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FINANCIAL ECONOMICS

Peter Ireland

Boston College

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Interpreting the Measures of Risk Aversion

Absolute risk aversion describes an investor's attitude towards absolute bets of plus or minus h.

A similar analysis shows that relative risk aversion describes an investor's attitude towards relative bets of plus or minus kY, so that now, k is a fraction of total income.

Interpreting the Measures of Risk Aversion

Consider an investor with initial income Y who is offered a bet: win kY with probability π and lose kY with probability $1 - \pi$.

A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

Let π^* be the probability that is just high enough to get the investor to accept the bet.

Now π^* must satisfy

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*) u(Y - Yk).$$

Interpreting the Measures of Risk Aversion

Take second-order Taylor approximations to $u(Y + Yk)$ and $u(Y - Yk)$:

$$u(Y + Yk) \approx u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$u(Y - Yk) \approx u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*) u(Y - Yk)$$

$$u(Y + Yk) \approx u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$u(Y - Yk) \approx u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

imply

$$\begin{aligned} u(Y) \;\approx\; & \pi^* \left[u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2\right] \\ & + (1 - \pi^*) \left[u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2\right] \end{aligned}$$

Interpreting the Measures of Risk Aversion

$$\begin{aligned} u(Y) &\approx \pi^* \left[u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2 \right] \\ &+ (1-\pi^*) \left[u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2 \right] \end{aligned}$$

implies

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$0 \approx (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$0 \approx (2\pi^* - 1)u'(Y) + \frac{1}{2}u''(Y)Yk$$

$$2\pi^* u'(Y) \approx u'(Y) - \frac{1}{2}u''(Y)Yk$$

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{Yu''(Y)}{u'(Y)}\right]k$$

Interpreting the Measures of Risk Aversion

Since

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion,}$$

it follows from these calculations that

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{Yu''(Y)}{u'(Y)}\right]k = \frac{1}{2} + \frac{1}{4}kR_R(Y) > \frac{1}{2}.$$

The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus kY relates directly to the coefficient of relative risk aversion.

Interpreting the Measures of Risk Aversion

Suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus one percent ($k = 0.01$) of your income?

And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $k = 0.01$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} k R_R(Y)$$

implies

$$0.75 \approx 0.50 + \frac{0.01}{4} R_R(Y)$$

$$0.25 \approx 0.0025 R_R(Y)$$

$$R_R(Y) \approx \frac{0.25}{0.0025} = 100.$$

Interpreting the Measures of Risk Aversion

Again, our notation $R_R(Y)$ allows relative risk aversion to depend on income Y.

On the other hand, since the coefficient of relative risk aversion describes aversion to risk over bets that are expressed relative to income, it is more plausible to assume that investors have constant relative risk aversion.

Interpreting the Measures of Risk Aversion

Suppose, therefore, that the Bernoulli utility function takes the form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

where $\gamma > 0$. For this function, Guillaume de l'Hôpital's (France, 1661-1704) rule implies that when $\gamma = 1$

$$\frac{Y^{1-\gamma} - 1}{1 - \gamma} = \ln(Y),$$

where ln denotes the natural logarithm. This was the form that Daniel Bernoulli used to describe preferences over payoffs.

Interpreting the Measures of Risk Aversion

With

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

it follows that

$$u'(Y) = Y^{-\gamma}$$

$$u''(Y) = -\gamma Y^{-\gamma-1}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \frac{Y\gamma Y^{-\gamma-1}}{Y^{-\gamma}} = \gamma,$$

so that this utility function displays constant relative risk aversion, which does not depend on income.

Interpreting the Measures of Risk Aversion

So if we were willing to make the assumption of constant relative risk aversion, we could use the results from our example, where an investor requires $\pi^* = 0.75$ to accept a bet with $k = 0.01$ to set $\gamma = 100$ in

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

and thereby tailor portfolio decisions specifically for this investor.

Certainty Equivalent

Suppose your income is $Y = 50000$ and you are offered a risky asset that pays 50000 with probability $1/2$ and 0 with probability $1/2$.

This asset has

$$E(\tilde{Z}) = (1/2)50000 + (1/2)0 = 25000.$$

Certainty Equivalent

Which would you choose: taking the risk with $\tilde{Z}$, or receiving $E(\tilde{Z})$ for sure?

No risk averse investor would give up $E(\tilde{Z})$ for sure to take the risk.

But suppose the risk-free alternative was less attractive? How much less would you accept to avoid risk?

Certainty Equivalent

The "certainty equivalent" $CE(\tilde{Z})$ is the smallest amount than an investor would accept, for sure, to avoid the risk associated with $\tilde{Z}$.

In this case, where $\tilde{Z}$ is a coin flip over 50000 versus nothing, $CE(\tilde{Z})$ satisfies

$$u[Y + CE(\tilde{Z})] = (1/2)u(Y + 50000) + (1/2)u(Y)$$

$$u[50000 + CE(\tilde{Z})] = (1/2)u(100000) + (1/2)u(50000)$$

Certainty Equivalent

$$u[50000 + CE(\tilde{Z})] = (1/2)u(100000) + (1/2)u(50000)$$

Suppose that

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

so that relative risk aversion equals γ.

Certainty Equivalent

$$u[50000 + CE(\tilde{Z})] = (1/2)u(100000) + (1/2)u(50000)$$

$$\frac{[50000 + CE(\tilde{Z})]^{1-\gamma} - 1}{1-\gamma}$$
$$= \quad \frac{1}{2}\left(\frac{100000^{1-\gamma} - 1}{1-\gamma}\right) + \frac{1}{2}\left(\frac{50000^{1-\gamma} - 1}{1-\gamma}\right)$$

$$[50000 + CE(\tilde{Z})]^{1-\gamma} = (1/2)(100000^{1-\gamma}) + (1/2)(50000^{1-\gamma})$$

$$CE(\tilde{Z}) = [(1/2)(100000^{1-\gamma}) + (1/2)(50000^{1-\gamma})]^{\frac{1}{1-\gamma}} - 50000$$

Assessing the Level of Risk Aversion

Certainty equivalent for an asset that pays 50000 with probability 1/2 and 0 with probability 1/2 when income is 50000 and the coefficient of relative risk aversion is γ.

γ	$CE(\tilde{Z})$	
0	25000	("risk neutrality," Pascal)
1	20711	(log utility, D Bernoulli)
2	16667	
3	13246	
4	10571	
5	8566	
10	3991	
20	1858	
50	712	