ECON 337901
FINANCIAL ECONOMICS

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Black-Scholes Option Pricing

The Black-Scholes formula is derived using methods in stochastic calculus developed by Kiyoshi Ito (Japan, 1915-2008) in the 1940s and early 1950s to "take the limit" as the number of subperiods and the number of final states become infinite and the distribution of final states becomes normal.

The derivation is tedious and the formula looks daunting at first glance. But we can get a feel for the intuition with reference to dynamic hedging and the binomial tree.

Black-Scholes Option Pricing

Let T denote the number of years until the option expires (can be a fraction) and σ denote the standard deviation of the stock's annual return (must be estimated).

Let r_b denote the "continuously compounded annual interest rate" on a risk-free discount bond that pays off one dollar T years from now and sells at price

$$q^b = e^{-r_b T}$$

today. This interest rate is assumed to be constant over the life of the option.

Black-Scholes Option Pricing

The Black-Scholes formula takes the form

$$q^o = N_1 q^s - N_2 K e^{-r_b T} = N_1 q^s - N_2 K q^b$$

Written in this way, the equation reveals that the option price q^o equals the cost of assembling a portfolio consisting of a long position $s = N_1$ shares and a short position of $b = -N_2 K$ bonds, at the beginning of a dynamic hedging routine with an infinite number of stages.

Black-Scholes Option Pricing

In the Black-Scholes formula

$$q^o = N_1 q^s - N_2 K e^{-r_b T} = N_1 q^s - N_2 K q^b,$$

the numbers $N_1 = F(d_1)$ and $N_2 = F(d_2)$ are computed with

$$d_1 = \frac{\ln(q^s/K) + (r_b + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

where F is the standard normal cumulative distribution function, so that $F(X)$ measures the probability that a random variable that is normally distributed with mean zero and variance one turns out to be less than or equal to X.

Black-Scholes Option Pricing

To use the Black-Scholes formula

$$q^o = N_1 q^s - N_2 K e^{-r_b T} = N_1 q^s - N_2 K q^b$$

where $N_1 = F(d_1)$ and $N_2 = F(d_2)$,

$$d_1 = \frac{\ln(q^s/K) + (r_b + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

you need an estimate of σ, the standard deviation of the return on the stock.

Black-Scholes Option Pricing

Alternatively, if you see the price of a traded option, you can use the Black-Scholes formula

$$q^o = N_1 q^s - N_2 K e^{-r_b T} = N_1 q^s - N_2 K q^b$$

where $N_1 = F(d_1)$ and $N_2 = F(d_2)$,

$$d_1 = \frac{\ln(q^s/K) + (r_b + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

to estimate σ, the standard deviation of the return on the stock. In fact, the VIX volatility index is based on the σ implied by the prices of options on the S&P 500.

Using Options to Infer Contingent Claims Prices

Notice that call options often have payoff structures that resemble those of contingent claims: making positive payoffs in "good" states and expiring worthless in "bad" states.

Douglas Breeden (US, b.1950) and Robert Litzenberger (US, b.1943) devised a way of using option prices to infer contingent claims prices, allowing for many states of the world.

Douglas Breeden and Robert Litzenberger, "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business* Vol.51 (October 1978): pp.621-651.

Using Options to Infer Contingent Claims Prices

Suppose there are N states, corresponding to different levels of the S&P 500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

with $\delta > 0$.

Using Options to Infer Contingent Claims Prices

For each state i, construct a "butterfly" portfolio of call options:

Buy one calls with strike price P^{i-1}

Write (sell short) two calls with strike price P^{i}

Buy one call with strike price P^{i+1}

If q_o^i is the price of an option with strike price P^i, this portfolio costs

$$q_o^{i-1} + q_o^{i+1} - 2q_o^i$$

Using Options to Infer Contingent Claims Prices

Now let's compute the portfolio's payoffs:

S&P 500	Long $K = P^{i-1}$	Short 2 $K = P^{i}$	Long $K = P^{i+1}$	Total
$P \leq P^{i-1}$	0	0	0	0
$P = P^{i}$	$P^{i} - P^{i-1}$	0	0	δ
$P \geq P^{i+1}$	$P - P^{i-1}$	$-2(P - P^{i})$	$P - P^{i+1}$	0

$$
\begin{aligned}
& (P - P^{i-1}) - 2(P - P^{i}) + (P - P^{i+1}) \\
= \ & 2P^{i} - P^{i-1} - P^{i+1} \\
= \ & 2P^{i} - (P^{i} - \delta) - (P^{i} + \delta) = 0
\end{aligned}
$$

Using Options to Infer Contingent Claims Prices

S&P 500	Long $K = P^{i-1}$	Short 2 $K = P^{i}$	Long $K = P^{i+1}$	Total
$P \leq P^{i-1}$	0	0	0	0
$P = P^{i}$	$P^{i} - P^{i-1}$	0	0	δ
$P \geq P^{i+1}$	$P - P^{i-1}$	$-2(P - P^{i})$	$P - P^{i+1}$	0

Since the portfolio's payoffs replicate those from δ claims for state i, the price q_{cc}^{i} of a contingent claim for state i must satisfy

$$q_{cc}^{i} = (1/\delta)(q_{o}^{i-1} + q_{o}^{i+1} - 2q_{o}^{i})$$

Using Options to Infer Contingent Claims Prices

$$q^i_{cc} = (1/\delta)(q^{i-1}_o + q^{i+1}_o - 2q^i_o)$$

Additional accuracy can be achieved by choosing smaller values of δ, that is, by using a "finer grid" to define the states.

Options trade on the S&P 500 with many strike prices, so data on q^i_o are readily available.

Note that you don't have to actually trade the options to price the contingent claims.