

# ECON 337901

# FINANCIAL ECONOMICS

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## Generalizing the Portfolio Problem

We can elaborate on our previous portfolio problem

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

by allowing the investor to allocate funds to  $N > 1$  risky assets.

$\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_N$

risky (random) returns

$a_1, a_2, \dots, a_N$

amounts allocated at the risky assets

$w_i = a_i/Y_0$

share of initial wealth allocated to each risky asset (portfolio **w**eight)

## Generalizing the Portfolio Problem

$$\begin{aligned}\tilde{Y}_1 &= \text{random terminal wealth} \\ &= (1 + r_f) \left( Y_0 - \sum_{i=1}^N a_i \right) + \sum_{i=1}^N a_i (1 + \tilde{r}_i) \\ &= (1 + r_f) Y_0 + \sum_{i=1}^N a_i (\tilde{r}_i - r_f) \\ &= (1 + r_f) Y_0 + \sum_{i=1}^N w_i Y_0 (\tilde{r}_i - r_f)\end{aligned}$$

## Generalizing the Portfolio Problem

With

$$\tilde{Y}_1 = (1 + r_f)Y_0 + \sum_{i=1}^N w_i Y_0 (\tilde{r}_i - r_f),$$

the generalized problem can be stated as

$$\max_{w_1, w_2, \dots, w_N} E \left\{ u \left[ Y_0(1 + r_f) + \sum_{i=1}^N w_i Y_0 (\tilde{r}_i - r_f) \right] \right\}$$

## Generalizing the Portfolio Problem

**Modern Portfolio Theory** examines the solution to this extended problem assuming that investors have **mean-variance utility**, that is, assuming that investors' preferences can be represented by a trade-off between the mean (expected value) and variance (or standard deviation) of terminal wealth.

MPT was developed by Harry Markowitz (US, 1927-2023, Nobel Prize 1990) in the early 1950s, the classic paper being his article "Portfolio Selection," *Journal of Finance* Vol.7 (March 1952): pp.77-91.

## Justifying Mean-Variance Utility

The mean-variance utility hypothesis seemed natural at the time the MPT first appeared, and it retains some intuitive appeal today. But viewed in the context of more recent developments in financial economics, particularly the development of vN-M expected utility theory, it now looks a bit peculiar.

A first question for us, therefore, is: Under what conditions will investors have preferences over the means and variances of asset returns?

# Justifying Mean-Variance Utility

Under what conditions will investors have preferences over the means and variances of asset returns?

There are three arguments:

1. Quadratic approximation to a general Bernoulli utility function.
2. Quadratic Bernoulli utility function.
3. Asset returns are normally distributed.

## Justifying Mean-Variance Utility

The first argument uses a quadratic approximation:

$$f(x + a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2$$

“Decompose” terminal wealth  $\tilde{Y}_1$  as

$$\tilde{Y}_1 = E(\tilde{Y}_1) + [\tilde{Y}_1 - E(\tilde{Y}_1)]$$

where  $x + a = \tilde{Y}_1$ ,  $x = E(\tilde{Y}_1)$ , and

$$a = \tilde{Y}_1 - E(\tilde{Y}_1)$$

is the size of the “bet.”

## Justifying Mean-Variance Utility

With  $x + a = \tilde{Y}_1$ ,  $x = E(\tilde{Y}_1)$ , and  $a = \tilde{Y}_1 - E(\tilde{Y}_1)$ ,

$$f(x + a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2$$

$$\begin{aligned} u(\tilde{Y}_1) &\approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)] \\ &\quad + \frac{1}{2}u''[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)]^2 \end{aligned}$$

## Justifying Mean-Variance Utility

If  $\tilde{X}$  is random and  $a$  is known, then  $E(a\tilde{X}) = aE(\tilde{X})$ .

Therefore

$$u(\tilde{Y}_1) \approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)] \\ + \frac{1}{2}u''[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)]^2$$

implies that an approximation to expected utility is

$$E[u(\tilde{Y}_1)] \approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)]E[\tilde{Y}_1 - E(\tilde{Y}_1)] \\ + \frac{1}{2}u''[E(\tilde{Y}_1)]E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2$$

## Justifying Mean-Variance Utility

Since

$$E[\tilde{Y}_1 - E(\tilde{Y}_1)] = 0 \text{ and } E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2 = \sigma^2(\tilde{Y}_1)$$

$$\begin{aligned} E[u(\tilde{Y}_1)] &\approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)]E[\tilde{Y}_1 - E(\tilde{Y}_1)] \\ &\quad + \frac{1}{2}u''[E(\tilde{Y}_1)]E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2 \end{aligned}$$

simplifies to

$$E[u(\tilde{Y}_1)] \approx u[E(\tilde{Y}_1)] + \frac{1}{2}u''[E(\tilde{Y}_1)]\sigma^2(\tilde{Y}_1)$$

## Justifying Mean-Variance Utility

$$E[u(\tilde{Y}_1)] \approx u[E(\tilde{Y}_1)] + \frac{1}{2}u''[E(\tilde{Y}_1)]\sigma^2(\tilde{Y}_1)$$

The right-hand side of this expression is in the desired form: if  $u$  is increasing, it rewards higher **mean returns** and if  $u$  is concave, it penalizes higher **variance** in returns.

So one possible justification for mean-variance utility is to assume that the size of the portfolio bet  $\tilde{Y}_1 - E(\tilde{Y}_1)$  is small enough to make this Taylor approximation a good one.

But is it safe to assume that portfolio bets are small?

## Justifying Mean-Variance Utility

A second argument is to assume that the Bernoulli utility function is **quadratic**, so that the quadratic approximation holds exactly, even for large bets.

With

$$u(Y) = a + bY + cY^2,$$

$$u'(Y) = b + 2cY$$

and

$$u''(Y) = 2c$$

mean that  $c < 0$  for risk aversion and  $b > 0$  if more is preferred to less.

## Justifying Mean-Variance Utility

Note, however, that  $u'(Y) = b + 2cY$  and  $u''(Y) = 2c$  imply

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = -\frac{2c}{b + 2cY} = -2c(b + 2cY)^{-1}$$

and therefore

$$R'_A(Y) = 2c(b + 2cY)^{-2}(2c) = \left(\frac{2c}{b + 2cY}\right)^2 > 0$$

so that quadratic utility implies **increasing** absolute risk aversion.