ECON 337901
FINANCIAL ECONOMICS

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Unconstrained Optimization

Recall also the “chain rule”

If

$$h(x) = f(g(x)),$$

then

$$h'(x) = f'(g(x))g'(x)$$

Unconstrained Optimization

Example:

$$f(y) = -\left(\frac{1}{2}\right) y^2$$

$$g(x) = 2 - x$$

Then

$$h(x) = f(g(x)) = -\left(\frac{1}{2}\right)(2 - x)^2$$

and

$$h'(x) = -\left(\frac{1}{2}\right) 2(2 - x)^{2-1}(-1) = 2 - x$$

Unconstrained Optimization: Example 2

Consider maximizing a function of three variables:

$$\max_{x_1, x_2, x_3} F(x_1, x_2, x_3)$$

Even if each variable can take on only 1,000 values, there are one billion possible combinations of (x_1, x_2, x_3) to search over!

This is an example of what Richard Bellman (US, 1920-1984) called the "curse of dimensionality."

Unconstrained Optimization: Example 2

Consider the problem:

$$\max_{x_1,x_2,x_3} \left(-\frac{1}{2}\right)(x_1-\tau)^2 + \left(-\frac{1}{2}\right)(x_2-x_1)^2 + \left(-\frac{1}{2}\right)(x_3-x_2)^2.$$

Now the three first-order conditions

$$-(x_1^*-\tau)+(x_2^*-x_1^*)=0$$
$$-(x_2^*-x_1^*)+(x_3^*-x_2^*)=0$$
$$-(x_3^*-x_2^*)=0$$

lead us to the solution: $x_1^*=x_2^*=x_3^*=\tau$.

Constrained Optimization

To find the value of x that solves

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

you can:

1. Try out every possible value of x.
2. Use calculus.

Since search could take forever, let's use calculus instead.

Constrained Optimization

A method for solving constrained optimization problems like

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

was developed by Joseph-Louis Lagrange (France/Italy, 1736-1813) and extended by Harold Kuhn (US, 1925-2014) and Albert Tucker (US, 1905-1995).

Constrained Optimization

Associated with the problem:

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Define the Lagrangian

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

where λ is the Lagrange multiplier.

Constrained Optimization

Then, look for a critical point of the full Lagrangian

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

instead of just the objective function F by itself.

That is, use the FOC

$$F'(x^*) - \lambda^* G'(x^*) = 0.$$

Constrained Optimization

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

Theorem (Kuhn-Tucker) If x^* maximizes $F(x)$ subject to $c \geq G(x)$, then there exists a value $\lambda^* \geq 0$ such that, together, x^* and λ^* satisfy the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

and the complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0.$$

Constrained Optimization

In the case where $c > G(x^*)$, the constraint is non-binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

requires that $\lambda^* = 0$.

And the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = 0$.

Constrained Optimization

In the case where $c = G(x^*)$, the constraint is binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

puts no further restriction on $\lambda^* \geq 0$.

Now the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = \lambda^* G'(x^*)$.

Constrained Optimization: Example 1

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 7 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 7$, and $G(x) = x$. The Lagrangian is

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x).$$

Constrained Optimization: Example 1

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x - 5)^2 + \lambda(7 - x),$$

the first-order condition

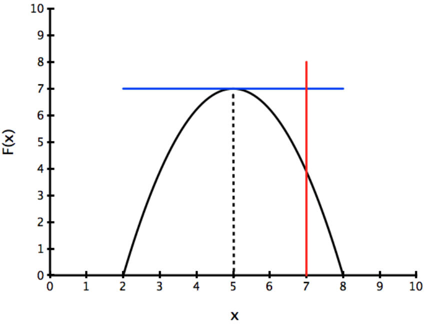
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(7 - x^*) = 0$$

are satisfied with $x^* = 5$, $F'(x^*) = 0$, $\lambda^* = 0$, and $7 > x^*$.

Constrained Optimization: Example 1



Here, the solution has $F'(x^*) = 0$ since the constraint is nonbinding.

Constrained Optimization: Example 2

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 4 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 4$, and $G(x) = x$. The Lagrangian is

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x).$$

Constrained Optimization: Example 2

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x),$$

the first-order condition

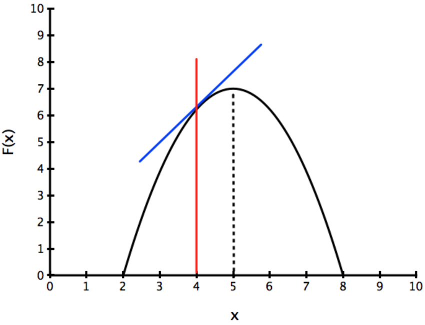
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(4 - x^*) = 0$$

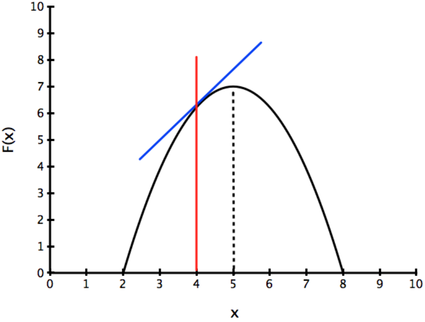
are satisfied with $x^* = 4$ and $F'(x^*) = \lambda^* = 1 > 0$.

Constrained Optimization: Example 2



Here, the solution has $F'(x^*) = \lambda^* G'(x^*) > 0$ since the constraint is binding. $F'(x^*) > 0$ indicates that we'd like to increase the value of x, but the constraint won't let us.

Constrained Optimization: Example 2



With a binding constraint, $F'(x^*) \neq 0$ but $F'(x^*) - \lambda^* G'(x^*) = 0$. The value x^* that solves the problem is a critical point, not of the objective function $F(x)$, but instead of the entire Lagrangian $F(x) + \lambda[c - G(x)]$.