# ECON 337901
# FINANCIAL ECONOMICS

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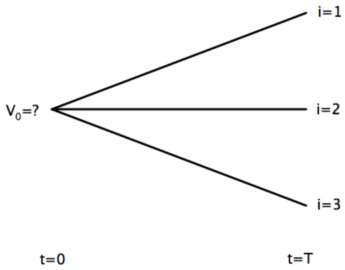
# Black-Scholes Option Pricing

Black and Scholes and Merton considered a more general setting, in which the option priced at $t = 0$ does not expire until $t = T$.

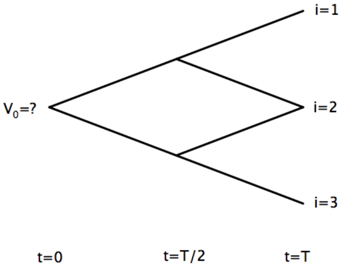
They also allowed for (many) more than two possible states at $t = T$.

# Black-Scholes Option Pricing



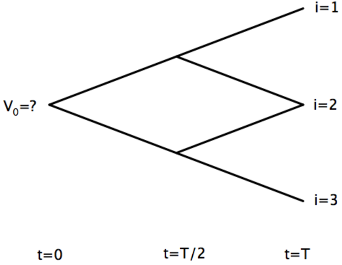

The technical problem is that with more than two states at $t = T$, more than two assets are needed to create a portfolio with the same payoffs as the option.

# Black-Scholes Option Pricing



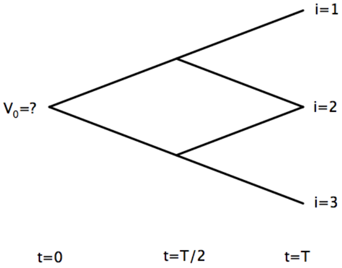

Black and Scholes and Merton realized that this problem can be solved by breaking the full period into sub-periods, so that there are only two states in each sub-period.

# Black-Scholes Option Pricing



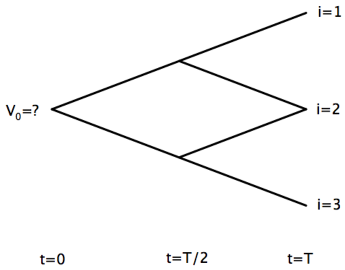

With three states at $t = T$, only two subperiods are needed, but with many states at $t = T$, many subperiods are needed.

# Black-Scholes Option Pricing




A dynamic hedging strategy can then be used to track the payoffs on the option using a portfolio consisting only of the stock and bond ...

# Black-Scholes Option Pricing



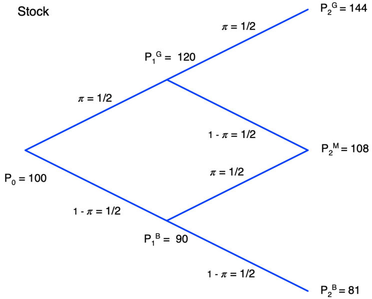

. . . but where the number of shares and the number of bonds must be adjusted in each subperiod so that the portfolio can continue to track the option's payoffs.

# Black-Scholes Option Pricing

As an example of how to implement dynamic hedging and price an option with more than two final states, set $T = 2$ and use two subperiods.

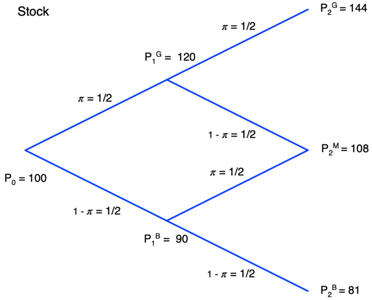
Then there will be three periods $t = 0$, $t = 1$, and $t = 2$, a good and bad state at $t = 1$, and a good, medium, and bad state at $t = 2$.

# Black-Scholes Option Pricing



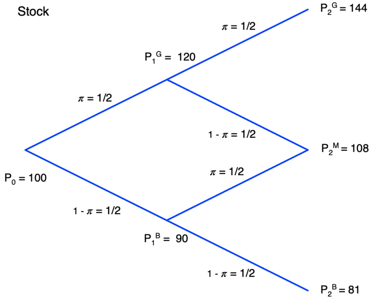

Suppose the stock price starts at $P_0 = 100$, and moves up by 20 percent or down by 10 percent with equal probability in each subperiod.

# Black-Scholes Option Pricing



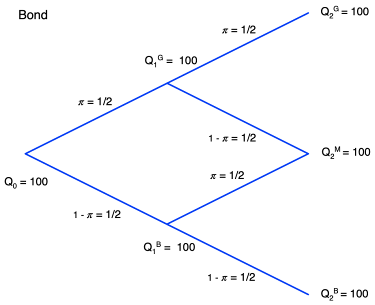

Notice that these assumptions make the middle state more likely than the good or bad at $t = 2$.

# Black-Scholes Option Pricing



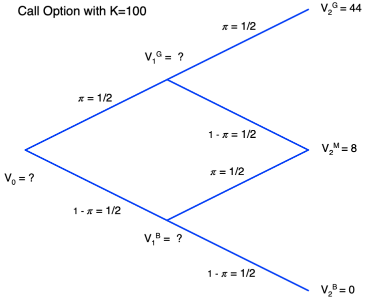

In fact, as the number of subperiods on the binomial tree grows larger, the distribution of final states will start to look more and more like the normal distribution.

# Black-Scholes Option Pricing



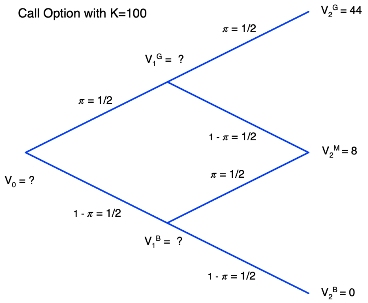

Assume for simplicity that the bond price stays constant at 100, that is, the interest rate is zero.

# Black-Scholes Option Pricing



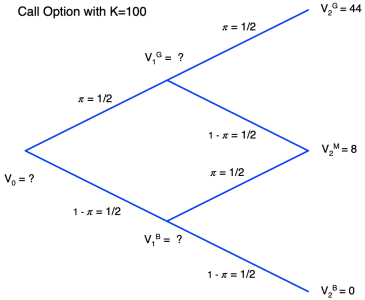

A call option with $K = 100$ and expiration $t = 2$ will be in the money in the good and medium states, but out of the money in the bad state.

# Black-Scholes Option Pricing




We can use dynamic hedging and "backwards recursion" to determine the option values $V_1^G$ and $V_2^B$ at $t = 1$ and then $V_0$ at $t = 0$.

# Black-Scholes Option Pricing




Focus first on the good state at $t = 1$.

# Black-Scholes Option Pricing

Focus first on the good state at $t = 1$:

The stock price is $P_1^G = 120$ and can rise to $P_2^G = 144$ or fall to $P_2^M = 108$.

The bond price is $Q_1^G = 100$ and remains at $Q_2^G = Q_2^M = 100$ no matter what.

The option price is $V_1^G = ?$ and can rise to $V_2^G = 44$ or fall to $V_2^M = 8$.

# Black-Scholes Option Pricing

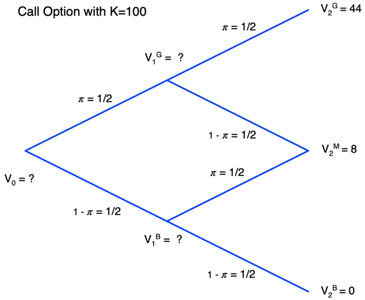
Form a portfolio of $s$ shares and $b$ bonds to replicate the option's payoffs going from the good state at $t = 1$ to either the good or medium state at $t = 2$:

$$44 = 144s + 100b$$
$$8 = 108s + 100b$$

No arbitrage then requires

$$V_1^G = P_1^G s + Q_1^G s = 120s + 100b.$$

# Black-Scholes Option Pricing




Now move down to the bad state at $t = 1$.

# Black-Scholes Option Pricing

Move down to the bad state at $t = 1$:

The stock price is $P_1^B = 90$ and can rise to $P_2^M = 108$ or fall to $P_2^B = 81$.

The bond price is $Q_1^B = 100$ and remains at $Q_2^M = Q_2^B = 100$ no matter what.

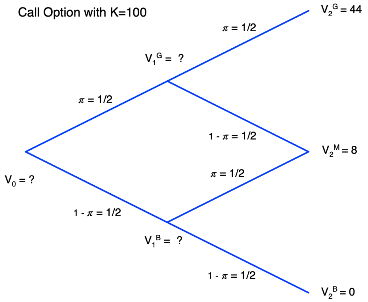
The option price is $V_1^B = ?$ and can rise to $V_2^M = 8$ or fall to $V_2^B = 0$.

# Black-Scholes Option Pricing

Form a portfolio of $s$ shares and $b$ bonds to replicate the option's payoffs going from the bad state at $t = 1$ to either the medium or bad state at $t = 2$.

Then find the option price $V_1^B$ in the bad state at $t = 1$ implied by no arbitrage.

# Black-Scholes Option Pricing




Finally, move back to $t = 0$, having filled in the values for $V_1^G$ and $V_1^B$.

# Black-Scholes Option Pricing

Move back to $t = 0$:

The stock price is $P_0 = 100$ and can rise to $P_1^G = 120$ or fall to $P_1^B = 90$.

The bond price is $Q_0 = 100$ and remains at $Q_1^G = Q_1^B = 100$ no matter what.

The option price is $V_0 = ?$ and can rise to $V_1^G$ or fall to $V_1^B$.

# Black-Scholes Option Pricing

Form a portfolio of $s$ shares and $b$ bonds to replicate the option's payoffs going from $t = 0$ to either the good or bad state at $t = 1$.

Then find the option price $V_0$ at $t = 0$ implied by no arbitrage.