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FINANCIAL ECONOMICS

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Expected Utility Functions

We have already seen that concavity of the Bernoulli utility function u in the expected utility function

$$U(p_G, p_B) = \pi u(p_G) + (1 - \pi) u(p_B)$$

captures the investor's risk aversion.

In fact, this insight was first made by Gabriel Cramer and Daniel Bernoulli in the 1700s.

Expected Utility Functions

Concavity of the Bernoulli utility function u in the expected utility function

$$U(p_G, p_B) = \pi u(p_G) + (1 - \pi)u(p_B)$$

captures the investor's risk aversion.

Since concavity is reflected by the condition

$$u''(p) < 0$$

is it the case that a "more concave" Bernoulli utility function as reflected in a "more negative" value of $u''(p)$ captures a greater degree of risk aversion?

Expected Utility Functions

Unfortunately, no. To see why, suppose that an investor's preferences are represented by the expected utility function U, so that

The investor says "I prefer p^1 to p^2"

if and only if

$$
\begin{aligned}
U(p_G^1, p_B^1) &= \pi u(p_G^1) + (1-\pi)u(p_B^1) \\
&> \pi u(p_G^2) + (1-\pi)u(p_B^2) \\
&= U(p_G^2, p_B^2)
\end{aligned}
$$

Expected Utility Functions

Since

$$\begin{aligned} V(p_G, p_B) &= \alpha U(p_G, p_B) \\ &= \alpha[\pi u(p_G) + (1-\pi)u(p_B)] \\ &= \pi\alpha u(p_G) + (1-\pi)\alpha u(p_B) \\ &= \pi v(p_G) + (1-\pi)v(p_B) \end{aligned}$$

where $v(p) = \alpha u(p)$ and therefore $v''(p) = \alpha u''(p)$.

By choosing α to be large or small, we can make $v''(p)$ "more" or "less" negative, yet the underlying degree of risk aversion – that is, the investor's preferences – are the same.

Expected Utility Functions

Because, in economics, utility is an ordinal, not a cardinal, concept, we will have to work a little harder to quantify risk aversion in the expected utility framework.

Measuring Risk Aversion

In the mid-1960s, Kenneth Arrow and John Pratt proposed two alternative measures of risk aversion that are immune to this problem:

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

where Y measures the investor's income level.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

Absolute risk aversion applies to bets over absolute dollar amounts: $\pm$ \$1000.

Relative risk aversion applies to bets expressed relative to (as a fraction of) income: $\pm$ 1 percent of Y.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

Since $v(p) = \alpha u(p)$ implies $v'(p) = \alpha u'(p)$ and $v''(p) = \alpha u''(p)$, these measures are invariant to arbitrary rescalings of the Bernoulli utility function.

Measuring Risk Aversion

Two alternative measures of risk aversion are

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

where Y measures the investor's income level.

And since both measures have a minus sign out in front, both are positive and increase when risk aversion rises.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

The notation $R_A(Y)$ and $R_R(Y)$ emphasizes that both measures of risk aversion can depend on the investor's income Y. A given bet can seem more or less risky, depending on the investor's income.

Interpreting the Measures of Risk Aversion

To interpret the two measures of risk aversion, it is helpful to recall from calculus the theorem stated by Brook Taylor (England, 1685-1731), regarding the approximation of a function f using its derivatives: the "first-order" approximation

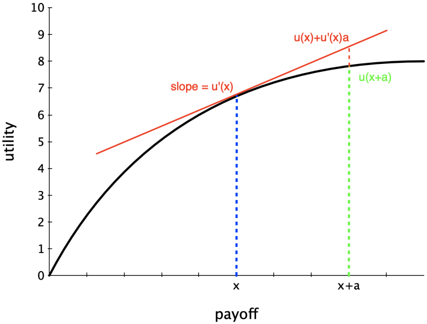
$$f(x+a) \approx f(x) + f'(x)a$$

and the "second-order" approximation

$$f(x+a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2.$$

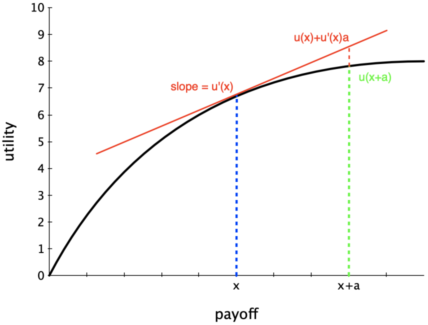
The second-order approximation is more accurate than the first, and both become more accurate as a becomes smaller.

Interpreting the Measures of Risk Aversion



The first-order (linear) approximation $u(x+a) \approx u(x) + u'(x)a$ overstates $u(x+a)$ when u is concave.

Interpreting the Measures of Risk Aversion



Since $u''(x) < 0$, the second-order (quadratic) approximation $u(x + a) \approx u(x) + u'(x)a + (1/2)u''(x)a^2$ will be more accurate.

Interpreting the Measures of Risk Aversion

Focusing first on the measure of absolute risk aversion, consider an investor with initial income Y who is offered a bet: win h with probability π and lose h with probability $1 - \pi$.

A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

Let π^* be the probability that is just high enough to get the investor to accept the bet.

Then π^* must satisfy

$$u(Y) = \pi^* u(Y + h) + (1 - \pi^*) u(Y - h).$$

Interpreting the Measures of Risk Aversion

Take second-order Taylor approximations to $u(Y + h)$ and $u(Y - h)$:

$$u(Y + h) \approx u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$u(Y - h) \approx u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) = \pi^* u(Y+h) + (1-\pi^*)u(Y-h)$$
$$u(Y+h) \approx u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2$$
$$u(Y-h) \approx u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2$$

imply

$$\begin{aligned} u(Y) \;\approx\; & \pi^* \left[u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2\right] \\ & + (1-\pi^*) \left[u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2\right] \end{aligned}$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx \pi^* \left[u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2 \right] + (1-\pi^*) \left[u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2 \right]$$

implies

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$0 \approx (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$0 \approx (2\pi^* - 1)u'(Y) + \frac{1}{2}u''(Y)h$$

$$2\pi^* u'(Y) \approx u'(Y) - \frac{1}{2}u''(Y)h$$

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{u''(Y)}{u'(Y)}\right]h$$

Interpreting the Measures of Risk Aversion

Since

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion,}$$

it follows from these calculations that

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{u''(Y)}{u'(Y)}\right] h = \frac{1}{2} + \frac{1}{4} h R_A(Y) > \frac{1}{2}.$$

The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus h relates directly to the coefficient of absolute risk aversion.

Interpreting the Measures of Risk Aversion

As an example, suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus $h = \$1000$?

And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $h = \$1000$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} h R_A(Y)$$

implies

$$0.75 \approx 0.50 + \frac{1000}{4} R_A(Y)$$

$$0.25 \approx 250 R_A(Y)$$

$$R_A(Y) \approx \frac{0.25}{250} = 0.001.$$

Interpreting the Measures of Risk Aversion

Realistically, a bet over \$1000 is probably going to seem more risky to someone who starts out with less income.

In general, $R_A(Y)$ can depend on Y. More specifically, it seems likely that $R_A(Y)$ decreases when Y goes up, so that

$$R_A'(Y) < 0.$$