long-Run Growth," *Journal of Political Economy*, Vol.94, October 1986), endogenizes the process of technological change to obtain a variant that is consistent with continuing long-run economic growth. A special case of Romer's model, considered here, is one in which the differential equations implied by the maximum principle can be solved, subject to boundary conditions supplied by the initial capital stock and the transversality condition, so as to characterize explicitly the equilibrium growth path.

Suppose that, as in the version of the Ramsey model we studied in class, the representative consumer's preferences are described by the utility function

$$\int_0^\infty e^{-\rho t} \ln(c(t)) \, \mathrm{d}t, \tag{1}$$

where $c(t)$ is consumption at time $t \in [0, \infty)$ and the discount rate satisfies $\rho > 0$. Suppose also that output produced by the representative consumer at each time t is a Cobb-Douglas function of the stock of capital $k(t)$ owned by the consumer and the economy-wide stock of technical knowledge $x(t)$, which each consumer takes as given. The representative consumer's capital stock is therefore governed by

$$k(t)^\alpha x(t)^{1-\alpha} - \delta k(t) - c(t) \geq \dot{k}(t) \tag{2}$$

for all $t \in [0, \infty)$, where the share parameter and the depreciation rate both lie between zero and one: $0 < \alpha < 1$ and $0 < \delta < 1$. Given his or her initial capital stock $k(0)$, the consumer chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize the utility function (1) subject to the constraint (2) for $t \in [0, \infty)$.

a. Define (write down) the maximized Hamiltonian for the consumer's problem, and use the maximum principle to derive the set of optimality conditions (the first-order condition and pair of differential equations) that describe the solution to this problem. *Note:* For this problem, you can use whichever form of the Hamiltonian – present or current value – you find most convenient. Either way, however, just make sure the optimality conditions you write down are consistent with your choice.

b. Now suppose, following Romer but departing from the original Ramsey model, that instead of growing exogenously, the stock of technical knowledge expands through a process of learning-by-doing, modeled here through the simple assumption that

$$x(t) = k(t) \tag{3}$$

for all $t \in [0, \infty)$. Note that in solving the representative consumer's problem, above, we assumed that the individual consumer takes $x(t)$ as given, ignoring its dependence on $k(t)$. Thus, with (2) and (3), we're assuming that private capital accumulation provides a positive externality, by increasing the stock of technical knowledge that all other consumers and producers benefit from. The presence of this positive externality means that equilibrium and optimal resource allocations no longer coincide. Here, we're focusing on equilibrium allocations. Substitute the equilibrium condition (3) into the differential equations you derived from the maximum principle in part (a), above. Then, as we did for the Ramsey model in class, use the first-order condition to eliminate reference to the multiplier in the differential equations. The result will be a pair of differential equations: the Euler equation describing the growth rate of consumption $\dot{c}(t)/c(t)$ and the capital accumulation constraint describing the behavior of $\dot{k}(t)$.

c. The Euler equation you just derived in part (b) should reveal that, in this version of Romer's model, equilibrium consumption growth is constant, with

$$\frac{\dot{c}(t)}{c(t)} = g \tag{4}$$

for all $t \in [0, \infty)$. Use the Euler equation to express the constant consumption growth rate g in terms of the model's parameters: ρ, α, and δ.

d. The constant growth rate for consumption summarized by equation (4) implies that

$$c(t) = c(0)e^{gt} \tag{5}$$

for $t \in [0, \infty)$. If you substitute (5) into the differential equation for capital that you derived in part (b), you should get

$$\dot{k}(t) = (1 - \delta)k(t) - c(0)e^{gt}. \tag{6}$$

It turns out that the differential equation in (6) has a general solution of the form

$$k(t) = Ae^{(1-\delta)t} + \left[\frac{c(0)}{1-\delta-g}\right] e^{gt}, \tag{7}$$

for any value of A. To see this, differentiate both sides of (7) with respect to time:

$$\begin{aligned}
\dot{k}(t) &= (1-\delta)Ae^{(1-\delta)t} + g\left[\frac{c(0)}{1-\delta-g}\right] e^{gt} \\
&= (1-\delta)\left\{k(t) - \left[\frac{c(0)}{1-\delta-g}\right] e^{gt}\right\} + g\left[\frac{c(0)}{1-\delta-g}\right] e^{gt} \\
&= (1-\delta)k(t) - (1-\delta-g)\left[\frac{c(0)}{1-\delta-g}\right] e^{gt} \\
&= (1-\delta)k(t) - c(0)e^{gt}
\end{aligned} \tag{8}$$

as required by (6). The general solutions (5) and (7) involve two as-yet unknown values, $c(0)$ and A, which must be pinned down using the model's two boundary conditions: the initial condition for the capital stock $k(0)$ and the transversality condition from the representative consumer's dynamic optimization problem. Regardless of whether you set up the Hamiltonian for the consumer's problem in present or current value form, the transversality condition, when combined with the first-order condition for $c(t)$, requires that

$$\lim_{t\to\infty} e^{-\rho t}\left[\frac{1}{c(t)}\right] k(t) = 0. \tag{9}$$

Substituting the general solutions (5) and (7) into (9) yields

$$\begin{aligned}
0 &= \lim_{t\to\infty}\left[\frac{1}{c(0)}\right] e^{-(g+\rho)t}\left\{Ae^{(1-\delta)t} + \left[\frac{c(0)}{1-\delta-g}\right] e^{gt}\right\} \\
&= \left[\frac{1}{c(0)}\right] \lim_{t\to\infty}\left\{Ae^{(1-\delta-g-\rho)t} + \left[\frac{c(0)}{1-\delta-g}\right] e^{-\rho t}\right\}
\end{aligned} \tag{10}$$

The second term inside brackets in (10), involving $e^{-\rho t}$, goes to zero as $t \to \infty$ since $\rho > 0$. Thus, (10) requires

$$0 = \frac{A}{c(0)} \lim_{t\to\infty} e^{(1-\delta-g-\rho)t} \tag{11}$$

Use the formula for the constant growth rate of consumption g that you derived in part (c), above, to verify that the exponential growth rate in (11) satisfies

$$1 - \delta - g - \rho > 0. \tag{12}$$

Note: We can conclude from (11) and (12) that the transversality condition requires $A = 0$. Now, it only remains to pin down $c(0)$. But this can be done by returning to (7), after setting $A = 0$ as required by the TVC. For $t = 0$, this condition requires

$$c(0) = (1 - \delta - g)k(0). \tag{13}$$

To summarize: with $k(0)$ given and $c(0)$ determined by (13), consumption and the capital stock both grow at the constant rate g is equilibrium, according to (5) and (7) with $A = 0$.

2. Stochastic Growth with Labor Supply and Serially Correlated Shocks

This problem asks you to use dynamic programming to solve a version of the stochastic growth model from problem set 14, extended to include an optimal labor supply decision and serially correlated productivity shocks. Maintaining the assumption that physical capital depreciates fully between periods continues to allow an explicit solution for the value function to be found using the guess-and-verify method.

In this version of the model, the representative consumer chooses contingency plans for consumption c_t and labor supply n_t (for "number of hours worked") for all $t = 0, 1, 2, \ldots$ and physical capital k_t for all $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln(c_t) - \gamma n_t], \tag{14}$$

where the discount factor and the constant disutility of labor satisfy $0 < \beta < 1$ and $\gamma > 0$, subject to the constraints k_0 given and

$$z_t k_t^\alpha n_t^{1-\alpha} \geq c_t + k_{t+1}, \tag{15}$$

where the share parameter in the Cobb-Douglas production function satisfies $0 < \alpha < 1$. The natural log of productivity shock z_t follows the autoregressive process

$$\ln(z_{t+1}) = \rho \ln(z_t) + \varepsilon_{t+1}, \tag{16}$$

with persistence parameter satisfying $0 \leq \rho < 1$, where the serially uncorrelated innovation ε_{t+1} satisfies $E_t(\varepsilon_{t+1}) = 0$ for all $t = 0, 1, 2, \ldots$. Hence, the value of z_t is known when c_t, n_t, and k_{t+1} are chosen during period t, but the value of z_{t+1}, though partly forecastable via (16), still contains the element of randomness due to ε_{t+1}. The constraint in (15) must hold for all $t = 0, 1, 2, \ldots$ and all possible realizations of z_t.

For this problem, the Bellman equation takes the general form

$$v(k_t, z_t) = \max_{c_t, n_t} \ln(c_t) - \gamma n_t + \beta E_t v(z_t k_t^\alpha n_t^{1-\alpha} - c_t, z_{t+1}).$$

Using the same conjectured form of the value function as in problem set 14, namely,

$$v(k_t, z_t) = E + F \ln(k_t) + G \ln(z_t)$$

the Bellman equation becomes

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t, n_t} \ln(c_t) - \gamma n_t + \beta E + \beta F \ln(z_t k_t^\alpha n_t^{1-\alpha} - c_t) + \beta G \rho \ln(z_t), \tag{17}$$

where, in the very last term on the right-hand side, use has been made of (16), which implies

$$E_t[\ln(z_{t+1})] = \rho \ln(z_t).$$

a. Starting from (17), derive (write down) the first-order conditions for c_t and n_t and the envelope condition for k_t that help characterize the solution to the consumer's problem.

b. Next, use the first-order condition for c_t, the envelope condition for k_t, and the binding constraint

$$k_{t+1} = z_t k_t^\alpha n_t^{1-\alpha} - c_t$$

to derive expressions that show how the constant F from the value function depends on the parameters α and β and how optimal consumption and savings are determined as constant fractions of output, with

$$c_t = (1 - s) z_t k_t^\alpha n_t^{1-\alpha}$$

and

$$k_{t+1} = s z_t k_t^\alpha n_t^{1-\alpha},$$

where s also depends on the parameters α and β.

c. Now use the first-order condition for n_t to show that optimal labor supply is a constant, which depends on the parameters γ, α, and β.

d. Finally, substitute your solutions for F, c_t, n_t, and k_t back into the Bellman equation (17) and, by observing that the equation must hold for all values of $\ln(z_t)$, use the result to find a solution for G in terms the parameters α, β, and ρ.

Note: It is also possible – but not necessary for the purposes of this exam – to use (17) to find the solution for E in terms of γ, α, and β.

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2024

Due Tuesday, December 17

1. Technical Knowledge Spillovers and Long-Run Economic Growth

The representative consumer takes $k(0)$ as given and chooses $c(t)$ for all $t \in [0, \infty)$ and $k(t)$ for all $t \in (0, \infty)$ to maximize

$$\int_0^\infty e^{-\rho t} \ln(c(t)) \; \mathrm{d}t, \tag{1}$$

subject to the constraint

$$k(t)^\alpha x(t)^{1-\alpha} - \delta k(t) - c(t) \geq \dot{k}(t) \tag{2}$$

for $t \in [0, \infty)$.

a. In present-value form, the maximized Hamiltonian for the consumer's problem is

$$H(k(t), \pi(t); t) = \max_{c(t)} e^{-\rho t} \ln(c(t)) + \pi(t)[k(t)^\alpha x(t)^{1-\alpha} - \delta k(t) - c(t)].$$

The maximum principle implies that the solution to the problem is described by the first-order condition

$$\frac{e^{-\rho t}}{c(t)} - \pi(t) = 0$$

and the pair of differential equations

$$\dot{\pi(t)} = -H_k(k(t), \pi(t); t) = -\pi(t)[\alpha k(t)^{\alpha-1} x(t)^{1-\alpha} - \delta]$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = k(t)^\alpha x(t)^{1-\alpha} - \delta k(t) - c(t).$$

b. Substituting the equilibrium condition

$$x(t) = k(t) \tag{3}$$

into the differential equations yields

$$\dot{\pi(t)} = -\pi(t)(\alpha - \delta)$$

and

$$\dot{k}(t) = H_\pi(k(t), \pi(t); t) = (1 - \delta)k(t) - c(t).$$

The first-order condition for $c(t)$, when rewritten

$$\pi(t)c(t) = e^{-\rho t}$$

and differentiated with respect to t, implies that

$$\dot{\pi}(t)c(t) + \pi(t)\dot{c}(t) = -\rho\pi(t)c(t).$$

Using the differential equation to eliminate $\dot{\pi}(t)$,

$$-\pi(t)(\alpha - \delta)c(t) + \pi(t)\dot{c}(t) = -\rho\pi(t)c(t).$$

Dividing through by $\pi(t)$ and rearranging then leads to the Euler equation

$$\dot{c}(t) = (\alpha - \delta - \rho)c(t),$$

which combines with the capital accumulation equation

$$\dot{k}(t) = (1 - \delta)k(t) - c(t)$$

to describe the equilibrium paths for consumption and the capital stock.

c. The Euler equation reveals that

$$\frac{\dot{c}(t)}{c(t)} = g \tag{4}$$

for all $t \in [0, \infty)$, where the constant consumption growth rate is

$$g = \alpha - \delta - \rho.$$

d. Using the solution for g just derived, the inequality

$$1 - \delta - g - \rho > 0 \tag{12}$$

must hold since

$$1 - \delta - g - \rho = 1 - \delta - \alpha + \delta + \rho - \rho = 1 - \alpha > 0.$$

We can now conclude from (11) and (12) that the transversality condition requires $A = 0$. And for $t = 0$, (7) then requires

$$c(0) = (1 - \delta - g)k(0). \tag{13}$$

To summarize: with $k(0)$ given and $c(0)$ determined by (13), consumption and the capital stock both grow at the constant rate g is equilibrium, according to (5) and (7) with $A = 0$.

2. Stochastic Growth with Labor Supply and Serially Correlated Shocks

The representative consumer chooses contingency plans for consumption c_t and labor supply n_t for all $t = 0, 1, 2, \ldots$ and physical capital k_t for all $t = 1, 2, 3, \ldots$ to maximize the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln(c_t) - \gamma n_t], \tag{14}$$

subject to the constraints k_0 given and

$$z_t k_t^{\alpha} n_t^{1-\alpha} \geq c_t + k_{t+1}, \tag{15}$$

for all $t = 0, 1, 2, \ldots$ and all possible realizations of z_t.

Using the conjectured form of the value function, the Bellman equation for this problem becomes

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t, n_t} \ln(c_t) - \gamma h_t + \beta E + \beta F \ln(z_t k_t^{\alpha} n_t^{1-\alpha} - c_t) + \beta G \rho \ln(z_t). \tag{17}$$

a. Starting from (17), the first-order condition for c_t is

$$\frac{1}{c_t} - \frac{\beta F}{z_t k_t^{\alpha} n_t^{1-\alpha} - c_t} = 0,$$

the first-order condition for n_t is

$$-\gamma + \frac{(1-\alpha)\beta F z_t k_t^{\alpha} n_t^{-\alpha}}{z_t k_t^{\alpha} n_t^{1-\alpha} - c_t} = 0,$$

and the envelope condition for k_t is

$$\frac{F}{k_t} = \frac{\alpha \beta F z_t k_t^{\alpha-1} n_t^{1-\alpha}}{z_t k_t^{\alpha} n_t^{1-\alpha} - c_t}.$$

b. The first-order condition for c_t can be rearranged to yield

$$c_t = \left(\frac{1}{1+\beta F}\right) z_t k_t^{\alpha} n_t^{1-\alpha}.$$

Substituting this expression for c_t into the envelope condition for k_t and simplifying yields

$$\frac{1}{1+\beta F} = 1 - \alpha\beta$$

Hence,

$$F = \frac{\alpha}{1-\alpha\beta},$$

$$c_t = (1-\alpha\beta) z_t k_t^{\alpha} n_t^{1-\alpha},$$

and, using the binding constraint,

$$k_{t+1} = \alpha\beta z_t k_t^\alpha n_t^{1-\alpha}.$$

These last two results confirm that the consumer optimally chooses to consume and save the fixed fractions $1 - s$ and s of output, where $s = \alpha\beta$.

c. Using the solutions for F and c_t, the first-order condition for n_t implies

$$(1 - \alpha\beta)\gamma\alpha\beta z_t k_t^\alpha n_t^{1-\alpha} = (1 - \alpha)\beta\alpha z_t k_t^\alpha n_t^{-\alpha}.$$

Simplifying yields the solution for the constant labor supply:

$$n_t = n = \frac{1}{\gamma}\left(\frac{1-\alpha}{1-\alpha\beta}\right).$$

d. Finally, substituting the solutions for F, c_t, n_t, and k_t back into the Bellman equation (17) yields

$$\begin{aligned} E + G\ln(z_t) \quad = \quad & \ln(1-\alpha\beta) + \ln(z_t) + (1-\alpha)\ln(n) - \gamma n + \beta E \\ & + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(z_t) \\ & + (1-\alpha)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(n) + \beta G\rho\ln(z_t) \end{aligned}$$

after cancelling out the terms in $\ln(k_t)$ that appear on both sides. Since this last equation must hold for all values of $\ln(z_t)$, it requires

$$G = 1 + \frac{\alpha\beta}{1-\alpha\beta} + \beta G\rho,$$

which provides the solution for

$$G = \frac{1}{(1-\alpha\beta)(1-\beta\rho)}.$$

For the sake of completeness, one can also use the Bellman equation to solve for

$$E = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta) - \gamma n + \left(\frac{1-\alpha}{1-\alpha\beta}\right)\ln(n)\right],$$

where n depends on γ, α, and β as shown above.

Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2025

Due Tuesday, October 28

This exam has three questions on five pages. Please check to make sure that your copy has all three questions and all five pages. The questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Marshallian Demands

Consider a utility-maximizing consumer, who solves

$$\max_{c_1,c_2} U(c_1,c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2,$$

where c_1 and c_2 denote consumption of two goods, U is a utility function that describes the consumer's preferences, I is the consumer's income, and p_1 and p_2 are the prices at which the consumer purchases the two goods in perfectly competitive markets.

a. As we discussed in class, the Marshallian demand curves $c_1(p_1,p_2,I)$ and $c_2(p_1,p_2,I)$ describe how the consumer's optimal choices of c_1 and c_2 depend on the prices p_1 and p_2 and income I. Suppose first that the consumer's utility function takes the form

$$U(c_1,c_2) = c_1^a c_2^b,$$

where $a > 0$ and $b > 0$, and derive expressions for the Marshallian demands $c_1(p_1,p_2,I)$ and $c_2(p_1,p_2,I)$. *Note:* Strictly speaking, the consumer's choices must satisfy the nonnegativity constraints $c_1 \geq 0$ and $c_2 \geq 0$ as well as the budget constraint. But since these nonnegativity constraints won't bind at the optimum, it's not necessary to consider them explicitly in any of the three cases described here and below.

b. Suppose instead that the consumer's utility function takes the form

$$U(c_1,c_2) = 125 c_1^{75} c_2^{50}.$$

What are the Marshallian demands $c_1(p_1,p_2,I)$ and $c_2(p_1,p_2,I)$ in this case? *Hint:* Before you solve the consumer's optimization problem all over again, think about the relationship between the preferences described by the utility function here, in part (b), and those described by the utility function above, in part (a).

c. As we also discussed in class, the indirect utility function $V(p_1, p_2, I)$ is the maximum value function defined by the solution to the consumer's utility-maximization problem:

$$V(p_1, p_2, I) = \max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2.$$

Suppose that the indirect utility function takes the form

$$V(p_1, p_2, I) = \frac{I}{\sqrt{p_1 p_2}}.$$

What are the Marshallian demand curves $c_1(p_1, p_2, I)$ and $c_2(p_1, p_2, I)$ in this case? *Hint:* Use the envelope theorem or, in this case more specifically, Roy's identity.

2. Market Work, Housework, and Leisure

In 1965, Gary Becker developed "A Theory of the Allocation of Time" (*Economic Journal*) in which a consumer combines goods purchased in the market with labor applied at home to produce the composite goods that enter the utility function describing his or her preferences. For example, the consumer might use groceries bought at the supermarket together with time spent in the kitchen preparing a lunch or dinner.

In a 2006 paper, Mark Aguiar and Erik Hurst ("Measuring Trends in Leisure: The Allocation of Time over Five Decades," Federal Reserve Bank of Boston Working Paper 06-2) presented a nice illustrative example that shows how, in a Beckerian model of household production, the consumer's optimal allocation of time depends on the parameters of a household technology as well as the parameters describing the consumer's preferences. This example was dropped from the published version of Aguiar and Hurst's paper (*Quarterly Journal of Economics* 2007), perhaps because it's implications don't quite match the data that are the paper's main focus. But the example is still an interesting one for us to consider here.

In this example, the household uses c units of a good purchased in the market (groceries) with h units of home labor (housework) to produce Z units of a composite commodity (lunches or dinners) according to the production function

$$Z = \left(c^{\frac{\sigma-1}{\sigma}} + h^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $\sigma > 0$ measures the constant elasticity of substitution between the market good and housework in producing the composite. The household also supplies m units of labor to the market, getting paid at the real wage w, and then enjoys

$$L = 1 - h - m \tag{2}$$

units of pure leisure. In (2), the consumer's time endowment is normalized to equal one; hence, h, m, and L should be interpreted as the fractions of total time spent doing housework, supplying labor to the market, and enjoying leisure. The consumer's preferences over the composite good Z and leisure L are described by the utility function

$$\alpha \ln(Z) + (1-\alpha)\ln(L), \tag{3}$$

where $0 < \alpha < 1$. Finally, the consumer's budget constraint requires his or her labor income to cover the cost of his or her market purchases:

$$wm \geq c,$$

where the price of c has also been normalized to one since w measures the real wage.

After substituting (1) and (2) into (3), the consumer's utility maximization problem can be written:

$$\max_{c,h,m} \ \alpha \left(\frac{\sigma}{\sigma-1}\right) \ln\left(c^{\frac{\sigma-1}{\sigma}} + h^{\frac{\sigma-1}{\sigma}}\right) + (1-\alpha)\ln(1-h-m) \text{ subject to } wm \geq c.$$

a. Derive (write down) the first-order conditions for c, h, and m that help describe the solution to the consumer's problem.

b. Use your first-order conditions, together with the binding constraint $wm^* = c^*$, to solve for the consumer's optimal choices of market purchases c^*, housework h^*, market work m^*, and leisure $L^* = 1 - h^* - m^*$ in terms of the preference parameter α, the technology parameter σ, and the real wage w.

c. Use your solutions to confirm that when the real wage w rises:

   i. Market purchases c^* increase,

   ii. Leisure L^* stays constant,

   iii. Housework h^* rises if $\sigma < 1$, stays constant if $\sigma = 1$, and falls if $\sigma > 1$, and

   iv. Market work m^* falls if $\sigma < 1$, stays constant if $\sigma = 1$, and rises if $\sigma > 1$.

3. Optimal Tariffs

A paper by Richard F. Kahn ("Tariffs and the Terms of Trade," *Review of Economic Studies* 1947-48) derives what has become a famous result in international economics, relating the welfare-maximizing level of tariffs to the supply elasticity of imported goods. This result can be re-derived here with the help of an example taken from Narayana Kocherlakota's "Optimal Tariffs when Labor Income Taxes are Distortionary" (National Bureau of Economic Research Working Paper 33759, May 2025), which follows our own work from class more closely by comparing the optimality conditions describing the solution to a social planner's problem to those describing the solution to a representative consumer's problem in a decentralized equilibrium.

In this example, each consumer in an open economy supplies n units of labor to produce y_D units of a domestic good according to the simple, linear production function $y_D = n$. Each consumer also purchases c_F units of a foreign good, imported at price $p(c_F)$ where, at the level of the economy as a whole, the import price rises as the quantity imported increases, so that $p'(c_F) > 0$. Each consumer's preferences over consumption of the domestic and foreign good and over labor supplied are described by the utility function

$$U(c_D, c_F) - V(n)$$

where U is strictly increasing in both of its arguments and V is strictly increasing in its single argument.

What distinguishes the social planner's problem from the representative consumer's problem in this economy is that the social planner explicitly accounts for the positive supply elasticity of the foreign good, that is, the fact that the supply price $p(c_F)$ increases when c_F rises. Each individual consumer, by contrast, takes the price of the foreign good $p_F = p(c_F)$ as given. This difference provides the role for welfare-enhancing tariffs.

a. Consider first the social planner's problem: Choose c_D, c_F, and n to maximize

$$U(c_D, c_F) - V(n)$$

subject to the economy-wide resource constraint

$$n \geq c_D + p(c_F)c_F,$$

which uses the domestic good as numeraire, so that $p(c_F)$ is the price of the foreign good relative to the domestic good. Accounting for the positively-sloped curve $p(c_F)$, derive (write down) the first-order conditions that, together with the binding resource constraint, determine the planner's optimal choices c_D^*, c_F^*, and n^*.

b. Next consider the problem solved by a single utility-maximizing consumer, who takes the price p_F of the foreign good as given. But suppose as well that this consumer must pay a tariff τ on his or her purchases of the foreign good and also receives a lump-sum transfer T from the government. Then the consumer chooses c_D, c_F, and n to maximize

$$U(c_D, c_F) - V(n)$$

subject to the budget constraint

$$T + n \geq c_D + (1 + \tau)p_F c_F.$$

Derive (write down) the first-order conditions that, together with the binding budget constraint, determine the consumer's optimal choices c_D^{**}, c_F^{**}, and n^{**}.

c. Finally, note that in the decentralized equilibrium from part (b), the price of the foreign good paid by all individual consumers must be $p_F = p(c_F^{**})$. Assume, as well, that to support this equilibrium, the government rebates all of its tax revenue, so that

$T = \tau p(c_F^{**})c_F^{**}$. Then, by comparing the conditions from part (a) describing the socially optimal allocations with the conditions from part (b) describing the individually optimal allocations, show that the optimal tariff satisfies

$$\tau = \tau^* = \frac{p'(c_F^*)c_F^*}{p(c_F^*)}.$$

Since $p(c_F)$ is the inverse supply curve (that is, the supply relationship expressing how price depends on quantity instead of the other way around), the right-hand side of this expression for the optimal tariff τ^* equals the inverse of the supply elasticity of the imported good: this reproduces Kahn's (1947) result.

Solutions to Midterm Exam

ECON 772001 - Math for Economists | Peter Ireland
Boston College, Department of Economics | Fall 2025

Due Tuesday, October 28

1. Marshallian Demands

The utility-maximizing consumer solves

$$\max_{c_1,c_2} U(c_1,c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2.$$

a. With

$$U(c_1,c_2) = c_1^a c_2^b,$$

the Lagrangian for the consumer's problem is

$$L(c_1,c_2,\lambda) = c_1^a c_2^b + \lambda(I - p_1c_1 + p_2c_2),$$

and the first-order conditions for the values c_1^* and c_2^* that solve this problem together with the associated value λ^* are

$$ac_1^{*a-1}c_2^{*b} - \lambda^* p_1 = 0$$

and

$$bc_1^{*a}c_2^{*b-1} - \lambda^* p_2 = 0.$$

Since the utility function is strictly increasing in both arguments, the budget constraint will bind at the optimum. Hence, c_1^*, c_2^*, and λ^* must also satisfy

$$I = p_1c_1^* + p_2c_2^*.$$

There are a variety of ways of using these optimality conditions to find the Marshallian demand curves, but probably the easiest is to divide the first-order condition for c_1^* by the first-order condition for c_2^* so as to eliminate λ^*:

$$\frac{ac_2^*}{bc_1^*} = \frac{p_1}{p_2}.$$

Solve this expression for

$$c_2^* = \left(\frac{bp_1}{ap_2}\right) c_1^*,$$

which can be substituted into the binding constraint to get

$$I = p_1c_1^* + p_2\left(\frac{bp_1}{ap_2}\right) c_1^* = \left(\frac{a+b}{a}\right) p_1c_1^*.$$

This last expression can be rearranged to obtain the Marshallian demand curve for good 1:

$$c_1^*(p_1, p_2, I) = \left(\frac{a}{a+b}\right)\frac{I}{p_1}.$$

Finally, substituting this result back into the previous solution for c_2^* yields the demand curve for good 2:

$$c_2^*(p_1, p_2, I) = \left(\frac{b}{a+b}\right)\frac{I}{p_2}.$$

b. Suppose instead that the consumer's utility function takes the form

$$U(c_1, c_2) = 125c_1^{75}c_2^{50}.$$

The 125 in this expression is irrelevant: it simply rescales the magnitude of the utility function without changing the preference ordering. Hence, the Marshallian demand curves for this case can be found simply by substituting $a = 75$ and $b = 50$ into the more general expressions derived in part (a), above:

$$c_1^*(p_1, p_2, I) = \left(\frac{3}{5}\right)\frac{I}{p_1}.$$

and

$$c_2^*(p_1, p_2, I) = \left(\frac{2}{5}\right)\frac{I}{p_2}.$$

c. Finally, suppose that the indirect utility function defined by the solution to the consumer's utility-maximization problem,

$$V(p_1, p_2, I) = \max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1c_1 + p_2c_2.,$$

takes the form

$$V(p_1, p_2, I) = \frac{I}{\sqrt{p_1p_2}}.$$

Roy's identity allows the Marshallian demand curves to be recovered as

$$c_1^*(p_1, p_2, I) = -\frac{\frac{\partial V(p_1,p_2,I)}{\partial p_1}}{\frac{\partial V(p_1,p_2,I)}{\partial I}} = -\frac{-\frac{1}{2}Ip_1^{-3/2}p_2^{-1/2}}{p_1^{-1/2}p_2^{-1/2}} = \left(\frac{1}{2}\right)\frac{I}{p_1}$$

and

$$c_2^*(p_1, p_2, I) = -\frac{\frac{\partial V(p_1,p_2,I)}{\partial p_2}}{\frac{\partial V(p_1,p_2,I)}{\partial I}} = -\frac{-\frac{1}{2}Ip_1^{-1/2}p_2^{-3/2}}{p_1^{-1/2}p_2^{-1/2}} = \left(\frac{1}{2}\right)\frac{I}{p_2}.$$

2. Market Work, Housework, and Leisure

The utility-maximizing consumer solves

$$\max_{c,h,m} \alpha\left(\frac{\sigma}{\sigma-1}\right)\ln\left(c^{\frac{\sigma-1}{\sigma}} + h^{\frac{\sigma-1}{\sigma}}\right) + (1-\alpha)\ln(1-h-m) \text{ subject to } wm \geq c.$$

a. The Lagrangian for the consumer's problem is

$$L(c, h, m, \lambda) = \alpha\left(\frac{\sigma}{\sigma-1}\right)\ln\left(c^{\frac{\sigma-1}{\sigma}} + h^{\frac{\sigma-1}{\sigma}}\right) + (1-\alpha)\ln(1-h-m) + \lambda(wm - c),$$

and the first-order conditions are

$$\frac{\alpha c^{*-\frac{1}{\sigma}}}{c^{*\frac{\sigma-1}{\sigma}} + h^{*\frac{\sigma-1}{\sigma}}} - \lambda^* = 0,$$

$$\frac{\alpha h^{*-\frac{1}{\sigma}}}{c^{*\frac{\sigma-1}{\sigma}} + h^{*\frac{\sigma-1}{\sigma}}} - \frac{1-\alpha}{1-h^*-m^*} = 0,$$

and

$$-\frac{1-\alpha}{1-h^*-m^*} + \lambda^* w = 0.$$

b. Substitute the first-order condition for m^* into the first-order condition for h^* and divide the result by the first-order condition for c^* to get

$$\left(\frac{c^*}{h^*}\right)^{\frac{1}{\sigma}} = w$$

or

$$c^* = w^\sigma h^*.$$

Next, substitute this result back into the first-order condition for h^* to get

$$\frac{\alpha h^{*-\frac{1}{\sigma}}}{h^{*\frac{\sigma-1}{\sigma}}(1+w^{\sigma-1})} - \lambda^* w = 0$$

or

$$h^* = \left(\frac{\alpha}{w\lambda^*}\right)\left(\frac{1}{1+w^{\sigma-1}}\right).$$

Substitute this result back into the previous expression for c^* to get

$$c^* = \left(\frac{\alpha w^{\sigma-1}}{\lambda^*}\right)\left(\frac{1}{1+w^{\sigma-1}}\right).$$

and into the first-order condition for m^* to get

$$m^* = 1 - \frac{1}{w\lambda^*}\left(\frac{\alpha}{1+w^{\sigma-1}} + 1 - \alpha\right).$$

Now use these expressions for c^* and m^* together with the binding constraint to solve for

$$\lambda^* = 1/w.$$

Finally, using this solution for $\lambda^* = 1/w$,

$$c^* = \alpha\left(\frac{w^\sigma}{1+w^{\sigma-1}}\right),$$

$$h^* = \alpha \left(\frac{1}{1 + w^{\sigma-1}} \right),$$

$$m^* = \alpha \left(\frac{w^{\sigma-1}}{1 + w^{\sigma-1}} \right),$$

and

$$L^* = 1 - \alpha.$$

c. These solutions imply that

$$\frac{\partial c^*}{\partial w} = \alpha \left[\frac{\sigma w^{\sigma-1} + w^{2\sigma-2}}{(1 + w^{\sigma-1})^2} \right] > 0,$$

$$\frac{\partial L^*}{\partial w} = 0,$$

$$\frac{\partial h^*}{\partial w} = -\alpha \left[\frac{(\sigma - 1)w^{\sigma-2}}{(1 + w^{\sigma-1})^2} \right],$$

and

$$\frac{\partial m^*}{\partial w} = \alpha \left[\frac{(\sigma - 1)w^{\sigma-2}}{(1 + w^{\sigma-1})^2} \right].$$

These last two results confirm that when w rises, housework rises if $\sigma < 1$, stays constant if $\sigma = 1$, and falls if $\sigma > 1$, while market work falls if $\sigma < 1$, stays constant if $\sigma = 1$, and rises if $\sigma > 1$.

3. Optimal Tariffs

The consumer's preferences are described by the utility function

$$U(c_D, c_F) - V(n),$$

the domestic good is produced with labor n according to the simple, one-for-one linear technology, and the foreign good is supplied at price $p(c_F)$, where $p'(c_F) > 0$.

a. The social planner explicitly accounts for the positively-sloped supply curve for the foreign good, solving

$$\max_{c_D, c_F, n} U(c_D, c_F) - V(n) \text{ subject to } n \geq c_D + p(c_F)c_F.$$

The Lagrangian for this problem is

$$L(c_D, c_F, n, \lambda) = U(c_D, c_F) - V(n) + \lambda[n - c_D - p(c_F)c_F].$$

The solution is described by the first-order conditions

$$U_1(c_D^*, c_F^*) - \lambda^* = 0,$$

$$U_2(c_D^*, c_F^*) - \lambda^*[p'(c_F^*)c_F^* + p(c_F^*)] = 0,$$

and

$$-V'(n^*) + \lambda^* = 0,$$

together with the binding constraint

$$n^* = c_D^* + p(c_F^*)c_F^*.$$

b. Each individual consumer takes the foreign good's price p_F as given, and solves

$$\max_{c_D, c_F, n} U(c_D, c_F) - V(n) \text{ subject to } T + n \geq c_D + (1+\tau)p_F c_F,$$

where τ is the tariff and T is the government transfer. The Lagrangian for this problem is

$$L(c_D, c_F, n, \lambda) = U(c_D, c_F) - V(n) + \lambda[T + n - c_D - (1+\tau)p_F c_F].$$

The solution is described by the first-order conditions

$$U_1(c_D^{**}, c_F^{**}) - \lambda^{**} = 0,$$

$$U_2(c_D^{**}, c_F^{**}) - \lambda^{**}(1+\tau)p_F = 0,$$

and

$$-V'(n^{**}) + \lambda^{**} = 0,$$

together with the binding constraint

$$T + n^{**} = c_D^{**} + (1+\tau)p_F c_F^{**}.$$

c. Substitute the equilibrium conditions $p_F = p(c_F^{**})$ and $T = \tau p(c_F^{**})c_F^{**}$ into the first-order conditions and constraint from part (b) describing the consumer's optimizing behavior to get

$$U_1(c_D^{**}, c_F^{**}) - \lambda^{**} = 0,$$

$$U_2(c_D^{**}, c_F^{**}) - \lambda^{**}(1+\tau)p(c_F^{**}) = 0,$$

$$-V'(n^{**}) + \lambda^{**} = 0,$$

and

$$n^{**} = c_D^{**} + p(c_F^{**})c_F^{**}.$$

These conditions will coincide with those from part (a), describing the social planner's optimal choices, if the tariff is set so that

$$(1+\tau)p(c_F^*) = p'(c_F^*)c_F^* + p(c_F^*)$$

or

$$\tau = \tau^* = \frac{p'(c_F^*)c_F^*}{p(c_F^*)}.$$

Final Exam

ECON 772001 - Math for Economists	Peter Ireland
Boston College, Department of Economics	Fall 2025

Due Thursday, December 18

This exam has two questions on three pages; before you begin, please check to make sure that your copy has both questions and all three pages. The two questions will be weighted equally in determining your overall exam score.

This is an open-book exam, meaning that it is fine for you to consult your notes, my notes, homeworks, textbooks, and other written or electronic references when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class. The answers you submit must be yours and yours alone.

1. The Community Lake

Michael Caputo (*Foundations of Dynamic Economic Analysis: Optimal Control Theory and Applications*, 2005) suggests the following example. Members of a community have access to a lake with water level $w(t)$. At each date $t \in [0, \infty)$, each community member gets utility from consuming $c(t)$ units of water from the lake, but also enjoys the amenity of the lake itself in proportion to the water level $w(t)$. In particular, the representative community member has preferences described by the utility function

$$\int_0^\infty e^{-\rho t}[\ln(c(t)) + \alpha \ln(w(t))]\, \mathrm{d}t$$

over the infinite horizon, where $\rho > 0$ is the discount rate and $\alpha > 0$ is the weight on the amenity flow from the lake itself relative to consumption of water from the lake. Given the initial stock of water $w(0)$ in the lake, the community planners choose $c(t)$ for all $t \in [0, \infty)$ and $w(t)$ for all $t \in (0, \infty)$ to maximize this infinite-horizon utility function subject to the constraint

$$R - c(t) \geq \dot{w}(t),$$

for all $t \in [0, \infty)$, where $R > 0$ is a parameter measuring the constant flow of rainfall that replenishes the water in the lake, even as consumption runs the water level down.

a. Define (write down) the maximized Hamiltonian for the community planners' problem, and use the maximum principle to derive the set of optimality conditions (the first-order condition and pair of differential equations) that describe the solution to this problem. *Note:* For this problem, you can use whichever form of the Hamiltonian – present or current value – you find most convenient. Either way, however, just make sure the optimality conditions you write down are consistent with your choice.

b. Next, as we did for the Ramsey model in class, use the first-order condition to eliminate reference to the multiplier that appears in the planners' optimality conditions. The result will be a pair of differential equations: the Euler equation describing the evolution of consumption $\dot{c}(t)$ and the binding constraint describing the evolution of the water level $\dot{w}(t)$.

c. The pair of differential equations you just derived in part (b) should reveal that, like the Ramsey model, this model has a unique nontrivial steady state in which consumption and the water level are both constant, with $c(t) = c^*$, $w(t) = w^*$, and hence $\dot{c}(t) = \dot{w}(t) = 0$. Use your equations to find expressions that show how the steady-state values c^* and w^* depend on the model's parameters: ρ, α, and R.

Note: It is not necessary for this exam to draw the phase diagram for this model. But if you do, the diagram will reveal that the model has the same stability properties as the Ramsey model. In particular, for any value $w(0) = w_0 > 0$, there exists a unique value $c(0) = c_0^*$ that puts the system on its stable manifold, converging to the unique steady state. Choices for $c(0)$ larger than c_0^* will exhaust the lake water in finite time; choices for $c(0)$ smaller than c_0^* lead to over-accumulation of lake water in violation of the transversality condition.

2. Investment and the Business Cycle

Jeremy Greenwood, Zvi Hercowitz, and Gregory Huffman ("Investment, Capital Utilization, the the Real Business Cycle," *American Economic Review*, 1988) develop a model in which stochastic fluctuations in the marginal efficiency of investment are a source of business cycles, that is, changes in consumption, investment, and output. This model is based on the observation that taking advantage of technological changes requires an economy to undertake new investment. The example presented here simplifies the model by assuming logarithmic utility, Cobb-Douglas production, and complete depreciation of physical capital; these simplifying assumptions make it possible to find the value function for the social planner's problem using the guess-and-verify method.

In this version of the model, the social planner takes the initial capital stock k_0 as given, and chooses contingency plans for consumption c_t and investment i_t for all $t = 0, 1, 2, \ldots$ and capital k_t for all $t = 1, 2, 3, \ldots$ to maximize a representative consumer's expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

where the discount factor satisfies $0 < \beta < 1$.

During each period $t = 0, 1, 2, \ldots$, the social planner divides aggregate output k_t^α, with $0 < \alpha < 1$, up into the amount c_t to be consumed and the amount i_t to be invested, subject to

$$k_t^\alpha \geq c_t + i_t$$

for all $t = 0, 1, 2, \ldots$. The amount of new productive capital k_{t+1} created through investment i_t depends on z_t, a random shock to the marginal efficiency of investment. In particular,

$$z_t i_t \geq k_{t+1}$$

for all $t = 0, 1, 2, \ldots$. The natural log of the investment-specific technology shock follows the autoregressive process

$$\ln(z_{t+1}) = \rho \ln(z_t) + \varepsilon_{t+1},$$

with persistence parameter $0 \leq \rho < 1$, where the serially uncorrelated innovation ε_{t+1} satisfies $E_t(\varepsilon_{t+1}) = 0$. Hence, the value of z_t is known when c_t, i_t, and k_{t+1} are chosen during period t, but the value of z_{t+1}, though partly forecastable via the autoregression, still contains the element of randomness due to ε_{t+1}.

To solve the social planner's problem using dynamic programming, it is helpful to combine the aggregate resource constraint and the capital accumulation constraint to obtain

$$z_t(k_t^\alpha - c_t) \geq k_{t+1}.$$

The Bellman equation then takes the general form

$$v(k_t, \ln(z_t)) = \max_{c_t} \ln(c_t) + \beta E_t v\left[z_t(k_t^\alpha - c_t), \rho \ln(z_t) + \varepsilon_{t+1}\right].$$

Based on past experience with problems of this kind, let's guess that the value function takes the form

$$v(k_t, \ln(z_t)) = E + F\ln(k_t) + G\ln(z_t).$$

The Bellman equation then becomes

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F\ln(z_t) + \beta F\ln(k_t^\alpha - c_t) + \beta G\rho\ln(z_t),$$

where, on the right-hand side, use has been made of the assumption that z_t is known time t and that

$$E_t[\ln(z_{t+1})] = \rho\ln(z_t).$$

a. Derive (write down) the first-order condition for c_t the envelope condition for k_t that help characterize the solution to the social planner's problem.

b. Next, use the first-order condition for c_t, the envelope condition for k_t, and the binding constraint

$$k_{t+1} = z_t(k_t^\alpha - c_t)$$

to derive expressions that show how the constant F from the value function depends on the parameters α and β, how period-t consumption c_t depends on the parameters α and β as well as the period-t capital stock k_t, and how period-$t+1$ capital k_{t+1} depends on the parameters α and β as well as the period-t capital stock k_t and the investment-specific productivity shock z_t.

c. Finally, substitute your solutions for F, c_t, and k_t back into the Bellman equation and, by observing that the equation must hold for all values of $\ln(z_t)$, find solutions for the constant E in terms of the parameters α and β and the constant G in terms the parameters α, β, and ρ.

Solutions to Final Exam

ECON 772001 - Math for Economists — Peter Ireland
Boston College, Department of Economics — Fall 2025

Due Thursday, December 18

1. The Community Lake

Given the initial stock of water $w(0)$ in the lake, the community planners choose $c(t)$ for all $t \in [0, \infty)$ and $w(t)$ for all $t \in (0, \infty)$ to maximize the utility function

$$\int_0^\infty e^{-\rho t}[\ln(c(t)) + \alpha \ln(w(t))]\, \mathrm{d}t,$$

subject to the constraint

$$R - c(t) \geq \dot{w}(t),$$

for all $t \in [0, \infty)$

a. With the maximized Hamiltonian for the planners' problem written in current value form as

$$H(w(t), \theta(t)) = \max_{c(t)} \ln(c(t)) + \alpha \ln(w(t)) + \theta(t)[R - c(t)],$$

the maximum principle implies that the solution to the problem is described by the first-order condition

$$\frac{1}{c(t)} - \theta(t) = 0$$

and the pair of differential equations

$$\dot{\theta}(t) = \rho\theta(t) - \frac{\alpha}{w(t)}$$

and

$$\dot{w}(t) = R - c(t).$$

b. Rewrite the first-order condition as

$$1 = \theta(t)c(t)$$

and differentiate with respect to t to get

$$0 = \dot{\theta(t)}c(t) + \theta(t)\dot{c}(t).$$

Now, use the differential equation to substitute out for $\dot{\theta}(t)$; this yields

$$0 = \rho\theta(t)c(t) - \frac{\alpha c(t)}{w(t)} + \theta(t)\dot{c}(t).$$

Finally, use the first-order condition again to replace $\theta(t)$ with $1/c(t)$:

$$0 = \rho - \frac{\alpha c(t)}{w(t)} + \frac{\dot{c}(t)}{c(t)}.$$

Rearranging this last expression yields the Euler equation

$$\dot{c}(t) = \left[\frac{\alpha c(t)}{w(t)} - \rho\right] c(t)$$

that combines with the water accumulation constraint

$$\dot{w}(t) = R - c(t)$$

to describe the behavior of consumption of water and the level of the lake water along their optimal paths.

c. The water accumulation constraint implies that in a steady-state with $\dot{w}(t) = 0$, consumption must be constant and equal to

$$c^* = R.$$

Intuitively, consuming water in line with the flow of rainfall leaves the water level in the lake unchanged. The Euler equation then provides the solution for the steady-state level of lake water:

$$w^* = \frac{\alpha R}{\rho}.$$

Intuitively, the optimal level of lake water depends positively on the weight α on $w(t)$ in utility, positively on the flow of rainfall R, and negatively on the discount rate ρ as a measure of impatience.

The phase diagram for this model reveals that if the initial water level $w(0) = w_0$ is less than the steady-state level w^*, initial consumption must also be below its steady-state level c^*. Along the stable manifold, consumption and the water level both rise as they converge to their steady-state levels. Conversely, if the initial water level $w(0) = w_0$ is above the steady-state level w^*, initial consumption must also be above its steady-state level c^*. Along the stable manifold, consumption and the water level both fall as they converge to their steady-state levels.

2. Investment and the Business Cycle

Using the conjectured form of the value, function the Bellman equation for the social planner's problem is

$$E + F\ln(k_t) + G\ln(z_t) = \max_{c_t} \ln(c_t) + \beta E + \beta F\ln(z_t) + \beta F\ln(k_t^\alpha - c_t) + \beta G\rho\ln(z_t).$$

a. The first-order condition for c_t is

$$\frac{1}{c_t} - \frac{\beta F}{k_t^\alpha - c_t} = 0$$

and the envelope condition for k_t is

$$\frac{F}{k_t} = \frac{\alpha\beta F k_t^{\alpha-1}}{k_t^\alpha - c_t}.$$

b. The first-order condition for c_t implies

$$c_t = \left(\frac{1}{1+\beta F}\right) k_t^\alpha.$$

Substituting this expression for c_t into the envelope condition for k_t yields the solution for the constant F:

$$F = \frac{\alpha}{1-\alpha\beta}.$$

Substituting this solution for F into the first-order condition for c_t and the binding constraint for k_{t+1} then yields

$$c_t = (1-\alpha\beta)k_t^\alpha$$

and

$$k_{t+1} = \alpha\beta z_t k_t^\alpha.$$

c. Finally, substituting the solutions for F, c_t, and k_t back into the Bellman equation and observing that the equation must hold for all values of $\ln(z_t)$ provides the solutions for the constants E and G:

$$E = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln(\alpha\beta)\right]$$

and

$$G = \left(\frac{1}{1-\beta\rho}\right)\left(\frac{\alpha\beta}{1-\alpha\beta}\right).$$