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FINANCIAL ECONOMICS

Peter Ireland

Boston College

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Consumer Optimization: The Risk Dimension

In the 1950s and 1960s, Kenneth Arrow (US, 1921-2017, Nobel Prize 1972) and Gerard Debreu (France, 1921-2004, Nobel Prize 1983) extended consumer theory to accommodate risk and uncertainty.

To do so, they drew on earlier ideas developed by others, but added important insights of their own.

Building Blocks of Arrow-Debreu Theory

1. Fisher's (1930) intertemporal model of consumer decision-making.
2. From probability theory: uncertainty described with reference to "states of the world." (Andrey Kolmogorov, 1930s).
3. Expected utility theory (John von Neumann and Oskar Morgenstern, 1947).
4. Contingent claims – stylized financial assets – a powerful analytic device of their own invention.

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To be more specific about the source of risk, let's suppose that there are two possible outcomes for income next year, good and bad:

$$Y_0 = \text{income today}$$

$$Y_1^G = \text{income next year in the "good" state}$$

$$Y_1^B = \text{income next year in the "bad" state}$$

where the assumption $Y_1^G > Y_1^B$ makes the "good" state good and where

$$\pi = \text{probability of the good state}$$

$$1 - \pi = \text{probability of the bad state}$$

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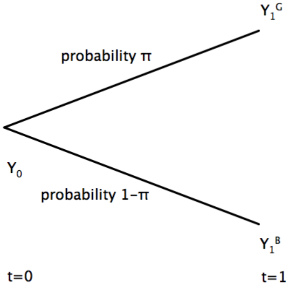

An event tree highlights randomness in income as the source of risk.

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Arrow and Debreu used the probabilistic idea of states of the world to extend Irving Fisher's work, recognizing that under these circumstances, the consumer chooses between three goods:

$$c_0 = \text{consumption today}$$
$$c_1^G = \text{consumption next year in the good state}$$
$$c_1^B = \text{consumption next year in the bad state}$$

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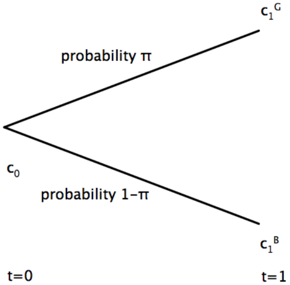

Under uncertainty, the consumer chooses consumption today and consumption in both states next year.

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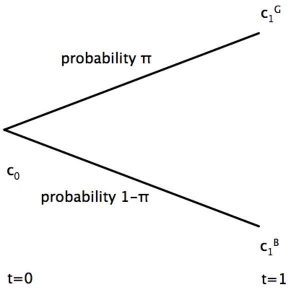

Uncertainty about future income "induces" randomness in future consumption as well.

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Suppose that the consumer's utility function is

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

so that the terms involving next year's consumption are weighted by the probability that each state will occur as well as by the discount factor β.

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In probability theory, if a random variable X can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of X is

$$E(X) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n.$$

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Hence, by assuming that the consumer's utility function is

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

we are assuming that the consumer's seeks to maximize expected utility

$$u(c_0) + \beta E[u(c_1)].$$

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But by writing out all three terms,

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

we can see that concavity of the function u, which in the standard microeconomic case represents a preference for diversity, represents here a preference for smoothness in consumption over time and across states in the future – the consumer is risk averse in the sense that he or she does not want consumption in the bad state to be too much different from consumption in the good state.

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To implement these state-contingent consumption plans, Arrow and Debreu imagined that the consumer would trade contingent claims for both future states.

A contingent claim for the good state costs q^G today, and delivers one unit of consumption next year in the good state and zero units of consumption next year in the bad state.

A contingent claim for the bad state costs q^B today, and delivers one unit of consumption next year in the bad state and zero units of consumption next year in the good state.

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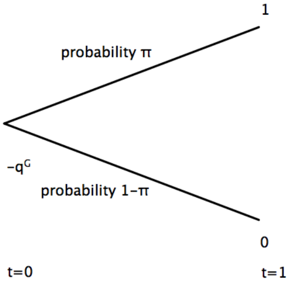

Payoffs for the contingent claim for the good state (a long position).

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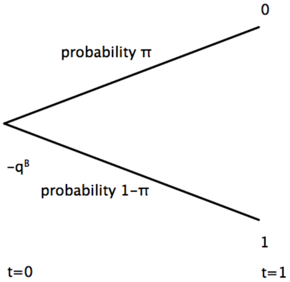

Payoffs for the contingent claim for the bad state (a long position).

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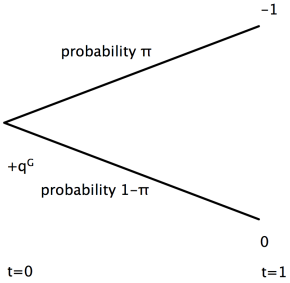

Payoffs for a short position in the contingent claim for the good state.

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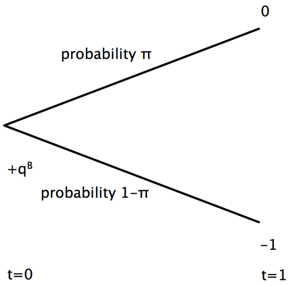

Payoffs for a short position in the contingent claim for the bad state.

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Trading Strategy	Claim	Cash Flow at $t = 0$	Cash Flow in Good State at $t = 1$	Cash Flow in Bad State at $t = 1$
Long	Good	$-q^G$	$+1$	0
Long	Bad	$-q^B$	0	$+1$
Short	Good	$+q^G$	-1	0
Short	Bad	$+q^B$	0	-1

Like a sophisticated form of saving and borrowing, where the investor can "fine-tune" the future state in which payments are received or made.